A well-balanced scheme for the shallow-water equations with topography and friction

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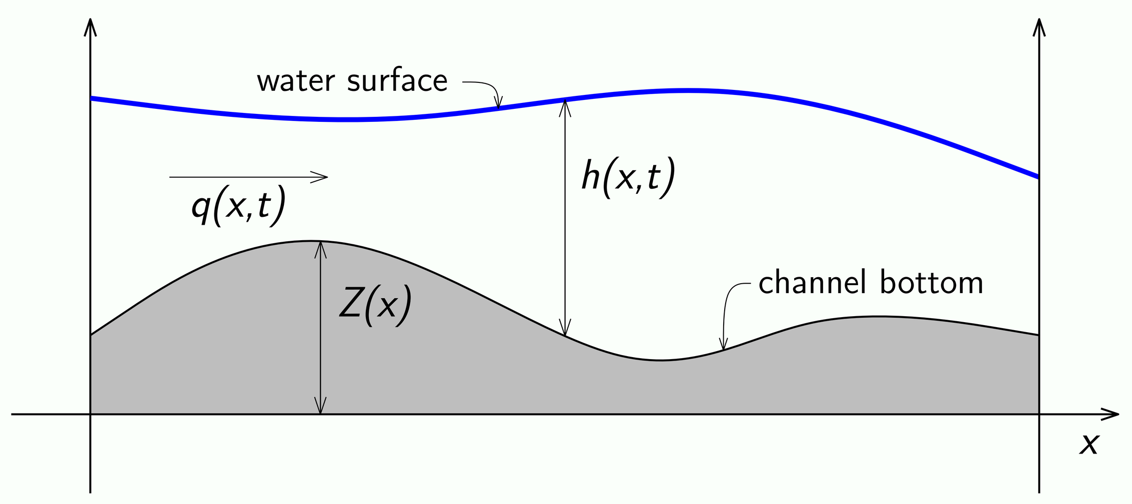
Introduction

- **model under consideration**: We study the shallow-water equations with the topography and Manning friction source terms:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - kq|q|h^{-\eta}. \end{cases}$$

  - $h \geq 0$ is the water height
  - q is the horizontal water discharge
  - $g > 0$ is the gravity constant
  - Z is the smooth topography
  - k is the friction coefficient and $\eta = 7/3$



- **steady state solutions**: They are time-independent solutions, governed by the shallow-water model with vanishing time derivatives:

$$\begin{cases} q = \text{cst} = q_0 \\ \partial_x \left(\dfrac{q_0^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - kq_0|q_0|h^{-\eta}. \end{cases}$$

- **objectives**: Propose a numerical scheme that:
  - is consistent with the shallow-water equations;
  - preserves all the steady states (*well-balance* property);
  - preserves the non-negativity of the height (*robustness* property);
  - provides a high order of accuracy.

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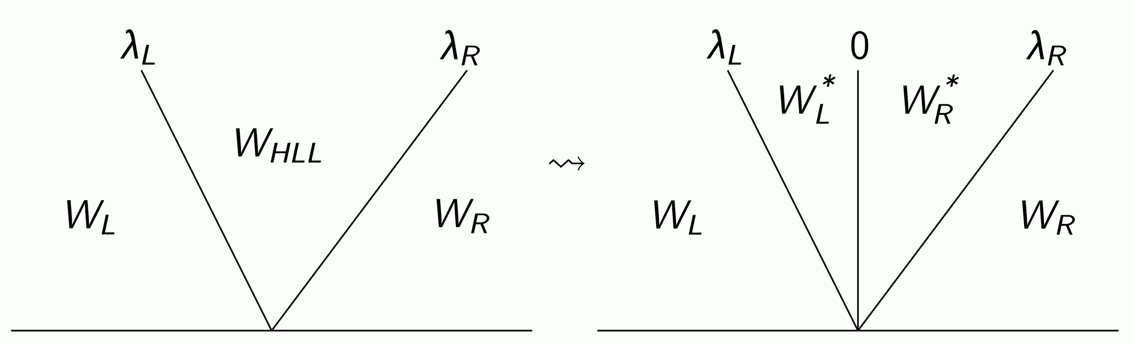
1. A generic well-balanced scheme

Structure of the Godunov-type scheme

Consider the shallow-water equations with a generic source term:

