A stability result for oblic interpolation

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Latu-M-Güclü-Ottaviani-Sonnendrücker, submitted
Collaboration with R. Ferretti (Roma Tre, Italy)
Physical context: the ITER project

- Tokamak construction at Cadarache in France
  - Aim: gain of energy by fusion of atoms with magnetic confinement
- Modelisation of plasma by PDE
  - MHD (fluid model)
    - Long time dynamic
    - Instabilities can destroy the machine
  - Multi-species Vlasov-Maxwell and gyrokinetic approximation
    - Short time dynamic
    - Micro-instabilities can degrade confinement quality
- Interest of numerical simulations:
  - Understand how heat flux due to turbulence vary with respect to the size of the plasma
Motivation

- Presence of strong magnetic field $B$
  - $\Rightarrow$ Alignment of the solution along direction of $B$
- Need to take this into account in the numerics
  - $\Rightarrow$ Design of a numerical method that can avoid to take too much poloidal planes without losing precision

Numerical tools: PIC/eulerian/semi-Lagrangian

New idea (Hariri–Ottaviani, 2013): aligned interpolation
1d constant advection

The semi-Lagrangian method for 1d constant advection

\[ \partial_t u + a \partial_x u = 0, \quad u = u(t, x) \]

\[ t_{\ell+1} = t_\ell + \Delta t \]

Characteristics are exact

Lagrange interpolation:

- Degree 1 (linear): \( x_{i*}, x_{i*+1} \)
- Degree 3 (cubic): \( x_{i*-1}, x_{i*}, x_{i*+1}, x_{i*+2} \)

Some known results

- \( L^2 \) stability \cite{Strang:1962}
- \( L^q, \quad q \geq 1 \) stability for odd degree \cite{Despres:2009}
- The scheme is equivalent to a Lagrange Galerkin scheme \cite{Pironneau:1982} for odd degree \( \leq 13 \) \cite{Ferretti:2010}

\[ \Rightarrow \text{other proof of } L^2\text{stability} \]
Oblic interpolation

Interpolation along a fixed oblic direction

⇒ Reconstruction of the values necessary by interpolation in $\theta$
⇒ Reconstruction in the aligned direction
Oblic advection

Constant advection along \( \mathbf{b} = (b_\theta, b_\varphi) \)

\[
\partial_t f + \mathbf{v} \mathbf{b} \cdot \nabla f = 0, \quad f(t=0, \theta, \varphi) = e^{(im\theta + in\varphi)}
\]

Lagrange interpolation of odd degree \( d_\theta \) in \( \theta \) and

- Standard method : Lagrange of odd degree \( d_\varphi \) in \( \varphi \)

\[
\left\| e^{(k)} \right\|_2 \leq C_{d_\theta} \frac{T (|m| \Delta \theta)^{d_\theta + 1}}{\Delta t} + C_{d_\varphi} \frac{T (|n| \Delta \varphi)^{d_\varphi + 1}}{\Delta t}
\]

- Aligned method : Lagrange of odd degree \( d_\varphi \) in aligned direction \( \mathbf{b} \)

\[
\left\| e^{(k)} \right\|_2 \leq G_{d_\theta} C_{d_\theta} \frac{T (|m| \Delta \theta)^{d_\theta + 1}}{\Delta t} + C_{d_\varphi} \frac{T \left( |n + \frac{b_\theta}{b_\varphi} m| \Delta \varphi \right)^{d_\varphi + 1}}{\Delta t}
\]

- Same accuracy for

\[
\Delta \varphi^{\text{aligned}} \simeq \left| \frac{b_\varphi}{\mathbf{b} \cdot \nabla f} \nabla \varphi \right| \Delta \varphi^{\text{standard}}.
\]
Stability of the aligned method

- We first treat the case of \( \lambda = \frac{b_\theta N_\theta}{b_\varphi N_\varphi} \) rational
  - 2\( d \) symbol writes as a convex combination of 1\( d \) symbols in aligned direction
    - coefficients are discrete Fourier transform
    - Discrete Fourier transform is real
    - Discrete Fourier transform is nonnegative
- Case of \( \lambda \) real by density
More precisely

