Stability of oblic interpolation

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Semi-Lagrangian numerical scheme

- distribution function $f$ is constant along characteristics
- find the origin of the characteristics ending at the grid points
- interpolate old value at origin of characteristics from known grid values
Example: constant transport equation

\[ \partial_t f + a_x \partial_x f + a_y \partial_y f = 0 \]

We have

\[ f(t^{n+1}, x_i, y_j) = f(t^n, x_i - a_x \Delta t, y_j - a_y \Delta t) \]

- \( f \) is only known at grid points \((x_i, y_j)\)
- We thus need to interpolate
**Classical interpolation : 1D case**

**Lagrange interpolation of degree 3**

We know \(f(x_{j-1}), f(x_{j+0}), f(x_{j+1})\) and \(f(x_{j+2})\)

Approximate \(f(x_{j+\alpha})\), \(0 < \alpha < 1\) with unique matching polynomial of degree \(\leq 3\)

![Graph showing the approximation of a function using Lagrange interpolation of degree 3.](image)
Classical interpolation : 2D case

Values are known at Blue circles. Objective is to find value at red square.

Step 1: Interpolate horizontally to find values at green triangles

Step 2: Interpolate vertically to find value at red square
Suppose now that function does not oscillate much in some fixed direction:

⇒ Can we interpolate better?
First solution

Use a aligned grid

Figure 1. The domain is periodic in cartesian directions $x$ and $y$. Periodicity in $y$ direction imposes a reconnection of different mesh lines to obtain the proper periodic boundary condition.

😊 It seems that error is reduced, when we follow such mesh (but why?)
😊 **Rational condition between mesh and oblic direction**
First solution

Some numerical results

**Figure 3.** Initial function, advection field vector \( (a_x = 1, a_y = 4) \) is aligned with dots but not aligned with mesh lines.
First solution
Some numerical results

\textbf{Figure 4.} $L_1$ and $L_\infty$ relative error between end and initial time, function of the angle between the advection velocity vector and the mesh lines. Angle $-20$ corresponds to the cartesian mesh.
Second solution

- **Classical mesh** (instead of aligned mesh)
- **Aligned interpolation** (instead of classical interpolation)
Description of the method

Interpolation along an oblic direction

⇒ Reconstruction of necessary values by interpolation in $\theta$
⇒ Reconstruction in the aligned direction
Convergence of the aligned method

Setting: advection of a wave function along $\mathbf{b} = (b_\theta, b_\varphi)$

$$\partial_t f + \mathbf{v} \mathbf{b} \cdot \nabla f = 0, \quad f(t = 0, \theta, \varphi) = e^{(im\theta + in\varphi)}$$

Lagrange interpolation of odd degree $d_\theta$ in $\theta$ and

- Standard method: Lagrange of odd degree $d_\varphi$ in $\varphi$
  $$\| e^{(k)} \|_2 \leq C_{d_\theta} \frac{T (|m| \Delta \theta)^{d_\theta + 1}}{\Delta t} + C_{d_\varphi} \frac{T (|n| \Delta \varphi)^{d_\varphi + 1}}{\Delta t}$$

- Aligned method: Lagrange of odd degree $d_\varphi$ for the aligned direction $\mathbf{b}$
  $$\| e^{(k)} \|_2 \leq G_{d_\theta} C_{d_\theta} \frac{T (|m| \Delta \theta)^{d_\theta + 1}}{\Delta t} + C_{d_\varphi} \frac{T (|n + b_\theta m| \Delta \varphi)^{d_\varphi + 1}}{\Delta t}$$

- Aligned method interesting when $f$ varies slowly along $\mathbf{b}$, while varying a lot along $\varphi$:
  $$|\nabla f \cdot \mathbf{b}| = |b_\theta m + b_\varphi n||f| \ll |n||f| = |\nabla \varphi f|$$