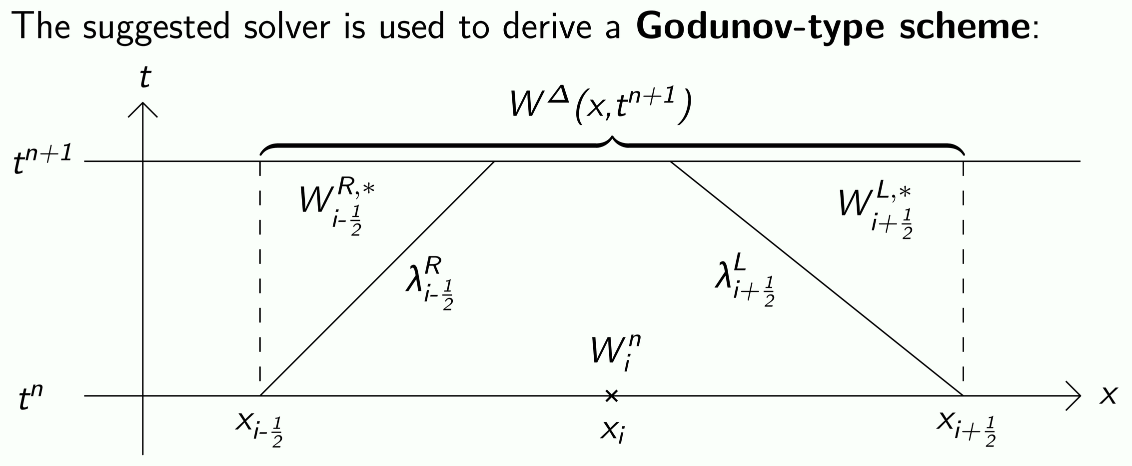
$$\partial_t W + \partial_x F(W) = \begin{pmatrix} 0 \\ S(W) \end{pmatrix}.$$

We use an **approximate Riemann solver** $\widetilde{W}$ based on the HLL solver.



HLL solver — suggested solver

The suggested solver is used to derive a **Godunov-type scheme**:



$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^L_{i+\frac12}\left(W^{L,*}_{i+\frac12} - W_i^n \right) - \lambda^R_{i-\frac12}\left(W^{R,*}_{i-\frac12} - W_i^n \right) \right].$$

Goal: determine the **intermediate states** $W_L^* = {}^t(h_L^*, q^*)$ and $W_R^* = {}^t(h_R^*, q^*)$ to get a *consistent*, *well-balanced* and *robust* scheme.

Consistency

We impose the following Harten-Lax integral **consistency** relation:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}\left(\frac{x}{\Delta t}; W_L, W_R \right) dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R \right) dx.$$

We assume known the following **source term average**:

$$\overline{S} \simeq \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} S\left(W_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R \right) \right) dx\, dt,$$

to finally get several relations governing the intermediate states:

$$\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L)\, h_{HLL}$$

$$q^* = q_{HLL} + \frac{\overline{S}\Delta x}{\lambda_R - \lambda_L}.$$

Well-balance and non-negativity

We seek the **well-balance property**:
$W_L^* = W_L$ and $W_R^* = W_R$ as soon as W_L and W_R satisfy the relation

$$q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S}\Delta x.$$

We thus impose the following relation on the intermediate heights:

$$\left(\frac{-(q^*)^2}{h_L h_R} + \frac{g}{2}(h_L + h_R) \right)(h_R^* - h_L^*) = \alpha(h_R^* - h_L^*) = \overline{S}\Delta x,$$

and we obtain their expressions, as follows:

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}. \end{cases}$$

Note that we do not have the **non-negativity**: instead, we set

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L} \right) h_{HLL} \right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R} \right) h_{HLL} \right). \end{cases}$$

2. Application to specific source terms

The topography source term $S^t = -gh\partial_x Z$

The topography steady states are governed by the following relations:

$$\partial_x\left(\frac{q_0^2}{h} + \frac{1}{2}gh^2 \right) = S^t \quad \text{and} \quad \partial_x\left(\frac{q_0^2}{2h^2} + g(h+Z) \right) = 0.$$

At the discrete level, they become:

$$q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S^t}\Delta x \quad \text{and} \quad \frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h+Z] = 0.$$

second equation ⇝ expression of q_0^2

first equation ⇝ $\overline{S^t} = -g\dfrac{2h_L h_R}{h_L + h_R}\dfrac{[Z]}{\Delta x} + \dfrac{g}{2\Delta x}\dfrac{[h]^3}{h_L + h_R}$

