

Weakly dispersive systems: an application to water wave models

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Abstract

In this paper, we study dispersive estimates for a class of weak dispersive systems. Then we apply our results to water wave models.

1 Introduction

We consider a class of dispersive equations under the form

$$\partial_t u = \pm i\varphi(\delta|D|)|D|u \quad (1)$$

where $\delta > 0$, $u(t, x) \in \mathbb{C}$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ with $d = 1$ or 2 . We introduce the function $g(y) = \varphi(|y|)y$ for any $y \in \mathbb{R}^*$ so that $\frac{d}{dr}(\varphi(\delta r)r) = g'(\delta r)$ for any $r > 0$. We assume in this paper one of the following assumption

(H0) φ is a real-valued smooth function defined on $\mathbb{R}_+^*$

or

(H1) φ is a real-valued smooth function defined on $\mathbb{R}_+$ and $\varphi(0) \neq 0$.

With Assumption (H0), the function φ is not necessary defined at 0. This assumption will be mostly used when $d = 1$. With Assumption (H1), Equation (1) can be seen as a perturbation of the half-wave equation when δ is small. Note that in that case, $g'(0) \neq 0$. One (but not the only) of the main motivation of this paper is to consider situations where $\varphi(y) \rightarrow 0$ as $y \rightarrow +\infty$ which corresponds to equations with weak dispersive effects. Another motivation is to consider equations under Assumption (H1) with $\delta \ll 1$. In the following we denote the unique solution of (1) with $u(0, x) = u_0$ as

$$u(t, x) = e^{\pm it\varphi(\delta|D|)|D|}u_0$$

using a Fourier multiplier notation.

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Remark 1.1. When $d = 1$, one can also consider dispersive equations under the form

$$\partial_t u = \pm \varphi(\delta|D|)\partial_x u$$

where $\delta > 0$, $u(t, x) \in \mathbb{R}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$. Indeed we have the relation

$$e^{it\varphi(\delta|D|)\partial_x} u_0 = e^{-it\varphi(\delta|D|)|D|} \mathbf{1}_{\{D < 0\}} u_0 + e^{it\varphi(\delta|D|)|D|} \mathbf{1}_{\{D > 0\}} u_0.$$

In this paper, we restrict our study to $d = 1$ or $d = 2$ since we will apply our results to water wave models. One could easily extend these results to $d \geq 2$ as in [GPW08]. The goal of this paper is to provide decay estimates, local-in-time Strichartz estimates and Morawetz estimates for Equation (1).

This problem is motivated by the study of shallow water-wave models. Introducing the shallowness parameter $\mu \in]0, 1]$, we consider here two models: the linearized irrotational water wave equations (under the Zakharov/craig-Sulem formulation)

$$\begin{cases} \partial_t \zeta - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}} |D| \psi = 0, \\ \partial_t \psi + \zeta = 0, \end{cases} \quad (2)$$

where $\delta = \sqrt{\mu}$ and

$$\varphi(r) = \sqrt{\frac{\tanh(r)}{r}}$$

and the dispersive part of the linearized abcd-system

$$\begin{cases} (1 - \mu b \Delta) \partial_t \zeta + (1 + \mu a \Delta) \nabla \cdot V = 0, \\ (1 - \mu b \Delta) \partial_t \nabla \cdot V + (1 + \mu c \Delta) \nabla \zeta = 0, \end{cases} \quad (3)$$

with $a + b + c + d = \frac{1}{3}$, under the condition that $b \geq 0$, $d \geq 0$, $a \leq 0$, $c \leq 0$ (in order to get the wellposedness), where $\delta = \sqrt{\mu}$ and

$$\varphi(r) = \sqrt{\frac{(1 - \mu a r^2)(1 - \mu c r^2)}{(1 + \mu b r^2)(1 + \mu d r^2)}}.$$

We also have in mind equations for $d = 1$. For $b \neq 0$ we consider the linearized Ostrovsky equation

$$\partial_t u = (-\partial_x^{-1} u \pm b \partial_x^3 u)$$

and, in the setting of Remark 1.1, $\delta = 1$ and $\varphi(\xi)\xi = \frac{1}{\xi} \pm b\xi^3$.

For $b \leq 0$, we consider a linearized BBM/KdV equation

$$\partial_t u + \mu b \partial_x^2 \partial_t u = \pm \partial_x u \pm \mu(b + \frac{1}{6}) \partial_x^3 u.$$

and, in the setting of Remark 1.1, $\delta = \sqrt{\mu}$ and $\varphi(\xi) = \frac{1 - \mu(b + \frac{1}{6})\xi^2}{1 + b\mu\xi^2}$.

1.1 Notations

- If u is a tempered distribution, we define $\mathcal{F}u$ or $\hat{u}$ as the Fourier transform of u by

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} u(x) dx.$$

- If f is a smooth function that is at most polynomial at infinity, we define the operator $f(D)$ such that, for any Schwartz class function u ,

$$f(D)u = \mathcal{F}^{-1}(f(\xi)\hat{u}(\xi)).$$

- In the whole paper φ is a smooth function defined at least on $\mathbb{R}_+^*$ and we define the function g defined at least on $\mathbb{R}^*$ as $g(y) = \varphi(|y|)y$.
- J_0 is a Bessel function defined in (4).
- For x_t and y_t positive numbers depending on a parameter t , we write $x_t \sim y_t$ if there exists $\Lambda, \lambda > 0$ independent from t such that $\lambda x_t \leq y_t \leq \Lambda x_t$.
- If g is a function and $l \in \mathbb{N}$, we denote by $g^{(l)}$ the l -th derivative of g .
- For an interval J , the subset $\mathbb{Z}_J$ of $\mathbb{Z}$ is defined in (5).

2 Dispersive estimates

2.1 Littlewood-Paley decomposition

Since the phase we consider is not necessary homogeneous, we consider a Littlewood-Paley decomposition. We refer to [BCD11] for a more precise description of such construction. We introduce a smooth nonnegative even function φ_0 supported in $[-1, 1]$ and that is equal to 1 in $[-\frac{1}{2}, \frac{1}{2}]$. Then we define for any $y \in \mathbb{R}$, $P(y) = \varphi_0(\frac{y}{2}) - \varphi_0(y)$ and for any $j \in \mathbb{Z}$, $P_j(y) = P(2^{-j}y)$. We note that P_j is a function supported in $\mathcal{C}(2^{j-1}, 2^{j+1})$. We have for any $\xi \in \mathbb{R}^d$,

$$\varphi_0(|\xi|) + \sum_{j \in \mathbb{N}} P_j(|\xi|) = 1, \quad \frac{1}{2} \leq \varphi_0^2(|\xi|) + \sum_{j \in \mathbb{Z}} P_j^2(|\xi|) \leq 2,$$

and for any $\xi \in \mathbb{R}^d$ with $\xi \neq 0$,

$$\sum_{j \in \mathbb{Z}} P_j(|\xi|) = 1, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} P_j^2(|\xi|) \leq 2.$$

Finally for $p, r \geq 1$, $s \in \mathbb{R}$, we define the homogeneous Besov norm $|\cdot|_{\dot{B}_{p,r}^s}$ by

$$|u|_{\dot{B}_{p,r}^s} = \left(\sum_{j \in \mathbb{Z}} 2^{jsr} |P_j(|D|)u|_{L^p}^r \right)^{\frac{1}{r}}$$

and the inhomogeneous Besov norm $|\cdot|_{B_{p,r}^s}$ by

$$|u|_{B_{p,r}^s} = \left(|\varphi_0(|D|)u|_{L^p}^r + \sum_{j \in \mathbb{N}} 2^{jsr} |P_j(|D|)u|_{L^p}^r \right)^{\frac{1}{r}}.$$

2.2 Decay estimates

2.2.1 Tools for the dispersive estimates

In this section, we introduce some basic tools in order to get our decay estimates. We first recall the van der Corput lemma (see for instance [Ste93]).

