Introduction to well-balanced numerical methods

Victor Michel-Dansac*

November 09, 2023

DU IRMIA++, Strasbourg

*TONUS (soon-to-be MACARON) project-team, Université de Strasbourg, CNRS, Inria, IRMA, France

Institut de Recherche Mathématique Avancée

Tsunami simulation

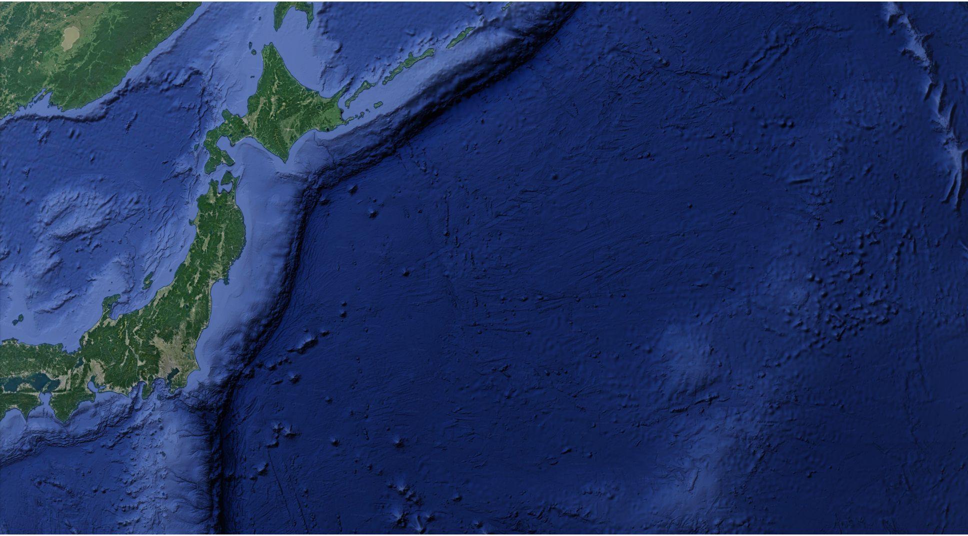
Ingredients for a tsunami simulation:

- a **physical model**: the shallow water (Saint-Venant) equations
- **applied mathematics**: developing *numerical methods* for solving these equations
- **data**: topography, mesh, water height measurements, ...

portrait of Adhémar Jean Claude Barré de Saint-Venant (1797-1886)

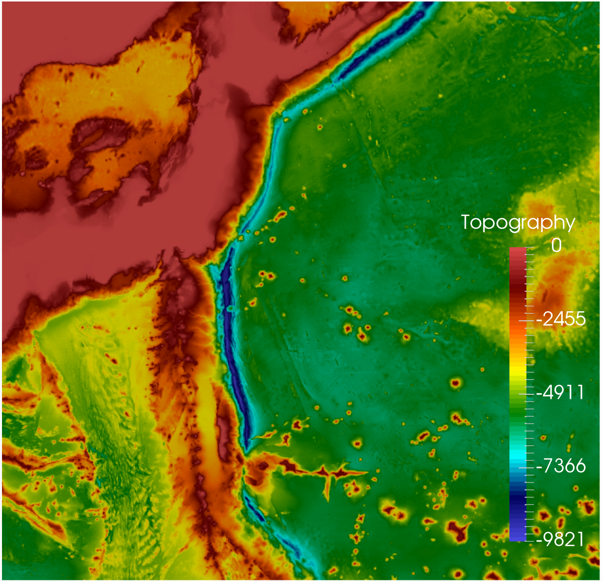
Ingredients required for a numerical simulation

How to perform a numerical simulation?



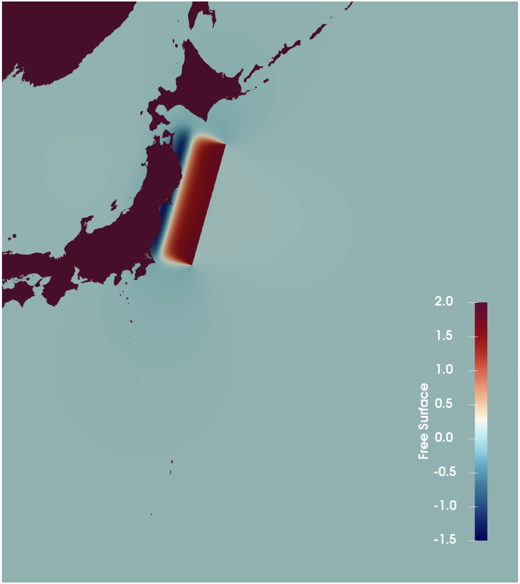
Ingredients required for a numerical simulation