The friction source term $S^f = -kq|q|h^{-\eta}$

The friction steady states are governed by the following relations:

$$\partial_x\left(\frac{q_0^2}{h} + \frac{gh^2}{2} \right) = -\frac{kq_0|q_0|}{h^\eta} \quad \text{and} \quad \partial_x\left(q_0^2\frac{h^{\eta-1}}{\eta-1} - g\frac{h^{\eta+2}}{\eta+2} \right) = kq_0|q_0|.$$

At the discrete level, we set $\overline{S}^f = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$, where:

- $\bar{q}$ is the harmonic mean of q_L and q_R;
- with $\mu_0 = \text{sgn}(q_0)$, the average $\overline{h^{-\eta}}$ is governed by:

$$q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = -k\mu_0 q_0^2 \Delta x\, \overline{h^{-\eta}},$$

$$-q_0^2\frac{\left[h^{\eta-1}\right]}{\eta-1} - g\frac{\left[h^{\eta+2}\right]}{\eta+2} = -k\mu_0 q_0^2\Delta x.$$

second equation ⇝ expression of q_0^2

first equation ⇝ $\overline{h^{-\eta}} = \dfrac{[h^2]}{2}\dfrac{\eta+2}{[h^{\eta+2}]} - \dfrac{\bar{\mu}}{k\,\Delta x}\left(\left[\dfrac{1}{h}\right] + \dfrac{[h^2]}{2}\dfrac{[h^{\eta-1}]}{\eta-1}\dfrac{\eta+2}{[h^{\eta+2}]} \right)$

Semi-implicitation of the scheme

The friction source term becomes **stiff** when the height is close to zero: we use a semi-implicit scheme (**splitting method**).
With both source terms, we exhibit the numerical flux:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac12} - \mathcal{F}^n_{i-\frac12} \right) + \Delta t\left(\begin{pmatrix} 0 \\ (\mathcal{S}^t)_i^n \end{pmatrix} + \begin{pmatrix} 0 \\ (\mathcal{S}^f)_i^n \end{pmatrix} \right).$$

first step: Solve $\partial_t W + \partial_x F(W) = {}^t(0, S^t(W))$ to get $W_i^{n+\frac12}$:

$$W_i^{n+\frac12} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac12} - \mathcal{F}^n_{i-\frac12} \right) + \Delta t \begin{pmatrix} 0 \\ (\mathcal{S}^t)_i^n \end{pmatrix}.$$

second step: Solve $\partial_t W = {}^t(0, S^f(W))$, to get $h_i^{n+1} = h_i^{n+\frac12}$ and:

$$q_i^{n+1} = \frac{(h_i^{n+1})^\eta q_i^{n+\frac12}}{(h_i^{n+1})^\eta + k\,\Delta t\,\left|q_i^{n+\frac12}\right|}.$$

Note that $q_i^{n+1} \neq q_i^n$ for a steady state: we replace $(h_i^{n+1})^\eta$ with a well-chosen average $(\widetilde{h^\eta})_i^{n+1}$ to ensure the well-balance.

3. High-order 2D extension

High-order strategy for the two-dimensional model

The goal is now to approximate the **2D shallow-water equations**:

$$\begin{cases} \partial_t h + \nabla \cdot \boldsymbol{q} = 0, \\ \partial_t \boldsymbol{q} + \nabla \cdot \left(\dfrac{\boldsymbol{q}\otimes\boldsymbol{q}}{h} + \dfrac{1}{2}gh^2\mathbb{I}_2 \right) = -gh\nabla Z - \dfrac{k\boldsymbol{q}\|\boldsymbol{q}\|}{h^\eta}. \end{cases}$$

To that end, we use the following **high-order** scheme:

$$W_i^{n+1} = W_i^n - \Delta t \sum_{j\in\nu_i}\frac{|e_{ij}|}{|c_i|}\sum_{r=0}^{R}\xi_r \mathcal{F}^n_{ij,r} + \Delta t\sum_{q=0}^{Q}\eta_q\left((\mathcal{S}^t)^n_{i,q} + (\mathcal{S}^f)^n_{i,q} \right),$$

which takes advantage of the following **polynomial reconstruction** (reconstruction of degree d $\implies$ scheme of order $d+1$):

$$\widehat{W}_i^n(x) = W_i^n + \sum_{|k|=1}^{d}\alpha_i^k\left[(x - x_i)^k - \frac{1}{|c_i|}\int_{c_i}(x - x_i)^k dx \right].$$

SSPRK methods are used as a high-order time integrator.