- Gain:
  $$\frac{|n|}{|n + b_\theta m|} \simeq \frac{|\nabla \varphi f|}{|\nabla f \cdot \mathbf{b}|}$$
Proof of the stability: about 1D classical interpolation degree 1
Proof of the stability: about 1D classical interpolation degree 3
Proof of the stability: about 1D classical interpolation degree 5
Proof of the stability: about 1D classical interpolation
degree 7
Proof of the stability: about $1D$ classical interpolation degree 17
Proof of the stability : about 1D classical interpolation

degree 17
Proof of the stability for the aligned method

Formula for the scheme

\[
f_{i,j}^{n+1} = \sum_{\ell = -d_\theta}^{d_\theta} \sum_{k = -d_b}^{d_b} L_{k=\ell}^{d_b} (\alpha_\varphi) L_{\ell}^{d_\theta} (\alpha_\theta, k) f_{i+\theta, k+\ell, j+\varphi+k}^{n}\]

Proof of the stability for the aligned method

Discrete 2D Fourier transform

\[ \sum_{i_2=0}^{N_\theta-1} \sum_{j_2=0}^{N_\varphi-1} f_{i_2,j_2}^{n+1} \exp \left( 2\pi i \frac{i_1 i_2}{N_\theta} \right) \exp \left( 2\pi i \frac{j_1 j_2}{N_\varphi} \right) = \]

\[ = \rho \left( \frac{2\pi i_1}{N_\theta}, \frac{2\pi j_1}{N_\varphi} \right) \sum_{i_2=0}^{N_\theta-1} \sum_{j_2=0}^{N_\varphi-1} f_{i_2,j_2}^n \exp \left( 2\pi i \frac{i_1 i_2}{N_\theta} \right) \exp \left( 2\pi i \frac{j_1 j_2}{N_\varphi} \right) \]
Proof of the stability for the aligned method

Fourier symbol

\[ \rho(\omega_\theta, \omega_\varphi) = \sum_{\ell=-d_\theta}^{d_\theta+1} \sum_{k=-d_b}^{d_b+1} L_{k}^{d_b}(\alpha_\varphi) L_{\ell}^{d_\theta}(\alpha_\theta, k) \exp(i(r_\theta, k + \ell)\omega_\theta) \exp(i(r_\varphi + k)\omega_\varphi). \]

modulus should be \( \leq 1 \)

\[ r_\theta, k + \alpha_\theta, k = (r_\varphi + k)\lambda, \quad \lambda = \frac{b_\theta N_\theta}{b_\varphi N_\varphi} \in \mathbb{R}. \]
Proof of the stability for the aligned method

- We define
  \[ \lambda = \frac{b_\theta N_\theta}{b_\varphi N_\varphi} \]

- We suppose that there exists \( q \in \mathbb{Z} \) such that
  \[ q \lambda \in \mathbb{Z}. \]
The expression of the Fourier symbol is then simplified and we have

\[ |\rho_{\lambda, r_\varphi, \alpha_\varphi}(\omega_\theta, \omega_\varphi)| \leq \left( \sum_{p=0}^{q-1} |t_p| \right) \left( \sup_{0 \leq \omega \leq 2\pi} \left| \sum_{k=-d_b}^{d_b} L_{k}^{d_b}(\alpha_\varphi) \exp(ik\omega) \right| \right) \]

\[ t_p = \frac{1}{q} \sum_{p_1=0}^{q-1} \sum_{\ell=-d_\theta}^{d_\theta} L_{\ell}^{d_\theta} \left( \frac{p_1}{q} \right) \exp \left( i \left( \ell - \frac{p_1}{q} \right) (\omega_\theta + 2\pi p) \right) \]

\[ \sum_{p=0}^{q-1} t_p = 1. \]
We have to prove that

\[ t(\omega) = \frac{1}{q} \sum_{p=0}^{q-1} \sum_{\ell=-d}^{d+1} L_{\ell} \left( \frac{p}{q} \right) \exp \left( i \left( \ell - \frac{p}{q} \right) \omega \right) \geq 0, \]

with \( L_{\ell}(x) = \prod_{-d \leq j \neq \ell \leq d+1} \frac{x-j}{\ell-j} \)
Proof of the stability for the aligned method