First step: Discretization (Lisbon University geophysicists)



Ingredients required for a numerical simulation

Second step : Tsunami initialization (Lisbon University geophysicists)

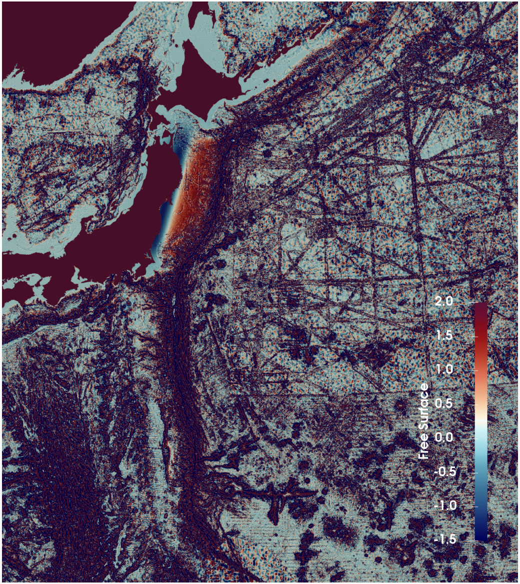


Ingredients required for a numerical simulation

Third step : Starting the simulation

Ingredients required for a numerical simulation

Third step : Starting the simulation



Ingredients required for a numerical simulation

... that did not work, the ocean at rest, far from the tsunami, starts spontaneously producing waves.

$\rightsquigarrow$ **The simulation is not usable!**

This comes from the non-preservation of stationary solutions:

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = s(u(x,t))$$

Ingredients required for a numerical simulation

... that did not work, the ocean at rest, far from the tsunami, starts spontaneously producing waves.

$\rightsquigarrow$ **The simulation is not usable!**

This comes from the non-preservation of stationary solutions:

$$\frac{\partial}{\partial x} f(u(x,t)) = s(u(x,t)) \qquad \text{if } \frac{\partial}{\partial t} u(x,t) = 0 \text{ (stationary solution)}$$

Ingredients required for a numerical simulation

... that did not work, the ocean at rest, far from the tsunami, starts spontaneously producing waves.

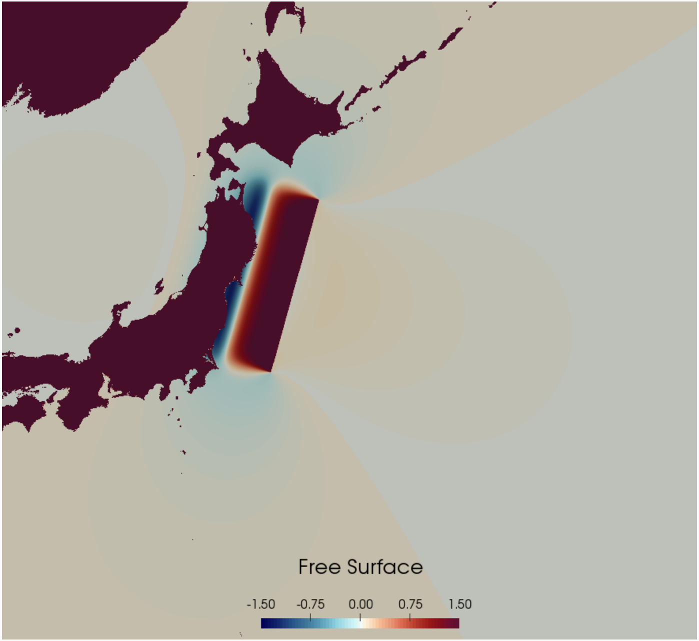
$\rightsquigarrow$ **The simulation is not usable!**

This comes from the non-preservation of stationary solutions:

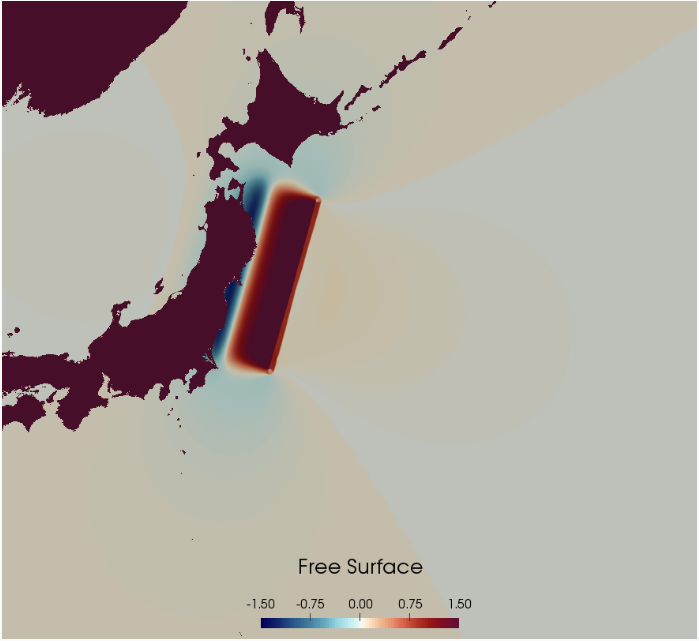
$$\frac{\partial}{\partial x} f(u(x,t)) = s(u(x,t)) \qquad \text{if } \frac{\partial}{\partial t} u(x,t) = 0 \text{ (stationary solution)}$$