Lemma 2.1 (Van der Corput). *Let $a < b$ be real numbers, $\lambda > 0$ and ϕ a smooth real-valued function defined on $(a, b]$. Assume that there exists $k \in \mathbb{N}$ with $k \geq 1$ such that $|\phi^{(k)}| \geq \lambda$ on $(a, b]$ and, if $k = 1$, that ϕ' is monotonic on $(a, b]$. Then, there exists a constant c_k , that only depends on k , such that for any $t \in \mathbb{R}^*$ and any smooth function f defined on $[a, b]$,*

$$\left| \int_a^b e^{it\phi(x)} f(x) dx \right| \leq c_k (\lambda |t|)^{-\frac{1}{k}} \left(|f(b)| + \int_a^b |f'(x)| dx \right).$$

We easily get a corollary in the case $k = 1$ which will be useful in the following.

Corollary 2.2. *Let $a < b$ be real numbers, $r \in \mathbb{N}$, $\lambda > 0$ and ϕ a smooth real-valued function defined on $(a, b]$. Assume that $|\phi'| \geq \lambda$ on $(a, b]$ and that ϕ'' has at most r zeros on (a, b) . Then, there exists a constant c such that for any $t \in \mathbb{R}^*$ and any smooth function f defined on $[a, b]$,*

$$\left| \int_a^b e^{it\phi(x)} f(x) dx \right| \leq \frac{c}{\lambda |t|} \left(r \|f\|_\infty + |f(b)| + \int_a^b |f'(x)| dx \right).$$

Then, we introduce the Bessel function J_0 : for any $s \in \mathbb{R}$,

$$J_0(s) = \int_0^{2\pi} e^{is \sin(\theta)} d\theta. \quad (4)$$

A direct application of Van der Corput's lemma gives

$$|J_0(s)| + |J_0'(s)| \lesssim \frac{1}{\sqrt{1 + |s|}}.$$

Furthermore, introducing a smooth nonnegative function χ defined on $[0, 2\pi]$ with $\chi(x) = 0$ for $x \in [\frac{5\pi}{4}, \frac{7\pi}{4}]$ and $\chi(x) = 1$ for any $x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, one can decompose J_0 as

$$\begin{aligned} J_0(s) &= e^{is} \int_0^{2\pi} \chi(\theta) e^{is(\sin(\theta)-1)} d\theta + e^{-is} \int_0^{2\pi} (1 - \chi(\theta)) e^{is(\sin(\theta)+1)} d\theta \\ &= e^{is} h_-(s) + e^{-is} h_+(s). \end{aligned}$$

Integrating by parts if necessary and using Van der Corput's lemma, one can get for any $s \in \mathbb{R}$ and any $k \in \mathbb{N}$

$$|h_{\pm}^{(k)}(s)| \lesssim \frac{1}{(1 + |s|)^{k+\frac{1}{2}}}.$$

2.2.2 General decays

2.2.3 d=1

We first give a easy consequence of Van der Corput's lemma in case a derivative of g does not vanish on an interval.

Lemma 2.3. *Let $\delta > 0$ and J an interval. Let $\lambda > 0$, $l \in \mathbb{N}$ with $l \geq 2$. Assume that φ satisfies (H0) and that $|g^{(l)}| \geq \lambda$ on J . Let χ be a smooth function on $\mathbb{R}^*$ whose support is a subset of J and such that $\chi' \in L^1(J)$. Then, for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}$, we have*

$$\left| \int_{\mathbb{R}} e^{ix\xi} e^{it\varphi(\delta|\xi|)|\xi|} \chi(\delta\xi) d\xi \right| \lesssim \frac{\delta^{\frac{1-l}{l}}}{|t|^{\frac{1}{l}}}.$$

Remark 2.4. When J is unbounded, the previous integral is well defined from integration by parts since $\frac{g''(y)}{g'(y)^2}$ has a sign for $|y|$ large enough and $\lim_{|y| \rightarrow +\infty} |g'(y)| = +\infty$.

Then we provide a consequence of Corollary 2.9 in [KPV91] that can be useful for low frequencies.

Lemma 2.5. *Let $\delta > 0$ and J an interval. Let $\lambda, \Lambda > 0$ and $\beta \in \mathbb{R}$ with $\beta \notin \{-2, -1\}$. Assume that φ satisfies (H0) and that $\Lambda|y|^\beta \geq |g''(y)| \geq \lambda|y|^\beta$. Let χ be a smooth function on $\mathbb{R}^*$ whose support is a subset of J and such that $\chi' \in L^1(J)$. Then, for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}$, we have*

$$\left| \int_{\mathbb{R}} e^{ix\xi} e^{it\varphi(\delta|\xi|)|\xi|} \sqrt{|g''(\delta|\xi|)|} \chi(\delta\xi) d\xi \right| \lesssim \frac{1}{\sqrt{\delta|t|}}.$$

Remark 2.6. As noted in [KPV91], the assumption $\beta \neq -2$ is essential since the estimate is not true for $g(\xi) = \ln(|\xi|)$. We chose to avoid the case $\beta = -1$ since we will not deal with such a phase in this paper.

The previous lemmas are particularly useful for homogeneous phases or low frequency estimates. However, in many situations, one has to be more accurate especially when $\varphi(y) \rightarrow 0$ as $y \rightarrow +\infty$. For this reason we use a Littlewood-Paley decomposition and introduce the quantity for any $t \in \mathbb{R}$, $x \in \mathbb{R}$, $s \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$I_{t,x,k}^s = \int_{\mathbb{R}} e^{ix\xi} e^{it\varphi(\delta|\xi|)|\xi|} P_k(\xi) |\xi|^s d\xi.$$

Using a change of variable, we can rewrite $I_{t,x,k}$

$$\begin{aligned} I_{t,x,k}^s &= 2^{(1+s)k} \int_{\frac{1}{2}}^2 e^{-i2^k x r} e^{it\varphi(\delta 2^k r) 2^k r} P(r) r^s dr + 2^{(1+s)k} \int_{\frac{1}{2}}^2 e^{i2^k x r} e^{it\varphi(\delta 2^k r) 2^k r} P(r) r^s dr \\ &:= I_{t,x,k,-}^s + I_{t,x,k,+}^s. \end{aligned}$$

For an interval J we define $\mathbb{Z}_J$ as

$$\mathbb{Z}_J = \{k \in \mathbb{Z}, \forall r \in [\frac{1}{2}, 2], 2^k \delta r \in J\}. \quad (5)$$

The following lemma gives a decay when a derivative of g does not vanish on an interval J .

