Second order Implicit-Explicit Total Variation Diminishing schemes for the Euler system in the low Mach regime

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Introduction

- **Model under consideration**. We study the compressible isentropic Euler system:

$$(M_\varepsilon)\begin{cases}\partial_t\rho+\nabla\cdot(\rho U)=0,\\ \partial_t(\rho U)+\nabla\cdot(\rho U\otimes U)+\dfrac{1}{\varepsilon}\nabla p(\rho)=0.\end{cases}$$

  - $\rho>0$ is the density of the fluid
  - $U\in\mathbb{R}$ is the velocity of the fluid
  - $p(\rho)=\rho^\gamma$ is the pressure
  - $\gamma\geq 1$ is the ratio of specific heats
  - ε is the squared Mach number

  This model introduces fast acoustic waves, governed by:

$$\partial_{tt}\rho-\frac{1}{\varepsilon}\Delta p(\rho)=\nabla^2:(\rho U\otimes U).$$

- **Incompressible limit**. With well-prepared initial and boundary conditions, the compressible Euler system tends to the following incompressible limit when ε tends to 0:

$$(M_0)\begin{cases}\rho=\rho_0,\\ \nabla\cdot U=0,\\ \rho_0\,\partial_t U+\rho_0\,\nabla\cdot(U\otimes U)+\nabla\pi_1=0,\end{cases}$$

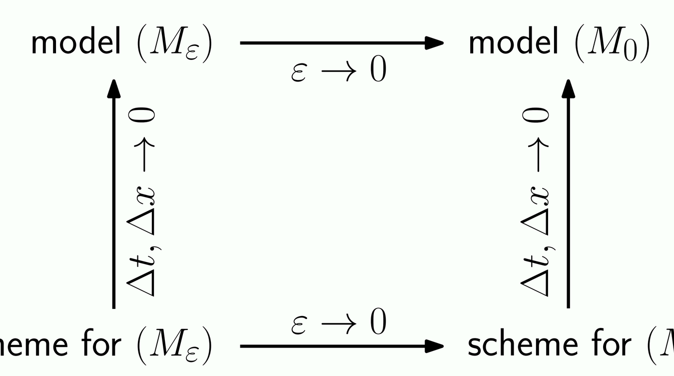
  where π_1 is the order one correction of the pressure.
  The time singularity of this limit is due to the propagation of the acoustic waves at a velocity proportional to $1/\sqrt{\varepsilon}$.

- **Numerical method**. Following [3], in [4], Dimarco, Loubère and Vignal propose a numerical scheme to preserve this asymptotic behavior. It is written below in semi-discrete form:

$$\frac{\rho^{n+1}-\rho^n}{\Delta t}+\nabla\cdot(\rho U)^{n+1}=0,$$
$$\frac{(\rho U)^{n+1}-(\rho U)^n}{\Delta t}+\nabla\cdot(\rho U\otimes U)^n+\frac{1}{\varepsilon}\nabla(p(\rho))^{n+1}=0.$$

  Thanks to the semi-implicitation, this scheme is:
  - *asymptotic preserving* (AP), i.e. it discretizes the incompressible Euler system when ε tends to 0;
  - *uniformly L^∞-stable* providing the space discretization is well-chosen.

- **AP property**. This scheme falls within the general framework of the AP schemes.



- **Objective**. Propose an *asymptotically accurate* extension of this numerical scheme. The following properties must be satisfied:
  - *higher accuracy* for all values of ε (including the asymptotic preserving property when $\varepsilon\to 0$);
  - ability to *control the oscillations* induced by the use of high accuracy space/time numerical schemes.