The symbol can be written as

$$\rho_{\lambda, r_{\phi}, \alpha_{\phi}}(\omega_\theta, \omega_\phi) = \sum_{p=0}^{q-1} t_p \exp(ir_{\phi}\omega_p) \sum_{k=-d_{\phi}}^{d_{\phi}+1} L_k^{d_{\phi}}(\alpha_{\phi}) \exp(ik\omega_p),$$

with $\omega_p = 2\pi p\lambda + \omega_\phi + \lambda\omega_\theta$, $q\lambda \in \mathbb{Z}$,

$$t_p = \frac{1}{q} \sum_{p_1=0}^{q-1} \sum_{\ell=-d_\theta}^{d_\theta} L_\ell^{d_\theta} \left( \frac{p_1}{q} \right) \exp \left( i \left( \ell - \frac{p_1}{q} \right) (\omega_\theta + 2\pi p) \right),$$

- $t_p$ is real by symmetry
- difficult part: $t_p$ is nonnegative
- $t_p$ is not real for even degree interpolation and we can find unstable situations (for $d_{\phi} \geq 1$)
- interpolation along the aligned direction can be changed
- the symbol is in a regular $q$-polygon
Corollary: proof of SL-LG equivalence

Ferretti, 2010

- SL and LG are equivalent for the 1d constant advection, if we can find a function $\phi$ such that

$$\int_{\mathbb{R}} \phi(\eta + y)\phi(y)dy = \psi(y) \quad \text{auto-correlation integral}$$

- $\psi$ describes the Semi Lagrangian (SL) scheme
- $\phi$ describes the Lagrange Galerkin (LG) scheme
- In Fourier

$$\hat{\psi}(\omega) = \left| \hat{\phi}(\omega) \right|^2$$

- Example: for degree 3, we have

$$\hat{\psi}(\omega) = \frac{8(6 + \omega^2)\sin(\omega/2)^4}{3\omega^4} \in \mathbb{R}^+$$

The result of stability in the oblic context implies the equivalence SL-LG of Ferretti, 2010 and is valid for an arbitrary odd degree
Direct proof of SL-LG equivalence

Ferretti-M, 2016  Algebraic form of the Fourier transform valid for arbitrary odd degree (conjectured in Ferretti, 2010)

- Aim: prove that \( S(\omega) = \int_0^1 \sum_{\ell=-d}^{d+1} L_\ell(x) \exp(i(\ell-x)\omega) \, dx \in \mathbb{R}^+ \)
- Compact formula for the derivative Boyer/Després’s lecture notes

\[
S'(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!} \sin^{2d+1} \left( \frac{\omega}{2} \right) \sigma(\omega)
\]

- Integration by parts for the factor

\[
\sigma(\omega) = \int_0^1 \cos \left( \left( x - \frac{1}{2} \right) \omega \right) w(x) \, dx, \quad w(x) = \prod_{j=-d}^{d+1} (x - j)
\]

- Recognise the primitive thanks to relation

\[
w^{(2k+1)}(0) = -\frac{d + 1}{2k + 2} w^{(2k+2)}(0), \quad k = 0, \ldots, d.
\]

- Final explicit form

\[
S(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!} \sin^{2d+2} \left( \frac{\omega}{2} \right) \sum_{k=0}^d \frac{w^{(2k+2)}(0)}{k+1} \frac{(-1)^k}{\omega^{2k+2}}
\]

⇒ New proof of \( L^2 \) stability of SL scheme for constant advection
Application: gyrokinetic simulation in the Selalib library

- drift kinetic model in cylinder geometry
- corresponds to Grandgirard et al 2006, when $b_\theta = 0$
- Poloidal cut $f(t, r, \theta, z = 0, v = 0)$
- Mode ($m = 10, n = -9$) the most unstable (aligned method, LAG17) $255 \times 512 \times 32 \times 128$ (256 proc), 4000 itérations $\Delta t = 2$, on helios, supercomputer, 21 heures.
Application: gyrokinetic simulation in Gysela

- Gain of factor 4 in Gysela (gyrokinetic code, CEA Cadarache)
- $256 \times 256 \times N_{\phi} \times 48$
- Initialization with a bath of modes
- Degree 5, and cubic splines in $\theta$ and other interpolations

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Potential energy
Torus geometry, $\rho_{*}=1/49$
Variable q(r) from 1 (rmin to 1.3 (rmax)
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N_{\phi} = 32, aligned
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N_{\phi} = 32, standard
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N_{\phi} = 128, standard
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Conclusion/Perspectives

- **Conclusion**
  - First results of numerical analysis of the new aligned method
  - New proof of convergence of SL scheme for constant advection
  - Validation of the aligned method in a gyrokinetic context

- **Perspectives**
  - Study of other reconstructions: splines, Hermite, SLDG
  - Scaling / more realistic configurations
  - Adaptation of the geometry / multi species
  - Convergence of SL schemes