Lemma 2.7. *Let $\delta > 0$ and $J \subset \mathbb{R}_+^*$ an interval. Let $\lambda > 0$, $\alpha \in \mathbb{R}$, $s \in \mathbb{R}$. Assume that φ satisfies (H0) and that $|g^{(l)}(y)| \geq \lambda y^\alpha$ on J . Then, for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}_J$*

$$|I_{t,x,k}^s| \lesssim \frac{2^{(s-\frac{\alpha}{l})k} \delta^{\frac{1-l-\alpha}{l}}}{|t|^{\frac{1}{l}}}.$$

Proof. Defining $\phi_\pm(r) = t\varphi(\delta 2^k r) 2^k r \pm 2^k x r$, we have $|\phi_\pm^{(l)}(r)| = 2^{lk} \delta^{l-1} |t g^{(l)}(\delta 2^k r)|$. Van der Corput's lemma gives

$$|I_{t,x,k,\pm}^s| \lesssim \frac{2^{(1+s)k}}{(2^{lk} \delta^{l-1} |t g^{(l)}(\delta 2^k r_1)|)^{\frac{1}{l}}}$$

with $r_1 \in [\frac{1}{2}, 2]$. The result follows. $\square$

As a direct consequence of the previous lemma we get, when $\delta = 1$ and $y_0 > 0$,

$$\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^s| \lesssim \frac{1}{|t|^{\frac{1}{l}}}$$

provided that $s - \frac{\alpha}{l} > 0$ and $J = (0, y_0)$ or $s - \frac{\alpha}{l} < 0$ and $J = (y_0, +\infty)$. The next lemma can be seen as an alternative to Lemma 2.3. We consider that $l = 2$ and $s - \frac{\alpha}{2} \leq 0$.

Lemma 2.8. *Let $\delta > 0$ and $J \subset \mathbb{R}_+^*$ an interval. Let $\Lambda, \lambda > 0$, $a, \alpha, s \in \mathbb{R}$. Assume that φ satisfies (H0) and that $\Lambda y^{\alpha+1} \geq |g'(y) - a| \geq \lambda y^{\alpha+1}$ and $|g''(y)| \geq \lambda y^\alpha$ on J . Then:*

(1) *If $-1 < s < \frac{\alpha}{2}$ and $\alpha \neq -1$, for any $t \in \mathbb{R}^*$, we have*

$$\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^s| \lesssim \frac{\delta^{-\frac{(1+\alpha)(s+1)}{2+\alpha}}}{|t|^{\frac{s+1}{2+\alpha}}}.$$

(2) *If $s = \frac{\alpha}{2}$ and $\alpha \notin \{-2, -1\}$, for any $t \in \mathbb{R}^*$, we have*

$$\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^{\frac{\alpha}{2}}| \lesssim \frac{\delta^{-\frac{1+\alpha}{2}}}{\sqrt{|t|}}.$$

Remark 2.9. When $J =]0, 1]$, $\delta = 1$ and $s = 0$, we get item (c) of Theorem 1 in [GPW08]. Point (2) is similar to Corollary 2.9 in [KPV91] (see also Lemma 2.5). Note that contrary to Lemma 2.3, if $l \geq 3$ we provide a way to get a decay of order $\frac{1}{|t|^{\frac{1}{l}}}$ without computing $g^{(l)}$.

Proof. We can assume that $a = 0$ since

$$e^{\pm i 2^k x r} e^{i t \varphi(\delta 2^k r) 2^k r} = e^{\pm i 2^k (x \mp a t) r} e^{i t (\varphi(\delta 2^k r) 2^k r - 2^k a r)}.$$

Note that the assumptions on s and α gives that $(s+1)$ and $(2+\alpha)$ agree in sign. We always have, for any $k \in \mathbb{Z}_J$, $|I_{t,x,k,\pm}^s| \lesssim 2^{(1+s)k}$.

Case $\frac{\lambda}{4} \delta^{\alpha+1} 2^{(\alpha+1)k} |t| \leq |x| \leq 4\Lambda \delta^{\alpha+1} 2^{(\alpha+1)k} |t|$: if $2^k (\delta^{\alpha+1} |t|)^{\frac{1}{\alpha+2}} \leq 1$, we have

$$|I_{t,x,k}^s| \lesssim 2^{(1+s)k} \lesssim \frac{\delta^{-\frac{(1+\alpha)(s+1)}{2+\alpha}}}{|t|^{\frac{s+1}{2+\alpha}}},$$

whereas if $2^k (\delta^{\alpha+1} |t|)^{\frac{1}{\alpha+2}} \geq 1$, we get from Lemma 2.7

$$|I_{t,x,k}^s| \lesssim \frac{2^{(s-\frac{\alpha}{2})k} \delta^{-\frac{1+\alpha}{2}}}{\sqrt{|t|}} = \frac{2^{(s-\frac{\alpha}{2})k} \delta^{-\frac{1+\alpha}{2}}}{|t|^{\frac{s+1}{2+\alpha}}} \frac{1}{|t|^{\frac{\alpha-2s}{2(2+\alpha)}}} \lesssim \frac{\delta^{-\frac{(1+\alpha)(s+1)}{2+\alpha}}}{|t|^{\frac{s+1}{2+\alpha}}}.$$

Note also that the size of $A_{t,x} := \{k \in \mathbb{Z}_J, \frac{\lambda}{4} \delta^{\alpha+1} 2^{(\alpha+1)k} |t| \leq |x| \leq 4\Lambda \delta^{\alpha+1} 2^{(\alpha+1)k} |t|\}$ is bounded by a number independent of t, x, δ (since $\alpha \neq -1$).

Case $\frac{\lambda}{4} \delta^{\alpha+1} 2^{(\alpha+1)k} |t| \geq |x|$ or $|x| \geq 4\Lambda \delta^{\alpha+1} 2^{(\alpha+1)k} |t|$: we get thanks to Corollary 2.2 with the phasis $\phi_{\pm}(r) = t\varphi(\delta 2^k r) 2^k r \pm 2^k x r$, since $|\phi'_{\pm}(r)| \geq \frac{\lambda}{4} \delta^{\alpha+1} 2^{(\alpha+2)k} |t|$,

$$|I_{t,x,k,\pm}^s| \lesssim \frac{2^{(s-\alpha-1)k} \delta^{-(\alpha+1)}}{|t|}.$$

Therefore gathering all previous estimates, we get $-1 < s \leq \frac{\alpha}{2}$ and $\alpha \neq -1$

$$\begin{aligned} \sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^s| &\lesssim \sum_{k \in A_{t,k}} |I_{t,x,k}^s| + \sum_{k \in \mathbb{Z}_J, k \notin A_{t,k}} |I_{t,x,k}^s| \\ &\lesssim \frac{\delta^{-\frac{(1+\alpha)(s+1)}{2+\alpha}}}{|t|^{\frac{s+1}{2+\alpha}}} + \sum_{2^k (\delta^{\alpha+1} |t|)^{\frac{1}{\alpha+2}} \leq 1} 2^{(1+s)k} + \sum_{2^k (\delta^{\alpha+1} |t|)^{\frac{1}{\alpha+2}} \geq 1} \frac{2^{(s-\alpha-1)k} \delta^{-(\alpha+1)}}{|t|}, \end{aligned}$$

whereas if $s = \frac{\alpha}{2} < -1$

$$\sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^{\frac{\alpha}{2}}| \lesssim \frac{\delta^{-\frac{(1+\alpha)}{2}}}{\sqrt{|t|}} + \sum_{2^k (\delta^{\alpha+1} |t|)^{\frac{1}{\alpha+2}} \leq 1} \frac{2^{-\frac{\alpha+2}{2}k} \delta^{-(\alpha+1)}}{|t|} + \sum_{2^k (\delta^{\alpha+1} |t|)^{\frac{1}{\alpha+2}} \geq 1} 2^{\frac{\alpha+2}{2}k}$$

and the results follow. □

Finally, we consider the situation where some derivatives of g do not vanish at the same time.