1. A model problem

We consider the following advection equation as a model problem:

$$\partial_t u+c_s\,\partial_x u+\frac{c_f}{\sqrt{\varepsilon}}\,\partial_x u=0,$$

where the slow and fast velocities c_s and $c_f/\sqrt{\varepsilon}$ are assumed to be non-negative and of order one.
Similarly to the Euler system, the characteristic velocity of the information is proportional to $1/\sqrt{\varepsilon}$. As a consequence, we consider the following semi-discrete scheme, mimicking the structure of the one proposed in [4]:

$$\frac{u^{n+1}-u^n}{\Delta t}+c_s\,(\partial_x u)^n+\frac{c_f}{\sqrt{\varepsilon}}\,(\partial_x u)^{n+1}=0.$$

Since $c_s\geq 0$ and $c_f\geq 0$, we use an upwind discretization in space:

$$\partial_x u\simeq\frac{u_j-u_{j-1}}{\Delta x}.$$

As a consequence, the fully discrete scheme reads:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+c_s\frac{u_j^n-u_{j-1}^n}{\Delta x}+\frac{c_f}{\sqrt{\varepsilon}}\frac{u_j^{n+1}-u_{j-1}^{n+1}}{\Delta x}=0.$$

Goal: Propose an **asymptotically accurate** extension of this scheme.

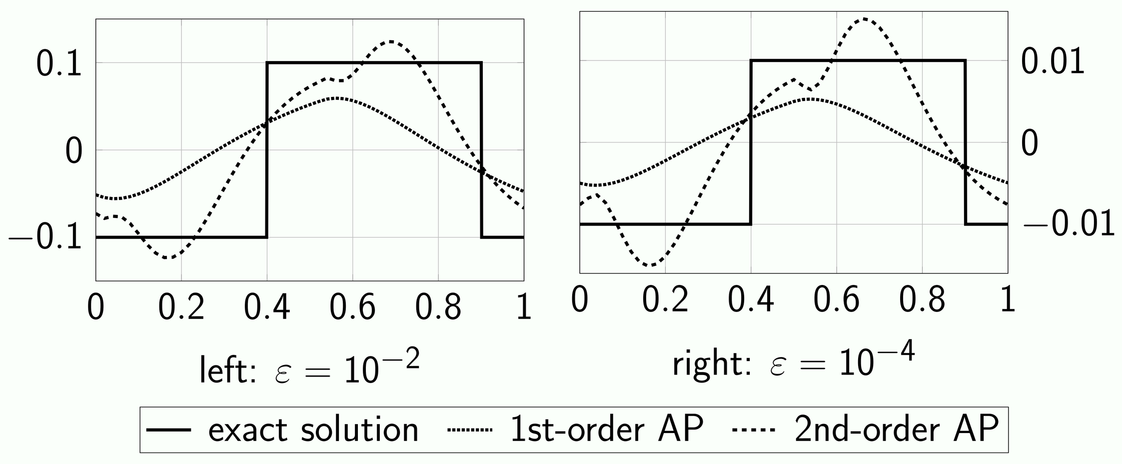
1.1. A more accurate time discretization

The above scheme uses an **IMEX** (IMplicit-EXplicit) time discretization (see [6] for instance). To improve its time accuracy, we use the two-step second-order in time **ARS(2,2,2)** discretization (see [1]):

$$\begin{cases}u_j^\star=u_j^n-\beta c_s\dfrac{\Delta t}{\Delta x}\Big(u_j^n-u_{j-1}^n\Big)-\beta\dfrac{c_f\,\Delta t}{\sqrt{\varepsilon}\,\Delta x}\Big(u_j^\star-u_{j-1}^\star\Big),\\ u_j^{n+1}=u_j^n-(\beta-1)c_s\dfrac{\Delta t}{\Delta x}\Big(u_j^n-u_{j-1}^n\Big)-(1-\beta)\dfrac{c_f\,\Delta t}{\sqrt{\varepsilon}\,\Delta x}\Big(u_j^\star-u_{j-1}^\star\Big)\\ \qquad-(2-\beta)c_s\dfrac{\Delta t}{\Delta x}\Big(u_j^\star-u_{j-1}^\star\Big)-\beta\dfrac{c_f\,\Delta t}{\sqrt{\varepsilon}\,\Delta x}\Big(u_j^{n+1}-u_{j-1}^{n+1}\Big).\end{cases}$$

1.2. A time limiting procedure

This discretization preserves the AP property of the scheme. However, it is oscillatory, as displayed below with the advection of a step function.