Recovering the well-balance and the robustness

Because of the reconstruction, the **well-balance** and the **robustness** properties are lost: to recover them, we suggest a **MOOD** method.

well-balance: We introduce a convex combination between the first-order scheme and the high-order scheme:

$$W_i^{n+1} = \theta_i^n (W_{\text{HO}})_i^{n+1} + (1 - \theta_i^n)(W_{\text{WB}})_i^{n+1}.$$

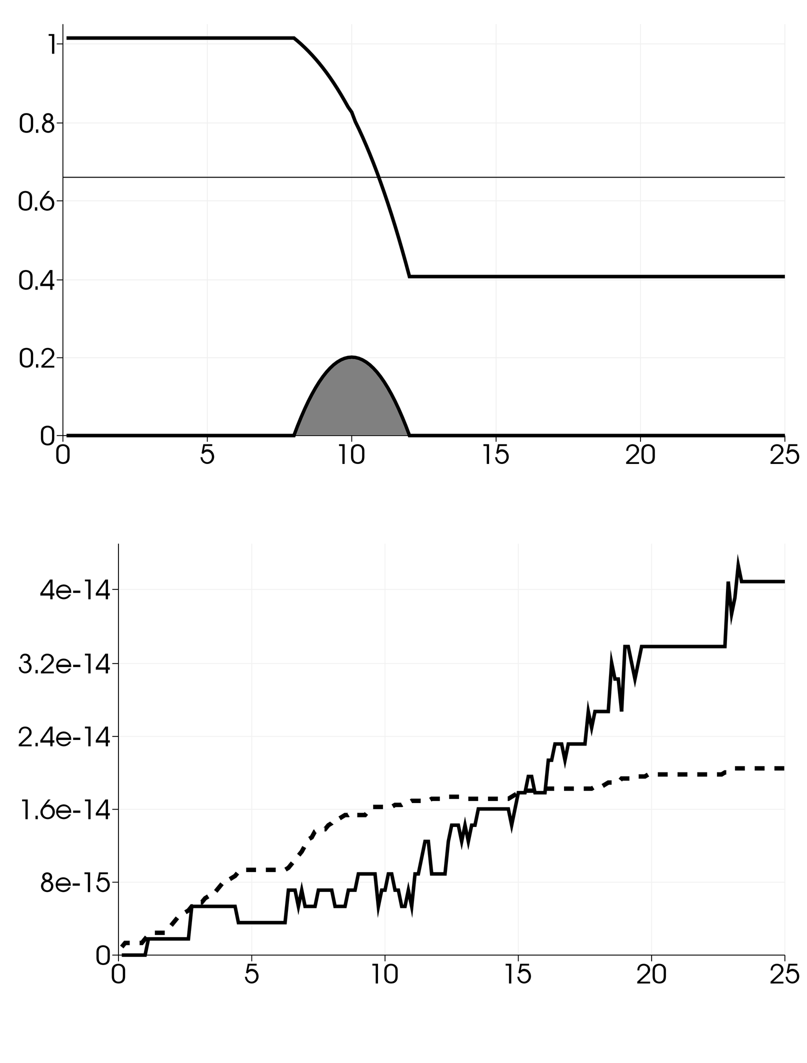
- $\theta_i^n = 0$ close to a steady solution ⇝ use the well-balanced scheme
- $\theta_i^n = 1$ far from a steady solution ⇝ use the high-order scheme

robustness: We use a classical MOOD method to lower the degree of the polynomial reconstruction until the robustness is recovered.