Lemma 2.10. *Let $\delta > 0$. Let $\lambda > 0$, $\alpha, \beta \in \mathbb{R}$, $y_0, y_1 > 0$ with $y_0 < y_1$ and $l \in \mathbb{N}$ with $l \geq 2$. Assume that φ satisfies (H0), that $|g''(y)| \geq \lambda y^\beta$ for any $y \in (0, y_0]$, that $\sum_{p=2}^l |g^{(p)}| \geq \lambda$ on $[y_0, y_1]$ and that $|g''(y)| \geq \lambda y^\alpha$ for any $y \geq y_1$. Then for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$,*

$$|I_{t,x,k}^s| \lesssim \begin{cases} \frac{2^{(s-\frac{\beta}{2})k} \delta^{-\frac{1+\beta}{2}}}{\sqrt{|t|}} & , \text{ if } 2^k \delta \leq \frac{y_0}{2}, \\ \frac{2^{(s+1-\frac{1}{l})k}}{|t|^{\frac{1}{l}}} + \frac{2^{\frac{k}{2}}}{\sqrt{|t|}} & , \text{ if } \frac{y_0}{2} < 2^k \delta < 2y_1, \\ \frac{2^{(s-\frac{\alpha}{2})k} \delta^{-\frac{\alpha+1}{2}}}{\sqrt{|t|}} & , \text{ if } 2^k \delta \geq 2y_1. \end{cases}$$

Furthermore, one also have for any $2^k \delta < 2y_1$,

$$|I_{t,x,k}^0| \lesssim \frac{2^{(1-\frac{\beta+2}{l})k} \delta^{-\frac{\beta+1}{l}}}{|t|^{\frac{1}{l}}},$$

and if $\alpha < -2$,

$$\sum_{2^k \delta \geq 2y_1} \left| I_{t,x,k}^{\frac{\alpha-l+2}{l}} \right| \lesssim \frac{\delta^{-\frac{\alpha+1}{l}}}{|t|^{\frac{1}{l}}}.$$

Proof. Adapting the proof of Lemma 2.7, one can control the integrals

$$\begin{aligned} I_0 &= 2^{(1+s)k} \int_{\frac{1}{2}}^2 e^{\pm i 2^k x r} e^{i t \varphi(\delta 2^k r)} 2^k r \mathbb{1}_{\{2^k \delta r \geq y_1\}} P(r) r^s dr, \\ I_1 &= 2^{(1+s)k} \int_{\frac{1}{2}}^2 e^{\pm i 2^k x r} e^{i t \varphi(\delta 2^k r)} 2^k r (\mathbb{1}_{\{2^k \delta r \leq y_0\}} + \mathbb{1}_{\{y_0 < 2^k \delta r < y_1\}} \mathbb{1}_{\{|g''| \geq \frac{\lambda}{l-1}\}}) P(r) r^s dr. \end{aligned}$$

It remains to control the integrals, for $p \in \{2, \dots, l\}$,

$$I_p = 2^{(1+s)k} \int_{\frac{1}{2}}^2 e^{\pm i 2^k x r} e^{i t \varphi(\delta 2^k r)} 2^k r \mathbb{1}_{\{y_0 < 2^k \delta r < y_1\}} \mathbb{1}_{\{|g^{(p)}| \geq \frac{\lambda}{l-1}\}} P(r) r^s dr.$$

Denoting $\phi_1(r) = t\varphi(\delta 2^k r) 2^k r \pm 2^k x r$, we have $|\phi_1^{(p)}(r)| = 2^k \delta^{p-1} |t g^{(p)}(\delta 2^k r)|$. Using Van der Corput's Lemma we get

$$|I_p| \lesssim 2^{sk} \frac{1}{\delta} \left(\frac{\delta}{|t|} \right)^{\frac{1}{p}} \lesssim 2^{sk} \frac{\delta^{\frac{1}{2}-1}}{\sqrt{|t|}} + 2^{sk} \frac{\delta^{\frac{1}{l}-1}}{|t|^{\frac{1}{l}}},$$

and using that $2^k \delta \sim 1$ in that case we get the first estimate. For the second estimate, we consider different cases. We assume first that $2^k \delta \leq \frac{y_0}{2}$. If $2^{(\beta+2)k} \delta^{\beta+1} |t| \leq 1$,

$$|I_{t,x,k}^0| \lesssim 2^k = 2^{(1-\frac{\beta+2}{l})k} 2^{\frac{\beta+2}{l}k} \leq 2^{(1-\frac{\beta+2}{l})k} \frac{\delta^{-\frac{1+\beta}{l}}}{|t|^{\frac{1}{l}}},$$

whereas if $2^{(\beta+2)k}\delta^{\beta+1}|t| \geq 1$,

$$2^{-\frac{\beta}{2}k} \frac{\delta^{-\frac{1+\beta}{2}}}{\sqrt{|t|}} = 2^{-\frac{\beta}{2}k} \frac{\delta^{-\frac{1+\beta}{2}}}{|t|^{\frac{1}{l}}} \frac{1}{|t|^{\frac{1}{2}-\frac{1}{l}}} \leq 2^{-\frac{\beta}{2}k} \frac{\delta^{-\frac{1+\beta}{2}}}{|t|^{\frac{1}{l}}} 2^{(\beta+2)(\frac{1}{2}-\frac{1}{l})k} \delta^{(\beta+1)(\frac{1}{2}-\frac{1}{l})}$$

and the first part of the second estimate follows. With the same computations, we can consider the case $\frac{y_0}{2} < 2^k\delta < 2y_1$ through the alternative $2^k|t| \leq 1$ and $2^k|t| \geq 1$ using that $2^k\delta \sim 1$. It remains to prove the last inequality. The case $l = 2$ directly follows from Lemma 2.8. If now $l \geq 3$ and $\alpha < -2$, taking $s = \frac{\alpha-l+2}{l}$, we have $s+1 < 0$, $s - \frac{\alpha}{2} = \frac{(\alpha+2)(2-l)}{2l} > 0$ and

$$\sum_{2^k\delta \geq 2y_1} |I_{t,x,k}^s| \lesssim \sum_{2^k\delta^{\frac{\alpha+1}{\alpha+2}}|t|^{\frac{1}{\alpha+2}} \leq 1} \frac{2^{(s-\frac{\alpha}{2})k} \delta^{-\frac{\alpha+1}{2}}}{\sqrt{|t|}} + \sum_{2^k\delta^{\frac{\alpha+1}{\alpha+2}}|t|^{\frac{1}{\alpha+2}} \geq 1} 2^{(s+1)k}.$$

The last inequality follows. $\square$

2.2.4 d=2

We begin with a series of results that provide a better decay compare to the wave equation. There are however not uniform with respect to $\delta \rightarrow 0$. First, we consider the quantity for $t \in \mathbb{R}$, $x \in \mathbb{R}^2$ and χ a smooth function,

$$I_{t,x,\chi} = \int_{\mathbb{R}^2} e^{ix \cdot \xi} e^{it\varphi(\delta|\xi|)|\xi|} \chi(\delta|\xi|) d\xi \quad (6)$$

We can rewrite the integral $I_{t,x,\chi}$, using polar coordinates and the functions $h_{\pm}$,

$$\begin{aligned} I_{t,x,\chi} &= \int_{\mathbb{R}^+} \int_0^{2\pi} e^{ir|x|\sin(\theta)} e^{it\varphi(\delta r)r} \chi(\delta r) r d\theta dr \\ &= \int_{\mathbb{R}^+} e^{i(t\varphi(\delta r)r + |x|r)} h_{-}(|x|r) \chi(\delta r) r dr + \int_{\mathbb{R}^+} e^{i(t\varphi(\delta r)r - |x|r)} h_{+}(|x|r) \chi(\delta r) r dr \\ &:= I_{t,x,\chi,-} + I_{t,x,\chi,+}. \end{aligned}$$

Our first result is a low and intermediate frequency estimate assuming that $g' \neq 0$ and some other derivatives of g do not vanish on a bounded interval.

