Lectures on gauge theory and integrable systems

Michèle AUDIN

Institut de Recherche Mathématique Avancée
Université Louis Pasteur et C.N.R.S.
7 rue René Descartes
F-67084 Strasbourg Cedex
France

Abstract

In these notes, I will describe the moduli space of flat connections on a principal bundle over a surface and its Poisson structure. I will then give examples of integrable systems on these spaces, following ideas of Goldman, Jeffrey and Weitsman, Fock and Roslyi, and Alekseev.

I will present here some examples of integrable systems, all of them defined on the moduli space of flat connections on a trivial bundle over a surface. These examples have been constructed by Goldman, Jeffrey and Weitsman, Fock, Alekseev, so that there will be nothing new in these notes. However, it seems to me that a general presentation is lacking in the literature, and that this is a pity, as so many beautiful ideas are involved.

There are various reasons why one is willing to consider such things, and the motives really depend on the author. My two starting points will be:

1. It is interesting to understand the geometry of the moduli space. There are many ways to motivate a study of this geometry (some of them coming from physics).

2. In order to understand the geometry of a Poisson manifold, it is very helpful to have an integrable system on it.

To relate points 1 and 2, notice that the moduli space is a "Poisson manifold" (or, at least, is Poisson) and that this is a very important part of its geometric structure: for instance, the moduli spaces associated with a sphere with three holes and a torus with one hole will have the same underlying topological space but of course, one feels that these spaces should be different... and actually, they are distinguished by their Poisson structures.

To comment on point 2, let me just recall the ideal situation for an integrable system on a symplectic manifold, one which is associated with a Hamiltonian torus action: then, Delzant's theorem asserts that once you know the image of the set of functions, you know the symplectic manifold (and even more!). Of course, the general situation is more complicated.

Now the theory of integrable systems is something very interesting and rich, and very extensive as well. What seems to be attractive in the case of the moduli spaces is that these spaces are complicated enough to carry some very different systems, for instance:

- Systems related to torus actions (as in Delzant's theorem), the Jeffrey-Weitsman examples.

J. Hurtubise and F. Lalonde (eds.), Gauge Theory and Symplectic Geometry, 1–48.

- Systems looking like the famous Calogero system (arising, moreover, in the very simple case of a torus with one hole), the Fock examples.

These examples belong to the same family, the family of Goldman functions. These are functions on the moduli space associated with curves drawn on the surface. Goldman has given us a lot of commuting functions, and we manage to have enough of them. There is, however, another family of examples:

- Systems (or at least families of functions) looking like algebraically integrable systems obtained by r-matrix constructions, which can be expected to relate the geometry of the moduli space to that of a family of Jacobians of algebraic curves, as in classical examples of integrable systems coming from mechanics, the Alekseev examples.

Contents of the lectures. — For the convenience of the reader, and to set the scene, I will recall a few basic facts (definitions, constructions and examples) from the theory of integrable systems. I will also explain some classical examples, in connection with the Arnold-Liouville theorem. I will then describe the moduli space and its Poisson structure from two points of view (infinite dimensional, *à la* Atiyah-Bott, finite dimensional, *à la* Fock-Roslyi). The last part is devoted to the description of Goldman's functions and to the examples.

I will not insist on giving complete proofs, but will try to give enough hints and references.

Notice that there are many other relations between integrable systems and moduli spaces, many of them very interesting and beautiful (e.g. Hitchin's systems [25]) and which will not be discussed in these notes.

Contents

References. — Although I will recall most of the basic definitions, these notes will neither try to be self-contained nor exhaustive. However, there will hopefully be enough references to guide the reader (moreover, I have added a bibliographical guide at the end of the paper).

Language. — I must apologize if this text is in English: recall that the Séminaire de Mathématiques Supérieures is supposed to be bilingual. The only serious (?) manifestation of this bilingualism could be the French joke in Figure 8.

Acknowledgements. — I cannot thank all the people who have helped me to understand various aspects of the material presented here, among whom Y. Kosmann-Schwarzbach and A. G. Reyman, all the participants of the "Séminaire Strasbourg-Lille sur les systèmes intégrables" in 1994-1995 and of the Symplectic Semester at the Isaac Newton Institute in 1994, especially E. Prato, P. Vanhaecke and T. Wurzbacher, with whom I have had many discussions on moduli spaces, and the very stimulating audience of the Séminaire de Mathématiques Supérieures.