The implicit part of this IMEX scheme is nothing but an implicit Runge-Kutta discretization. Unfortunately, the following negative result holds.

Theorem ([5]): There are no implicit Runge-Kutta schemes of order higher than one which preserves the TVD property.

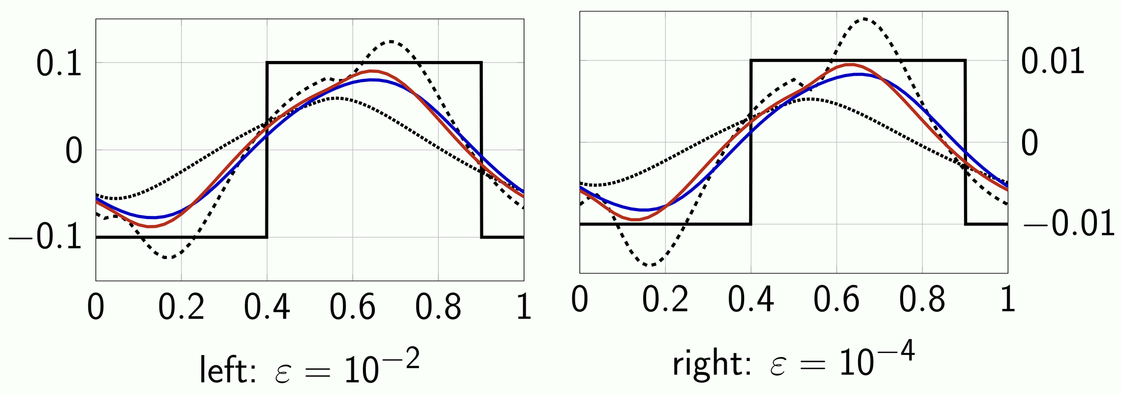
To obtain a scheme more accurate than the first-order one and still TVD, we introduce a **limiting procedure**. It consists in a **convex combination**, of parameter θ, between the second-order discretization and the first-order discretization, as follows:

$$\begin{aligned}u_j^{n+1}=u_j^n&-\theta(\beta-1)c_s\frac{\Delta t}{\Delta x}\Big(u_j^n-u_{j-1}^n\Big)-\theta(1-\beta)\frac{c_f\,\Delta t}{\sqrt{\varepsilon}\,\Delta x}\Big(u_j^\star-u_{j-1}^\star\Big)\\&-\theta(2-\beta)c_s\frac{\Delta t}{\Delta x}\Big(u_j^\star-u_{j-1}^\star\Big)-\theta\beta\frac{c_f\,\Delta t}{\sqrt{\varepsilon}\,\Delta x}\Big(u_j^{n+1}-u_{j-1}^{n+1}\Big)\\&-(1-\theta)c_s\frac{\Delta t}{\Delta x}\Big(u_j^n-u_{j-1}^n\Big)-(1-\theta)\frac{c_f\,\Delta t}{\sqrt{\varepsilon}\,\Delta x}\Big(u_j^{n+1}-u_{j-1}^{n+1}\Big).\end{aligned}$$

Theorem: With $\theta=\beta/(1-\beta)$, the above scheme is TVD.

Then, to further improve the scheme, we propose a **MOOD**-like technique (see [2]). It consists in using the above TVD-AP scheme only if oscillations are detected, to use the second-order scheme whenever possible. The following procedure is thus applied at each time step:

1. compute a **candidate solution** u^{n+1} with the original ARS(2,2,2) discretization, i.e. with $\theta=1$;
2. detect if this candidate solution satisfies the following global **maximum principle**: $\|u^{n+1}\|_\infty\leq\|u^n\|_\infty$;
3. if this maximum principle is not satisfied, then take $\theta=\beta/(1-\beta)$ and compute a **new solution** u^{n+1} with the above TVD-AP scheme.



The approximation provided by the **TVD-AP scheme** (blue curve) is in-bounds and more accurate than the first-order discretization.
The **MOOD procedure** (red curve) further improves this result.

2. Application to the Euler system

The strategy developed for the model problem is now applied to the Euler system. For the second-order space-time accuracy, we use:

- the *ARS(2,2,2) time discretization*;
- a *linear MUSCL reconstruction*.