Lemma 2.11. *Let $\delta > 0$ and $J \subset \mathbb{R}_+^*$ a bounded interval. Let $\lambda > 0$ and $l \in \mathbb{N}$ with $l \geq 2$. Assume that φ satisfies (H0), that $|g'| \geq \lambda$ and $\sum_{p=2}^l |g^{(p)}| \geq \lambda$ on J . Let χ be a smooth function whose support is a subset of J . Then, for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}^2$, we have*

$$|I_{t,x,\chi}| \lesssim \frac{\delta^{\frac{2-3l}{2l}}}{|t|^{\frac{1}{2}+\frac{1}{l}}}.$$

Remark 2.12. Such configuration typically occurs when there is a coupling between a high dispersive operator and a wave operator. Note that if we do not assume that $g' \neq 0$, one can only get

$$|I_{t,x,\chi}| \lesssim \frac{\delta^{\frac{1-2l}{l}}}{|t|^{\frac{1}{l}}}.$$

Proof. By a change of variables, one can assume that $\delta = 1$. We have to control, for $p \in \{2, \dots, l\}$, the integrals

$$I_{p,\pm} = \int_{\mathbb{R}^+} e^{i(t\varphi(r)r \mp |x|r)} h_{\pm}(|x|r) \chi(r) r \mathbf{1}_{\{|g^{(p)}| \geq \frac{\lambda}{l-1}\}}(r) dr.$$

Since χ is compactly supported, note that $|I_{p,\pm}| \lesssim 1$. Then we introduce the phase $\phi_{\pm}(r) = t\varphi(r)r \mp |x|r$. We consider two cases.

Case 1: $|x| \geq \frac{\lambda}{2}|t|$

Noticing that $|\phi_{\pm}^{(p)}(r)| = |tg^{(p)}(r)| \geq \frac{\lambda}{l-1}|t|$, by Van der Corput's lemma, the properties on the functions $h_{\pm}$ and since $|I_{p,\pm}| \lesssim 1$, we get

$$|I_{p,\pm}| \lesssim \min \left(1, \frac{1}{|t|^{\frac{1}{p}}} \frac{1}{\sqrt{|x|}} \right) \lesssim \min \left(1, \frac{1}{|t|^{\frac{1}{2} + \frac{1}{p}}} \right) \lesssim \frac{1}{|t|^{\frac{1}{2} + \frac{1}{l}}}.$$

Case 2: $|x| \leq \frac{\lambda}{2}|t|$

Noticing that $|\phi'_{\pm}(r)| \geq \frac{\lambda}{2}|t|$, using Corollary 2.2 (note $\phi''_{\pm}$ has a finite number of zeros since $\phi_{\pm}^{(l-1)}$ is monotonic), the properties on the functions $h_{\pm}$ and since $|I_{p,\pm}| \lesssim 1$, we get

$$|I_{p,\pm}| \lesssim \min \left(1, \frac{1}{|t|} \right) \lesssim \frac{1}{(1 + |t|)^{\frac{1}{2} + \frac{1}{l}}} \lesssim \frac{1}{|t|^{\frac{1}{2} + \frac{1}{l}}}.$$

□

Then we provide a low frequency estimate assuming that $g'(0) \neq 0$, $g''(0) = 0$ and $g'''(0) \neq 0$. As we will see later, the water wave phase or most of the Boussinesq phases satisfy these assumptions.

Lemma 2.13. *Let $\delta > 0$. Let $\Lambda, \lambda, y_0 > 0$. Assume that φ satisfies (H1), that $g''(0) = 0$ and that $\Lambda \geq |g'| \geq \lambda$ and $|g'''| \geq \lambda$ on $[0, y_0]$. Let χ be a smooth function such that $\chi(y) = 0$ for any $y \geq y_0$. Then, for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}^2$,*

$$|I_{t,x,\chi}| \lesssim \frac{1}{\delta|t|}.$$

Remark 2.14. This decay is better than the one provided by Lemma 2.11.

Proof. By a change of variables, one can assume that $\delta = 1$. We consider two cases.

Case 1: $|x| \leq \frac{\lambda}{2}|t|$ or $|x| \geq 2\Lambda|t|$

Defining $\phi_{\pm}(r) = t\varphi(r)r \mp |x|r$, we have in that case $|\phi'_{\pm}(r)| \geq \frac{\lambda}{2}|t|$ so that, using Corollary 2.2 and the properties on the functions $h_{\pm}$, we get

$$|I_{t,x,\chi,\pm}| \lesssim \frac{1}{|t|}.$$

Case 2: $\frac{\lambda}{2}|t| \leq |x| \leq 2\Lambda|t|$

Introducing $\tilde{\chi}$ a smooth compactly supported function that is equal to 1 on the support of χ , we notice that $I_{t,x,\chi,\pm}$ is the evaluation of a Fourier transform (with respect to the variable r) at $\mp|x|$

$$I_{t,x,\chi,\pm} = (2\pi)\mathcal{F}^{-1} \left(e^{it(\varphi(r)r - g'(0)r)} h_{\pm}(|x|r)\chi(r)r \times e^{ig'(0)tr} \tilde{\chi}(r) \right) (\mp|x|)$$

so that

$$\begin{aligned} |I_{t,x,\chi,\pm}| &\lesssim \sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} e^{iry} e^{it(\varphi(r)r - g'(0)r)} h_{\pm}(|x|r)\chi(r)r dr \right| \|\mathcal{F}^{-1}(\tilde{\chi})(\cdot + tg'(0))\|_{L^1(\mathbb{R})} \\ &\lesssim \sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} e^{iry} e^{it(\varphi(r)r - g'(0)r)} h_{\pm}(|x|r)\chi(r)r dr \right|. \end{aligned}$$

Then we notice that $|g'(y) - g'(0)| \geq \frac{\lambda}{2}y^2$ and $|g''(y)| \geq \lambda y$ for any $y \in [0, y_0]$. Using Lemma 2.8 (or Corollary 2.9 in [KPV91], see Lemma 2.5) and the properties on the functions $h_{\pm}$, we get for any $y \in \mathbb{R}$,

$$\left| \int_{\mathbb{R}} e^{iry} e^{it(\varphi(r)r - g'(0)r)} \sqrt{|g''(r)|} \times h_{\pm}(|x|r)\chi(r) \frac{r}{\sqrt{|g''(r)|}} dr \right| \lesssim \frac{1}{\sqrt{|t|}} \frac{1}{\sqrt{|x|}}$$

and the result follows in that case. $\square$

Secondly, we provide estimates for more general phases. For this reason we use a Littlewood-Paley decomposition and introduce the quantity for any $t \in \mathbb{R}$, $x \in \mathbb{R}^2$, $s \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$I_{t,x,k}^s = \int_{\mathbb{R}^2} e^{ix \cdot \xi} e^{it\varphi(\delta|\xi|)|\xi|} P_k(|\xi|) |\xi|^s d\xi.$$