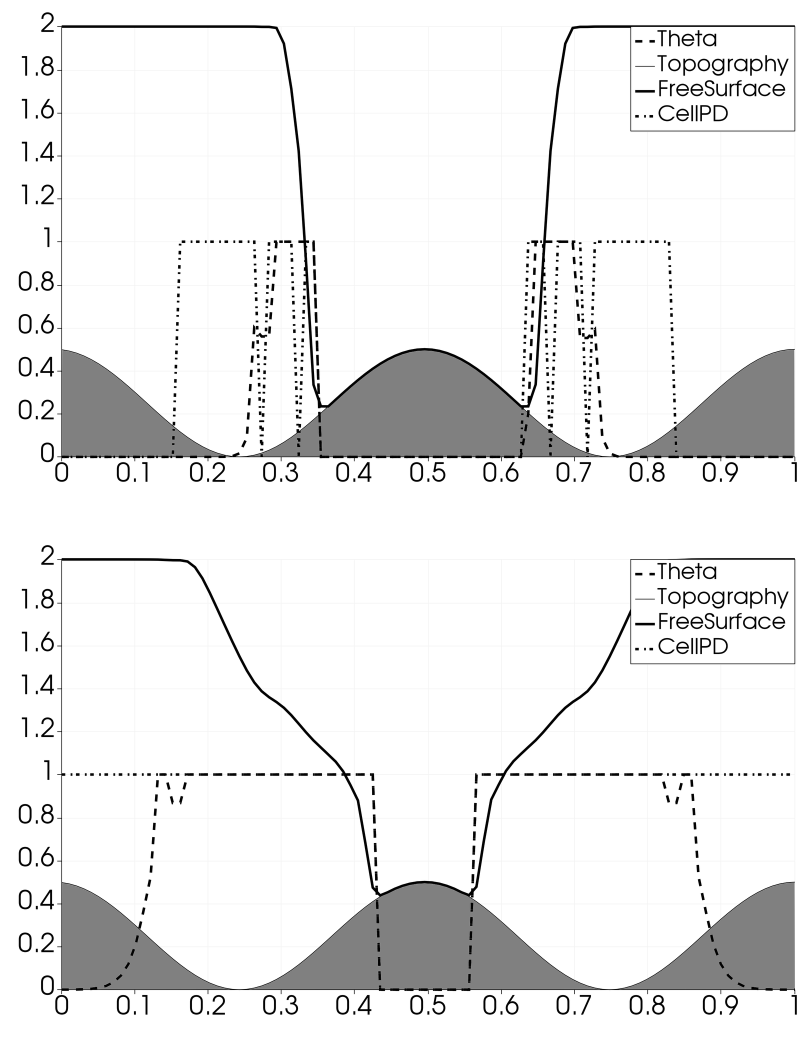
References

- S. Clain, S. Diot, and R. Loubère.
  A high-order finite volume method for systems of conservation laws—Multi-dimensional Optimal Order Detection (MOOD).
  J. Comput. Phys., 230(10):4028–4050, 2011.
- V. Michel-Dansac, C. Berthon, S. Clain, and F. Foucher.
  A well-balanced scheme for the shallow-water equations with topography.
  Comput. Math. Appl., 72(3):568–593, 2016.

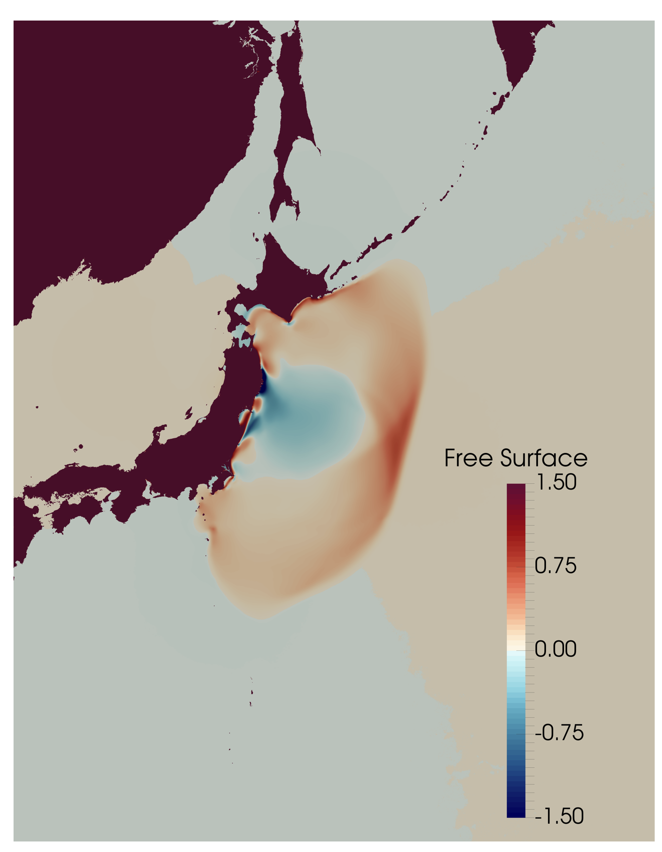
Transcritical flow without shock



Dry dam-break on a sinusoidal bottom



Simulation of the 2011 Japan tsunami



see experiments in motion by scanning the QR codes
← Japan ‖ dam-break →