Using the derivative, we define

$$f_p = (-1)^d w_d \left( \frac{p}{q} \right), \quad w_d(x) = \prod_{k=-d}^{d+1} (x - k)$$

We use a discrete integration by part:

$$\sum_{p=1}^{q-1} \cos(2n\pi \frac{p}{q}) f_p = \sum_{p=1}^{q-1} \frac{1 - \cos(2n\pi \frac{p}{q})}{4 \sin^2 \left( \frac{n\pi}{q} \right)} (f_{p+1} + f_{p-1} - 2f_p) \in \mathbb{R}^+.$$  

Conclusion: with monotonicity arguments (Rolle's theorem; degree of polynomial)
Numerical context: Semi-Lagrangian numerical scheme

- distribution function \( f = f(t, x, v) \) is constant along characteristics
- find the origin of the characteristics ending at the grid points
- interpolate old value at origin of characteristics from known grid values
Numerical context: Particle-in-Cell numerical scheme

Classical first order Cloud-in-Cell method

- macro-particles initialized randomly and follow characteristics
- linear splines for charge deposition and electric field interpolation
Comparison SL/PIC

- Semi-Lagrangian (SL)
  - 😊 no noise
  - 😊 no CFL condition (constraint on time step)
  - 😋 mesh of phase-space (6D for 3D applications)

- Particle-In-Cell (PIC)
  - 😋 no mesh of phase space
  - 😋 no CFL condition (constraint on time step)
  - 😋 numerical noise
A physical context : the ITER project

- Tokamak construction at Cadarache in France
  - Aim : gain of energy by fusion of atoms with magnetic confinement
- Modelisation of plasma by PDE
  - MHD (fluid model)
    - Long time dynamic
    - Instabilities can destroy the machine
  - Multi-species Vlasov-Maxwell and gyrokinetic approximation
    - Short time dynamic
    - Micro-instabilities can degrade confinement quality
- Interest of numerical simulations:
  - Understand how heat flux due to turbulence vary with respect to the size of the plasma
Physical context

In a tokamak, the distribution function $f(t, x, v)$ representing the density of particles is strongly influenced by the strong magnetic field $B(t, x)$ externally imposed, which helps to confine the particles.

**GYSELA**

**Figure** – 3D view of a plasma simulation in GYSELA code

GYSELA : GYro-kinetic **SEmi-LAgrangian** code (CEA-Cadarache, France)
Physical context

For simple configurations, magnetic field lines are near *straight lines* in the periodic plan representing the magnetic surfaces.

**Figure** – 3D view of a plasma simulation in GYRO, a *Eulerian* code (General Atomics, San Diego, USA) [Picture of Vlasovia, 2016: Copanello (Calabria), Italy, May 30 - June 2, 2016]
In this context, we should take benefit of this fact when designing numerical methods.

**Figure** – 3D view of a plasma simulation in ORB5/NEMORB, a Particle-In-Cell code (EPFL, Switzerland/IPP Garching, Germany)
Application: gyrokinetic simulation in Selalib

- drift kinetic model in cylinder geometry
- corresponds to Grandgirard et al 2006, when $b_\theta = 0$
- Poloidal cut $f(t, r, \theta, z = 0, v = 0)$
- Mode $(m = 10, n = -9)$ the most unstable (aligned method, LAG17) $255 \times 512 \times 32 \times 128$ (256 proc), 4000 itérations $\Delta t = 2$, on helios, supercomputer, 21 heures.
Application: gyrokinetic simulation in Gysela

- Gain of factor 4 in Gysela (gyrokinetic code, CEA Cadarache)
- $256 \times 256 \times N_\phi \times 48$
- Initialization with a bath of modes
- Degree 4, and cubic splines in $\theta$ and other interpolations