Using a change of variables and polar coordinates we can rewrite $I_{t,x,k}$

$$\begin{aligned}
I_{t,x,k}^s &= 2^{(2+s)k} \int_{\frac{1}{2}}^2 \int_0^{2\pi} e^{i2^k r |x| \sin(\theta)} e^{it\varphi(\delta 2^k r) 2^k r} P(r) r^{1+s} d\theta dr \\
&= 2^{(2+s)k} \int_{\frac{1}{2}}^2 e^{i(t\varphi(\delta 2^k r) 2^k r + 2^k |x| r)} h_-(2^k |x| r) P(r) r^{1+s} dr \\
&\quad + 2^{(2+s)k} \int_{\frac{1}{2}}^2 e^{i(t\varphi(\delta 2^k r) 2^k r - 2^k |x| r)} h_+(2^k |x| r) P(r) r^{1+s} dr \\
&:= I_{t,x,k,-}^s + I_{t,x,k,+}^s.
\end{aligned}$$

We recall that

$$\mathbb{Z}_J := \{k \in \mathbb{Z}, \forall r \in [\frac{1}{2}, 2], 2^k \delta r \in J\}.$$

Lemma 2.15. *Let $\delta > 0$ and $J \subset \mathbb{R}_+^*$ an interval. Let $\lambda > 0$, $\alpha, \theta, s \in \mathbb{R}$. Assume that φ satisfies (H0) and that $|g'(y)| \geq \lambda y^\theta$ and $|g''(y)| \geq \lambda y^\alpha$ on J . Then for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}^2$ and $k \in \mathbb{Z}_J$,*

$$|I_{t,x,k}^s| \lesssim \frac{2^{(s+\frac{1-\alpha-\theta}{2})k} \delta^{-\frac{\alpha+\theta+1}{2}}}{|t|} + \frac{2^{(s+1-\theta)k} \delta^{-\theta}}{|t|}.$$

Remark 2.16. Note that when $\theta = 0$, $\alpha = 1$ and J is bounded (water-wave-type behavior at low frequencies) Lemma 2.13 provides a better result in the sense that we actually have

$$\left| \sum_{k \in \mathbb{Z}_J} I_{t,x,k}^0 \right| \lesssim \frac{1}{\delta |t|}.$$

Furthermore, as we will see in Lemma 2.17, when $\theta = 1$, $\alpha = 0$ (Schrödinger-type behavior), we will prove under more technical assumptions that

$$\sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^0| \lesssim \frac{1}{\delta |t|}.$$

Proof. We introduce the phase $\phi_\pm(r) = t\varphi(\delta 2^k r) 2^k r \mp 2^k |x| r$. We consider two different cases.

Case 1: $|x| \geq \frac{|t|}{2} \min_{r \in [1/2, 2]} |g'(2^k \delta r)|$

Noticing that $|\phi_\pm''(r)| = 2^{2k} \delta |tg''(\delta 2^k r)|$, we use Van der Corput's lemma and the properties on the functions $h_\pm$, we get

$$|I_{t,x,k,\pm}^s| \lesssim \frac{2^{(s+2)k}}{\sqrt{2^{2k} |\delta tg''(\delta 2^k r_1)|}} \frac{1}{\sqrt{1 + 2^k |x|}} \lesssim \frac{2^{(s+2)k}}{\sqrt{2^{3k} \delta |tg'(\delta 2^k r_1) tg'(2^k \delta r_2)|}}$$

with $r_1, r_2 \in [\frac{1}{2}, 2]$. The result follows in that case.

Case 2: $|x| \leq \frac{|t|}{2} \min_{r \in [1/2, 2]} |g'(2^k \delta r)|$

Noticing that $|\phi'_\pm(r)| \gtrsim 2^k |tg'(2^k \delta r_1)|$ for some $r_1 \in [\frac{1}{2}, 2]$ and using Corollary 2.2 and the properties on the functions $h_\pm$, we get

$$|I_{t,x,k,\pm}^s| \lesssim \frac{2^{(s+2)k}}{2^k |tg'(\delta 2^k r_1)|}.$$

The result follows in that case. □

The next lemma has to be seen as a generalization of Lemma 2.3 in the 2d case.

Lemma 2.17. *Let $\delta > 0$ and $J \subset \mathbb{R}_+^*$ an interval. Let $\Lambda, \lambda > 0$, $\alpha, s \in \mathbb{R}$. Assume that φ satisfies (H0) and that $\Lambda y^{\alpha+1} \geq |g'(y)| \geq \lambda y^{\alpha+1}$ and $|g''(y)| \geq \lambda y^\alpha$ on J . Then:*

(1) *If $-2 < s < \alpha$ and $\alpha \neq -1$, for any $t \in \mathbb{R}^*$, we have*

$$\sup_{x \in \mathbb{R}^2} \sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^s| \lesssim \frac{\delta^{-\frac{(\alpha+1)(s+2)}{2+\alpha}}}{|t|^{\frac{s+2}{2+\alpha}}}.$$

(2) *If $s = \alpha$ and $\alpha \notin \{-2, -1\}$, assuming furthermore that $|g''(y)| \leq \Lambda y^\alpha$ and $|g'''(y)| \leq \Lambda y^{\alpha-1}$ on J , for any $t \in \mathbb{R}^*$, we have*

$$\sup_{x \in \mathbb{R}^2} \sum_{k \in \mathbb{Z}_J} |I_{t,x,k}^\alpha| \lesssim \frac{\delta^{-(\alpha+1)}}{|t|}.$$

Remark 2.18. This result can be read as follows. We assume by simplicity that $\delta = 1$. Let $\Lambda, \lambda > 0$, $l \in \mathbb{N}$ with $l \geq 2$. We recall that we define the quantity $I_{t,x,\chi}$ in (6).

(1) Assume that $|g^{(l)}| \geq \lambda$ on a bounded interval $J \subset \mathbb{R}^+$ containing 0 with $g^{(k)}(0) = 0$ for any $k \in \{1, \dots, l-1\}$, then for any smooth function χ whose support is a subset of J , we have

$$|I_{t,x,\chi}| \lesssim \frac{1}{|t|^{\frac{2}{l}}}.$$

(2) Assume that $\Lambda \geq |g^{(l)}| \geq \lambda$ on an unbounded interval $J \subset \mathbb{R}^+$ with $\inf(J) > 0$, then there exists $y_1 \in J$ such that for any smooth function χ whose support is a subset of $[y_1, +\infty)$ and with χ' compactly supported, we have

$$|I_{t,x,\chi}| \lesssim \frac{1}{|t|^{\frac{2}{l}}}.$$

For instance, when $l = 2$ (Schrödinger-type behavior) we get a decay of order $\frac{1}{|t|}$.

Proof. Note that the assumptions on s and α gives that $(s+2)$ and $(2+\alpha)$ agree in sign. We always have, for any $k \in \mathbb{Z}_J$, $|I_{t,x,k,\pm}^s| \lesssim 2^{(2+s)k}$. We consider several cases.