To control the oscillations, we introduce:

- the Euler analogue of the TVD-AP scheme;
- the MC limiter on the MUSCL reconstruction slopes.

Remark: The Euler variables no longer satisfy a maximum principle. Indeed, for most initial data, $\|\rho(t,\cdot)\|_\infty\leq\|\rho(0,\cdot)\|_\infty$ and $\|(\rho U)(t,\cdot)\|_\infty\leq\|(\rho U)(0,\cdot)\|_\infty$ are **false**.

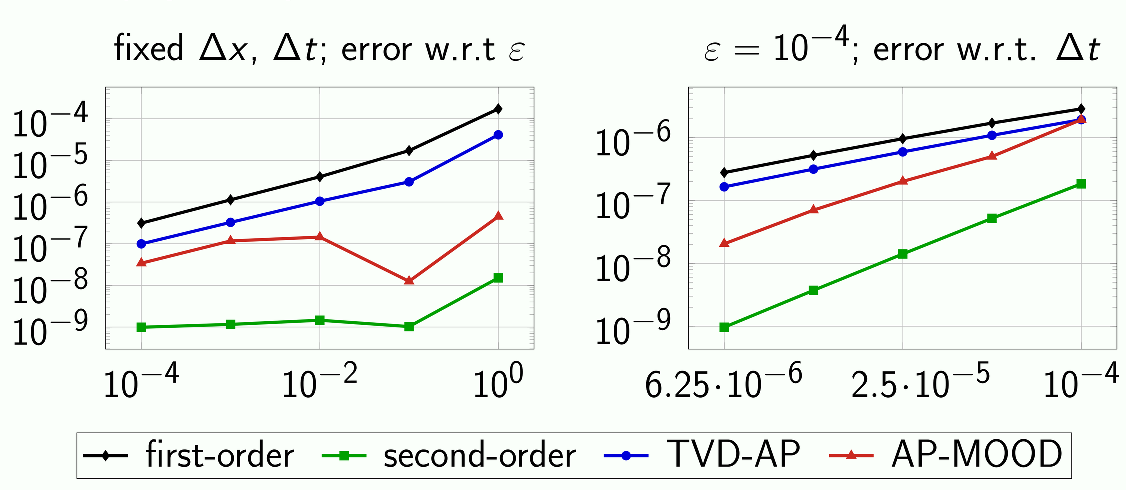
As a consequence, we cannot apply the same detection criterion as in the transport case. Instead, we turn to the **Riemann invariants**, defined by

$$u\mp\frac{2}{\gamma-1}\sqrt{\frac{p'(\rho)}{\varepsilon}}.$$

Even for non-smooth solutions, in a Riemann problem, at least one Riemann invariant satisfies a **maximum principle** (see J. A. Smoller and J. L. Johnson, 1969).

Error curves in 1D

We display density error curves in L^∞ norm for a smooth 1D solution.