I have used lecture notes taken by Sophie Gérardy to write this text and the English grammar was improved by Daniel Gatien; thanks to both of them.

Special thanks are due to Volodya Fock, who helped me understand the Poisson structure and allowed me to explain some of his (not yet published) results (all the contents of § 2.2, § 2.3 and § 3.4 come from his papers and/or explanations) and to Lisa Jeffrey who read a first draft of these notes and helped me to improve it.

1 Generalities on integrable systems

1.1 Symplectic and Poisson manifolds, integrable systems

I will recall quickly some basic and classical definitions (see the bibliographical guide).

Symplectic manifolds. — A *symplectic manifold* is a manifold W endowed with a non-degenerate closed 2-form ω. Any non-degenerate 2-form allows to associate, with any function f, a vector field X_f dual to the differential i.e. such that

$$\omega(X_f, \cdot) = df(\cdot).$$

This in turn allows us to define, by $\{f,g\} = X_f \cdot g$, a skew symmetric bracket $\{\,,\,\}$ on $C^\infty(W)$ which is a derivation in each entry (i.e. satisfies the Leibniz rule).

When the 2-form is closed, this bracket satisfies the Jacobi identity, so that it endows $C^\infty(W)$ with the structure of a Lie algebra. It is then called a *Poisson bracket.*

Poisson manifolds. — A *Poisson manifold* W is a manifold endowed with a Poisson bracket $\{\,,\,\}$, namely the structure of a Lie algebra on $C^\infty(W)$ obeying the Leibniz rule

$$\{f,gh\} = \{f,g\}\,h + g\,\{f,h\}\,.$$

Notice that $\{f,\cdot\}$ is then a derivation and thus a vector field, called the *Hamiltonian vector field* of f, and denoted X_f.

Recall that the Poisson bracket can also be defined as a bivector P, that is, a section of $\Lambda^2 TW$:

$$\{f,g\}(x) = \langle P_x, df(x)\wedge dg(x)\rangle.$$

Symplectic manifolds are, of course, Poisson manifolds but there are many more Poisson manifolds than symplectic manifolds.

The basic example of a Poisson manifold is the dual $\mathfrak{g}^\star$ of a Lie algebra $\mathfrak{g}$. A Poisson structure is actually what the dual vector space $\mathfrak{g}^\star$ gets from the Lie bracket of $\mathfrak{g}$, by Kirillov's magic formula

$$\langle \xi, [X,Y]\rangle$$

and more precisely, for $f, g \in \mathcal{C}^\infty(\mathfrak{g}^\star)$ (and $\xi \in \mathfrak{g}^\star$) by

$$\{f,g\}(\xi) = \langle \xi, [df(\xi), dg(\xi)]\rangle$$

where $\langle\ ,\ \rangle$ is the canonical pairing between $\mathfrak{g}^\star$ and $\mathfrak{g}$, $df(\xi)$ and $dg(\xi)$ are linear forms on $\mathfrak{g}^\star$, thus elements of the bidual, which is identified with $\mathfrak{g}$.

There is no reason why $\mathfrak{g}^\star$ should be a symplectic manifold (half of the time, its dimension does not even have the right parity).

The symplectic foliation. — In general the Poisson bracket of a Poisson manifold W defines nothing on a submanifold $V\subset W$. If it defines a Poisson bracket, that is, if $\{f,g\}|_V$ depends only on $f|_V$ and $g|_V$, then V is said to be a Poisson submanifold. If, moreover, the restricted Poisson bracket on V can be defined by a symplectic form, then V is a symplectic submanifold.

Now, any Poisson manifold has a *symplectic foliation*, a (singular) foliation whose leaves are the maximal symplectic submanifolds. I will not discuss this in great generality, but refer the reader to [31] or [39] for instance. However, in the case of the dual $\mathfrak{g}^\star$ of the Lie algebra of a group G, this is quite simple: the symplectic leaves are just the coadjoint orbits. This is an easy exercise, one simply checks that the magic formula $\langle \xi, [X,Y]\rangle$ can be used to define a non-degenerate 2-form on the orbit through ξ.

Poisson Lie groups. — An important class of Poisson manifolds is that of *Poisson Lie groups*, i.e. Lie groups with a Poisson structure such that the multiplication is a Poisson map. Although I will use Poisson Lie groups in what follows, I will introduce the (rather cumbersome) formalism only when I will need it (see § 1.4 and § 2.3).