Case 1: $2^k|x| \geq 1$

Sub-case 1: $\frac{\lambda}{4}\delta^{\alpha+1}2^{(\alpha+1)k}|t| \leq |x| \leq 4\Lambda\delta^{\alpha+1}2^{(\alpha+1)k}|t|$

The size of the set $A_{t,x} := \{k \in \mathbb{Z}, \frac{\lambda}{4}\delta^{\alpha+1}2^{(\alpha+1)k}|t| \leq |x| \leq 4\Lambda\delta^{\alpha+1}2^{(\alpha+1)k}|t|\}$ is bounded by a number independent of t, x, δ . If $2^k(\delta^{\alpha+1}|t|)^{\frac{1}{2+\alpha}} \leq 1$, we have

$$|I_{t,x,k}^s| \lesssim 2^{(2+s)k} \leq \frac{\delta^{-\frac{(\alpha+1)(s+2)}{2+\alpha}}}{|t|^{\frac{s+2}{2+\alpha}}},$$

whereas if $2^k(\delta^{\alpha+1}|t|)^{\frac{1}{2+\alpha}} \geq 1$, we get from Lemma 2.15

$$|I_{t,x,k}| \lesssim \frac{2^{(s-\alpha)k}\delta^{-(\alpha+1)}}{|t|} = \frac{2^{(s-\alpha)k}\delta^{-(\alpha+1)}}{|t|^{\frac{2+s}{2+\alpha}}} \frac{1}{|t|^{\frac{\alpha-s}{2+\alpha}}} \lesssim \frac{\delta^{-\frac{(\alpha+1)(s+2)}{2+\alpha}}}{|t|^{\frac{s+2}{2+\alpha}}}.$$

Sub-case 2: $\frac{\lambda}{4}\delta^{\alpha+1}2^{(\alpha+1)k}|t| \geq |x|$ or $|x| \geq 4\Lambda\delta^{\alpha+1}2^{(\alpha+1)k}|t|$.

Defining $\phi_{\pm}(r) = t\varphi(\delta 2^k r)2^k r \mp 2^k|x|r$, we notice that $|\phi'_{\pm}(r)| \geq \frac{\lambda}{4}\delta^{\alpha+1}2^{(\alpha+2)k}|t|$. If $s < \alpha$, use directly Corollary 2.2 and the properties on $h_{\pm}$, we get.

$$|I_{t,x,k,\pm}^s| \lesssim \frac{2^{(s-\alpha)k}\delta^{-(\alpha+1)}}{|t|},$$

If now $s = \alpha$, integrating by parts, we get

$$I_{t,x,k,\pm}^{\alpha} = 2^{(2+\alpha)k}i \int_{\frac{1}{2}}^2 e^{i(t\varphi(\delta 2^k r)2^k r \mp 2^k|x|r)} \frac{d}{dr} \left(\frac{1}{\phi'_{\pm}(r)} h_{\pm}(2^k|x|r) P(r) r \right) dr$$

so that using Corollary 2.2, the properties on $h_{\pm}$ and the controls on g', g'', g''' , we get

$$|I_{t,x,k,\pm}^{\alpha}| \lesssim \frac{2^{-(\alpha+2)k}\delta^{-2(\alpha+1)}}{t^2}.$$

Case 2: $2^k|x| \leq 1$

In that case, we set $A_{t,x} = \emptyset$. Adapting the strategy used in the previous sub-case to the phase $\phi_{\pm}(r) = t\varphi(2^k\delta r)2^k r$, we similarly get, with no restriction on k , for $s < \alpha$,

$$|I_{t,x,k,\pm}^s| \lesssim \frac{2^{(s-\alpha)k}\delta^{-(\alpha+1)}}{|t|} \text{ and } |I_{t,x,k,\pm}^{\alpha}| \lesssim \frac{2^{-(\alpha+2)k}\delta^{-2(\alpha+1)}}{t^2}.$$

Gathering all the previous cases and sub-cases, we conclude as in Lemma 2.8 when $s < \alpha$. If now $s = \alpha$ with $\alpha \neq 2$ we get

$$\begin{aligned}
\left| \sum_{k \in \mathbb{Z}_J} I_{t,x,k}^\alpha \right| &\lesssim \sum_{k \in \mathbb{Z}_J, k \in A_{t,k}} |I_{t,x,k}^\alpha| + \sum_{k \in \mathbb{Z}_J, k \notin A_{t,k}} |I_{t,x,k}^\alpha| \\
&\lesssim \frac{\delta^{-(\alpha+1)}}{|t|} + \sum_{k \in \mathbb{Z}_J, k \notin A_{t,k}, 2^k(\delta^{\alpha+1}|t|)^{\frac{1}{2+\alpha}} \leq 1} |I_{t,x,k}^\alpha| \\
&\quad + \sum_{k \in \mathbb{Z}_J, k \notin A_{t,k}, 2^k(\delta^{\alpha+1}|t|)^{\frac{1}{2+\alpha}} \geq 1} |I_{t,x,k}^\alpha| \\
&\lesssim \frac{\delta^{-(\alpha+1)}}{|t|} + \sum_{2^k(\delta^{\alpha+1}|t|)^{\frac{1}{2+\alpha}} \leq 1} 2^{(2+\alpha)k} + \sum_{2^k(\delta^{\alpha+1}|t|)^{\frac{1}{2+\alpha}} \geq 1} \frac{2^{-(\alpha+2)k} \delta^{-2(\alpha+1)}}{t^2}
\end{aligned}$$

and the result follows. $\square$

We then give two other lemmas with weaker decays compare to the previous lemma. In both situations, we assume that φ satisfies (H1) so that we see the problem as a perturbation of the half-wave equation. The following lemma gives a decay when g' and g'' do not vanish at the same time.

Lemma 2.19. *Let $\delta > 0$. Let $\alpha \in \mathbb{R}$, $\lambda, y_0 > 0$. Assume that φ satisfies (H1), that $|g'| \geq \lambda$ on $[0, y_0]$, that g'' has a finite number of zeros on $[0, y_0]$ and that $|g''(y)| \geq \lambda y^\alpha$ for any $y \geq y_0$. Then, for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}^2$ and $k \in \mathbb{Z}$,*

$$|I_{t,x,k}| \lesssim \begin{cases} \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}} & , \text{ if } 2^k \delta < 2y_0, \\ \frac{2^{(1-\frac{\alpha}{2})k} \delta^{-\frac{\alpha+1}{2}}}{\sqrt{|t|}} & , \text{ if } 2^k \delta \geq 2y_0. \end{cases}$$

Remark 2.20. Note that by taking $\alpha \leq -1$ (which is always possible), this estimate is uniform with respect to $\delta \in (0, 1]$. Note also that when $\delta = 0$, we exactly get the estimate of the wave equation.