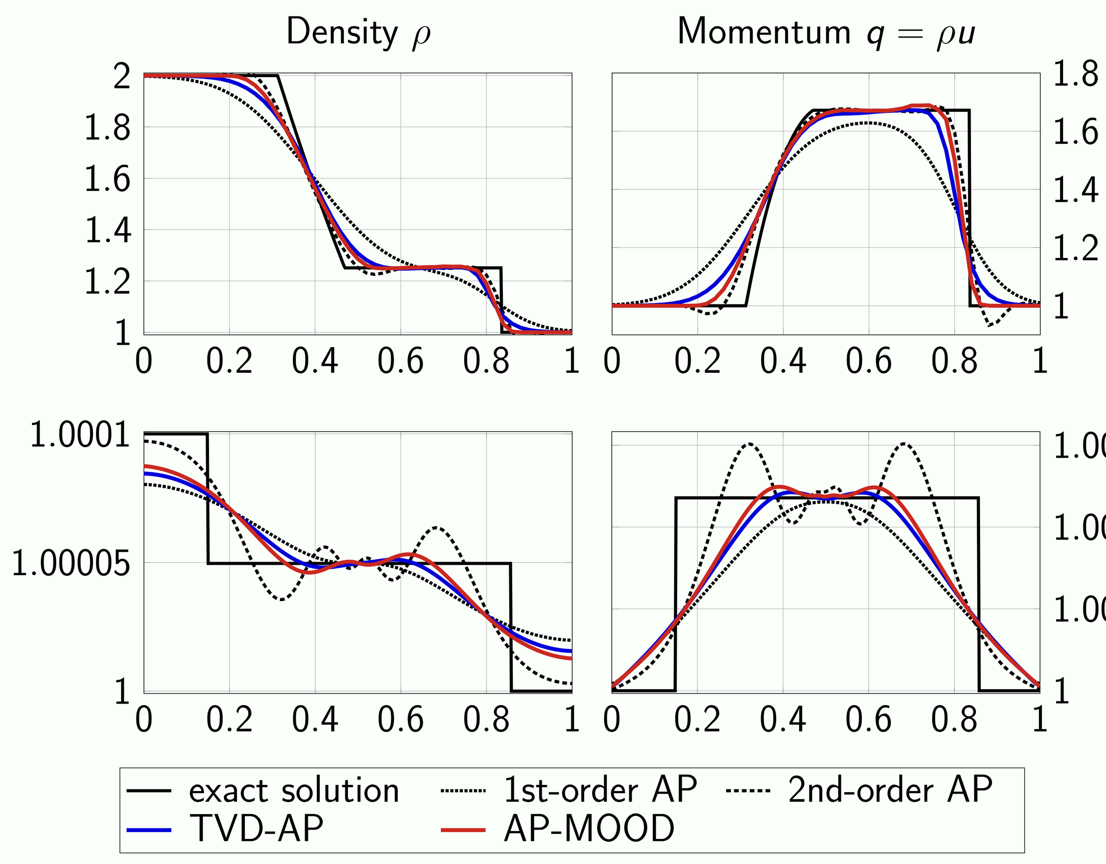
As expected, the TVD-AP scheme is more accurate than the first-order one, and the MOOD procedure further improves its accuracy.

Riemann problem

We consider a Riemann problem with the following initial data:

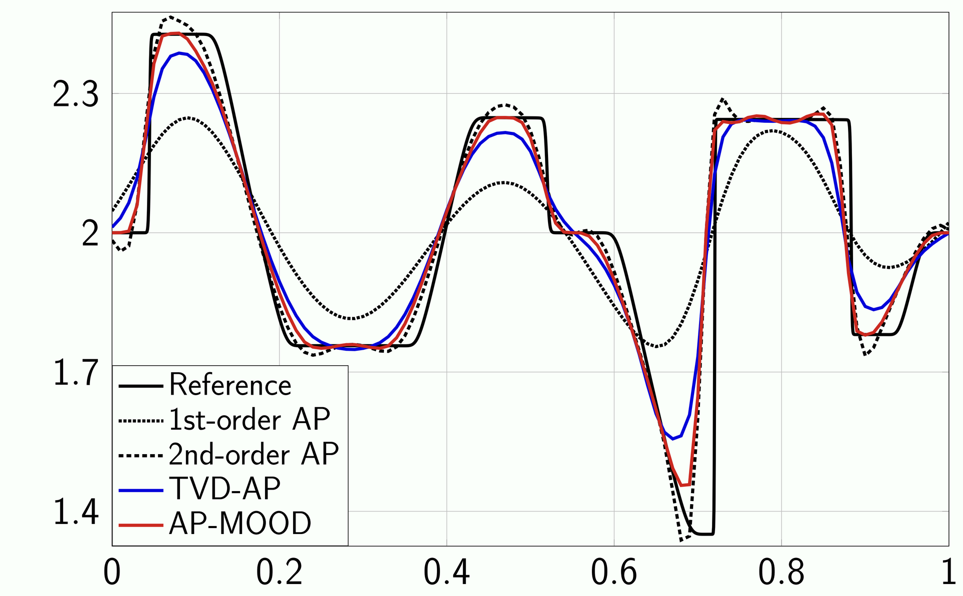
$$\begin{cases}\rho_L=1+\varepsilon,\\ \rho_R=1,\end{cases}\qquad\begin{cases}q_L=h_Lu_L=1,\\ q_R=h_Ru_R=1,\end{cases}$$

with $\varepsilon=1$ (top) and $\varepsilon=10^{-4}$ (bottom). We get a left rarefaction wave and a right shock wave, with characteristic velocities $\sim 1/\sqrt{\varepsilon}$.