Momentum mappings. — Let G be a Lie group acting on the Poisson manifold W. The action is *Hamiltonian* if there exists a *momentum mapping*, that is, a Poisson mapping

$$\mu : W \longrightarrow \mathfrak{g}^\star$$

such that, for all $X\in\mathfrak{g}$, the Hamiltonian vector field of the function $\mu_X : W\to\mathbf{R}$ defined by

$$\mu_X(w) = \langle \mu(w), X\rangle$$

is $\widetilde{X}$, the fundamental vector field associated with X. Recall that to require that μ preserve the Poisson structures is equivalent to asking that the map

$$\begin{array}{ccc}\mathfrak{g} & \longrightarrow & \mathcal{C}^\infty(W)\\ Y & \longmapsto & \mu_Y\end{array}$$

be a morphism of Lie algebras (see e.g. [8]).

Integrable systems. — On a $2n$ dimensional symplectic manifold W, an ***integrable system*** is a set of n functionally independent functions $f_1,\dots,f_n$ which pairwise commute, i.e. such that

$$\forall i,\quad \forall j \qquad \{f_i, f_j\} = 0.$$

That the functions are *functionally independent* means that the Hamiltonian vector fields $X_{f_1},\dots,X_{f_n}$ are independent at the points of an open dense subset of W. At any such point x, notice that they generate an isotropic subspace of T_xW. Thus n is the maximum possible number of commuting independent functions on W.

Why is this called a "system"? One could be interested in the differential (Hamiltonian) system defined by one of the Hamiltonian vector fields (say X_{f_1}) and view the other functions as playing an auxiliary role: as $\{f_i,f_j\}=0$, they will be constant along the trajectories of X_{f_1}—they are said to be first integrals of X_{f_1}.

Although it gives a symmetric role to all the f_i's, the definition I have given is certainly not the best possible: very often, one is only interested in the subalgebra of $\mathcal{C}^\infty(W)$ generated by the f_i's and not in a specific set of generators. This definition will become even worse when adapted to the case of a Poisson manifold: it is not quite clear how to require that there be as many commuting functions as possible. This can be formalised, but I will not do it here. In all the examples I will consider, we will deal with symplectic leaves and integrable systems (in the above sense) on them. For a formally better definition, see [41].

Let us consider once again the case of $\mathfrak{g}^\star$. It is quite easy to check that the Casimir functions (i.e. the functions f such that $X_f\equiv 0$ or $\{f,\cdot\}\equiv 0$) are the $\mathrm{Ad}^\star$-invariant functions. In the good cases (for instance if the Lie group is compact or the Lie algebra semi-simple, which are the cases we shall consider in these notes) there are enough invariant functions to describe the symplectic leaves, which thus appear as the connected components of the common level sets of the Casimir functions. Here the maximum number of commuting functions is $n+r$ where $r=\operatorname{rk}\mathfrak{g}$ is the number of independent Casimir functions and $2n+r=\dim\mathfrak{g}$, so that $2n$ is the dimension of the generic leaves.

Examples. — A lot of examples come from classical mechanics, such as the equations of motion of a spinning top or of other rigid bodies (see e.g. [10]), or the differential equation for the geodesics on the ellipsoid and their generalisations. Some come from the mechanics of particles with various potentials, such as the Toda lattice or the Calogero systems described below (§ 1.4 and § 3.4). A family of examples (Hamiltonian torus actions) will be described in § 1.2, some others (coming from mechanics) in § 1.3, a general construction (Lax equations given by r-matrices) will be given in § 1.4. The whole of § 3 will be devoted to examples.

Action-angle coordinates. — The Arnold-Liouville theorem (see [4]) describes the situation semi-locally, in the neighborhood of a regular common level set of the commuting functions f_i's.

Theorem 1.1.1 *Let $a = (a_1, \dots, a_n) \in \mathbf{R}^n$ be a regular value of the mapping $f = (f_1, \dots, f_n) : W \to \mathbf{R}^n$. Let $\mathcal{T}_a$ be the corresponding regular level, so that $\mathcal{T}_a$ is a Lagrangian submanifold.*