Proof. We rewrite $I_{t,x,k}$ as

$$\begin{aligned}
I_{t,x,k} &= I_0 + I_1 \\
&= 2^{2k} \int_0^{2\pi} \int_{\frac{1}{2}}^2 e^{i(2^k|x|r \sin(\theta) + t\varphi(\delta 2^k r) 2^k r)} \mathbb{1}_{\{2^k \delta r \geq y_0\}}(r) P(r) r dr d\theta \\
&\quad + 2^{2k} \int_{\frac{1}{2}}^2 e^{i(\operatorname{sgn}(tg'(0)) 2^k|x|r + t\varphi(\delta 2^k r) 2^k r)} \mathbb{1}_{\{2^k \delta r < y_0\}} P(r) r J_0(2^k|x|r) e^{-i \operatorname{sgn}(tg'(0)) 2^k|x|r} dr.
\end{aligned}$$

We introduce here the phase $\phi_0(r) = 2^k|x|r \sin(\theta) + t\varphi(\delta 2^k r) 2^k r$ with

$$|\phi_0''(r)| = 2^{2k} \delta |tg''(\delta 2^k r)| \gtrsim 2^{(2+\alpha)k} \delta^{1+\alpha} |t|.$$

Using Van der Corput's Lemma we get

$$|I_0| \lesssim \frac{2^{(1-\frac{\alpha}{2})k} \delta^{-\frac{\alpha+1}{2}}}{\sqrt{|t|}}.$$

Note that when $\frac{y_0}{2} \leq 2^k \delta \leq 2y_0$,

$$|I_0| \lesssim \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}}.$$

Then, we introduce the phase $\phi_1(r) = \text{sgn}(tg'(0))2^k|x|r + t\varphi(\delta 2^k r)2^k r$ and

$$|\phi_1'(r)| = 2^k|x| + 2^k|tg'(\delta 2^k r)|.$$

Using Corollary 2.2 and some properties of the Bessel function J_0 we get

$$|I_1| \lesssim \frac{2^{2k}}{2^k|x| + 2^k|t|} \sqrt{1 + 2^k|x|}.$$

If $2^k|x| \geq 1$,

$$|I_1| \lesssim 2^{2k} \frac{1}{\sqrt{2^k|t|}} \lesssim \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}},$$

whereas if $2^k|x| \leq 1$ and $2^k|t| \geq 1$,

$$|I_1| \lesssim 2^{2k} \frac{1}{2^k|t|} \lesssim \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}}$$

or if $2^k|x| \leq 1$ and if $2^k|t| \leq 1$,

$$|I_1| \lesssim 2^{2k} \lesssim \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}}.$$

□

Then, we consider the situation where some derivatives of g do not vanish at the same time.

Lemma 2.21. *Let $\delta > 0$. Let $\lambda > 0$, $\alpha \in \mathbb{R}$, $y_0, y_1 > 0$ with $y_0 < y_1$ and $l \in \mathbb{N}$ with $l \geq 2$. Assume that φ satisfies (H1), that $|g'| \geq \lambda$ on $[0, y_0]$, $\sum_{s=1}^l |g^{(s)}| \geq \lambda$ on $[y_0, y_1]$, that g'' has a finite number of zeros on $[0, y_0]$ and that $|g''(y)| \geq \lambda y^\alpha$ for any $y \geq y_1$. Then for any $t \in \mathbb{R}^*$, $x \in \mathbb{R}^2$ and $k \in \mathbb{Z}$,*

$$|I_{t,x,k}| \lesssim \begin{cases} \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}} & , \text{ if } 2^k \delta \leq \frac{y_0}{2}, \\ \frac{2^{(2-\frac{1}{l})k}}{|t|^{\frac{1}{l}}} + \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}} & , \text{ if } \frac{y_0}{2} < 2^k \delta < 2y_1, \\ \frac{2^{(1-\frac{\alpha}{2})k} \delta^{-\frac{\alpha+1}{2}}}{\sqrt{|t|}} & , \text{ if } 2^k \delta \geq 2y_1. \end{cases}$$

Furthermore, one also have

$$|I_{t,x,k}| \lesssim \begin{cases} \frac{2^{(2-\frac{1}{l})k}}{|t|^{\frac{1}{l}}} & , \text{ if } 2^k \delta < 2y_1, \\ \frac{2^{(2-\frac{\alpha+2}{l})k} \delta^{-\frac{\alpha+1}{l}}}{|t|^{\frac{1}{l}}} & , \text{ if } 2^k \delta \geq 2y_1. \end{cases}$$

Proof. Adapting the proof of the previous Lemma, one can control the integrals

$$I_0 = 2^{2k} \int_0^{2\pi} \int_{\frac{1}{2}}^2 e^{i(2^k|x|r \sin(\theta) + t\varphi(\delta 2^k r) 2^k r)} \mathbb{1}_{\{2^k \delta r \geq y_1\}} P(r) r dr d\theta,$$

$$I_1 = 2^{2k} \int_0^{2\pi} \int_{\frac{1}{2}}^2 e^{i(2^k|x|r \sin(\theta) + t\varphi(\delta 2^k r) 2^k r)} (\mathbb{1}_{\{2^k \delta r \leq y_0\}} + \mathbb{1}_{\{y_0 < 2^k \delta r < y_1\}} \mathbb{1}_{\{|g'| \geq \frac{\lambda_1}{l}\}}) P(r) r dr d\theta.$$

It remains to control the integrals, for $s \in \{2, \dots, l\}$,

$$I_s = 2^{2k} \int_0^{2\pi} \int_{\frac{1}{2}}^2 e^{i(2^k|x|r \sin(\theta) + t\varphi(\delta 2^k r) 2^k r)} \mathbb{1}_{\{y_0 < 2^k \delta r < y_1\}} \mathbb{1}_{\{|g^{(s)}| \geq \frac{\lambda_1}{l}\}} P(r) r dr d\theta.$$

For $\phi_1(r) = 2^k|x|r \sin(\theta) + t\varphi(\delta 2^k r) 2^k r$, we have $|\phi_1^{(s)}(r)| = 2^{sk} \delta^{s-1} |tg^{(s)}(\delta 2^k r)|$. Using Van der Corput's Lemma we get

$$|I_s| \lesssim \frac{2^{2k}}{\delta} \left(\frac{\delta}{2^{ks}|t|} \right)^{\frac{1}{s}} \lesssim \frac{2^k}{\sqrt{\delta|t|}} + \frac{2^k \delta^{\frac{1}{l}-1}}{|t|^{\frac{1}{l}}},$$

and since $2^k \delta \sim 1$ in that case,

$$|I_s| \lesssim \frac{2^{\frac{3}{2}k}}{\sqrt{|t|}} + \frac{2^{(2-\frac{1}{l})k}}{|t|^{\frac{1}{l}}}.$$

Therefore we get the first estimate. For the second estimate, we consider different cases. We assume first that $2^k \delta < 2y_1$. If $2^k |t| \leq 1$, we have

$$|I_{t,x,k}| \lesssim 2^{2k} = 2^{(2-\frac{1}{l})k} 2^{\frac{k}{l}} \leq \frac{2^{(2-\frac{1}{l})k}}{|t|^{\frac{1}{l}}},$$

whereas if $2^k |t| \geq 1$, we get

$$\frac{2^{\frac{3}{2}k}}{\sqrt{|t|}} = \frac{2^{\frac{3}{2}k}}{|t|^{\frac{1}{l}} |t|^{(\frac{1}{2}-\frac{1}{l})}} \leq \frac{2^{\frac{3}{2}k}}{|t|^{\frac{1}{l}}} 2^{(\frac{1}{2}-\frac{1}{l})k} = \frac{2^{(2-\frac{1}{l})k}}{|t|^{\frac{1}{l}}}.$$

Finally the case $2^k \delta \geq 2y_1$ follows from the same controls but with the alternative $2^{(2+\alpha)k} \delta^{(1+\alpha)} |t| \leq 1$ and $2^{(2+\alpha)k} \delta^{(1+\alpha)} |t| \geq 1$. $\square$

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