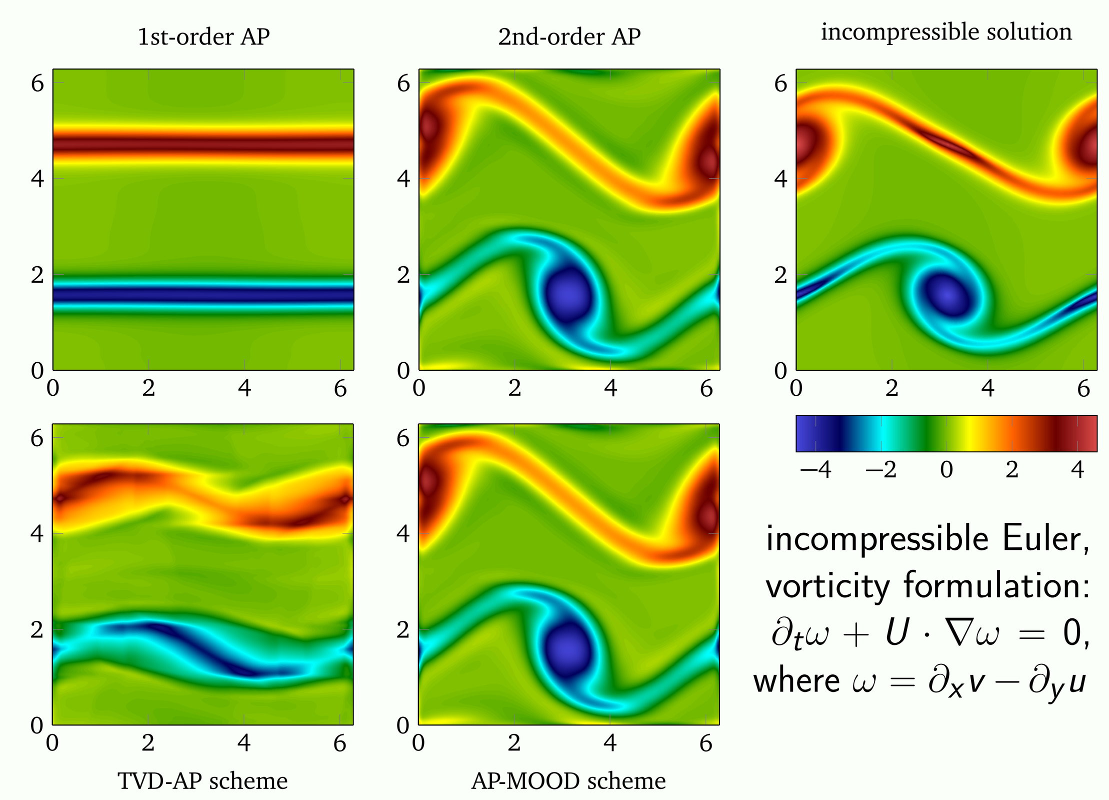
For both values of ε, the TVD-AP scheme and the MOOD procedure yield a better approximation than both other schemes: they are less diffusive than the first-order one and less oscillatory than the second-order one.

Degond-Tang numerical experiment from [3], $\varepsilon=1$



Comparison with an incompressible solution

As a last experiment, in 2D, we compare an incompressible reference solution to the solutions of our compressible schemes; we take $\varepsilon=10^{-5}$ and 200×200 cells. We compare the vorticity $\omega=\partial_x v-\partial_y u$.



Ongoing work and perspectives

- validate and verify the schemes on the full Euler system
- develop a relevant criterion to determine a local θ
- change time discretization to maximize the optimal θ
- domain decomposition with respect to ε

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