(a) *Let x be a point in $\mathcal{T}_a$. If the flows of the vector fields $X_{f_1}, \dots, X_{f_n}$ starting at x are complete, the connected component of x in $\mathcal{T}_a$ is a homogeneous space of $\mathbf{R}^n$. In particular, it has coordinates $(\varphi_1, \dots, \varphi_n)$ in which the vector field X_{f_i} can be written*

$$\sum_{j=1}^{n} A_j^i(a) \frac{\partial}{\partial \varphi_j}.$$

(b) *Assume the connected component of x in $\mathcal{T}_a$ is compact. Then it is a torus and there are coordinates $(\varphi_1, \dots, \varphi_n, I_1, \dots, I_n)$ in a neighbourhood of this component such that the symplectic form can be written as*

$$\omega = \sum dI_i \wedge d\varphi_i$$

and the vector field X_{f_i} as

$$\sum_{j=1}^{n} A_j^i(I) \frac{\partial}{\partial \varphi_j}.$$

Comments. — In other words, there are "linear" *angle* coordinates φ_i, for the flows of the f_i's and dual *action* coordinates I_i depending only on the f_j's.

If one insists on compactness, there is a more beautiful way to state this theorem, due to Duistermaat [17].

It is often thought that the Arnold-Liouville theorem states that the regular levels of an integrable system are tori. What the "tori" actually are is a very interesting question[1]. Some are indeed honest tori, for instance because the system is related to a torus action (see § 1.2), some are unions of tori, for instance because they are real parts of complex Abelian varieties (see § 1.3 and § 1.4), some are discs or cylinders, some are non-compact but have natural compactifications either as Abelian varieties or as toric varieties (see § 1.4 again) or as symmetric products of curves (see [40]), some even have nothing to do with any algebraic variety...

1.2 The momentum mapping of a torus action

One good reason to discuss this family of examples is that this is the paradigm of the Arnold-Liouville theorem, as everything (regular levels, action and angle coordinates...) is both easy and explicit from this point of view.

[1] For all these topological aspects, see [10].

Hamiltonian torus actions. — Let T be a torus acting on a symplectic $2n$-manifold (W, ω). Assume that the action is Hamiltonian with momentum mapping

$$\mu : W \xrightarrow{\ \mu\ } \mathfrak{t}^*.$$

Recall that the Atiyah [5] and Guillemin-Sternberg [24] convexity theorem asserts that if W is compact and connected the image of μ is a convex polyhedron, the convex hull of the images of the fixed points of the torus action.

Recall moreover that the main argument in the proof uses the linearization of the torus action near the fixed points to show, as noticed by Frankel [21], that any component of the momentum mapping is a Morse (-Bott) function whose indices are all even, and that this allows us to prove that all non-empty levels of μ are connected.

Half the dimension. — Notice that all the vector fields $\tilde{Y}$ (for $Y \in \mathfrak{t}$) commute: $\mathfrak{t}$ is abelian, so that the μ_Y generate an abelian subalgebra of $C^\infty(W)$. Notice also that the subspace generated by the $\tilde{Y}_w$ in $T_w W$ is isotropic, so that its dimension is less than or equal to n.

Let us assume for simplicity that the torus action is effective. Then $\dim T \leq \frac{1}{2} \dim W = n$. Assume now that the torus T, acting effectively, has the maximal possible dimension, that is n. We have now an integrable system on W. If one insists on giving n functions: pick a basis $(Y_1, \dots, Y_n)$ in $\mathfrak{t}$, so that $\mu_{Y_1}, \dots, \mu_{Y_n}$ are n commuting Hamiltonians. The fact that they are independent follows from the basic

Proposition 1.2.1 *The rank of the tangent map $T_w\mu$ is the dimension of the T-orbit through w.*

Proof The tangent map $T_w\mu : T_w W \to \mathfrak{t}$ is the transpose of the map $\varphi_w : \mathfrak{t} \to T_w^* W$

$$Y \overset{\varphi_w}{\longmapsto} \left(i_{\tilde{Y}} \omega\right)_w$$

so that the image of $T_w\mu$ is the annihilator of $\operatorname{Ker} \varphi_w$. But, ω being non-degenerate,

$$\operatorname{Ker} \varphi_w = \left\{ Y \in \mathfrak{t} \mid \tilde{Y}_w = 0 \right\}$$

and this is the Lie algebra of the stabilizer T_w of w. Thus

$$\operatorname{rk} T_w\mu = \dim \operatorname{Im} T_w\mu = \operatorname{codim} \operatorname{Ker} \varphi_w = \operatorname{codim} \mathfrak{t}_w = \dim(T \cdot w). \quad \square$$

As T is abelian and the action is effective, there is an open dense subset in W consisting of orbits with trivial stabilizers, so that in particular the interior of P is non-empty[2].

[2] Discrete stabilisers would be enough.