Hence the need to develop numerical methods that **exactly preserve stationary solutions**: so-called **well-balanced** methods.

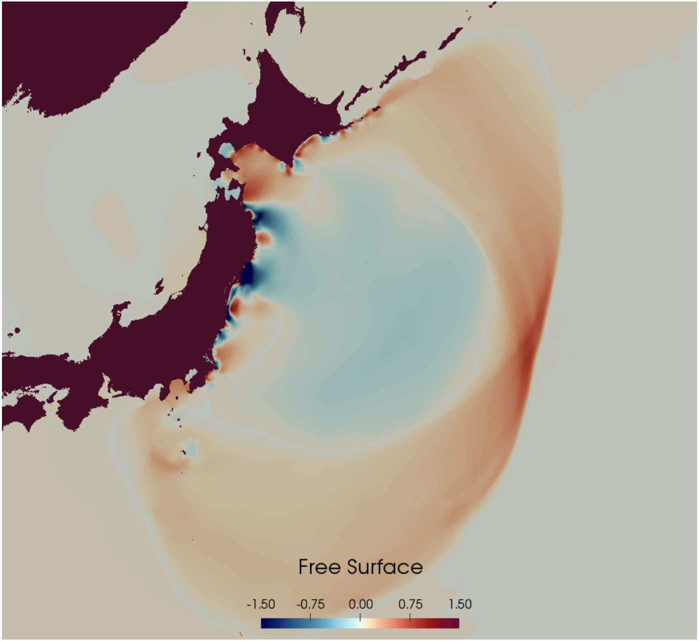
Ingredients required for a numerical simulation



Ingredients required for a numerical simulation

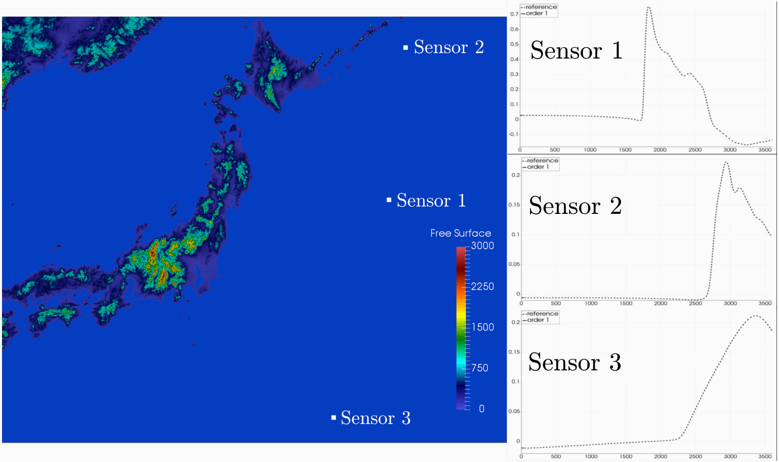


Ingredients required for a numerical simulation



Ingredients required for a numerical simulation

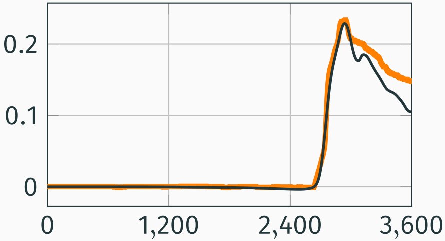
Fourth step: Verification of the numerical results



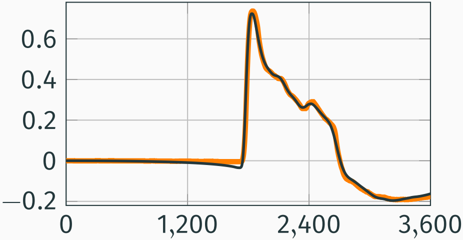
Water depth at sensors:

- #1: 5700 m;
- #2: 6100 m;
- #3: 4400 m.

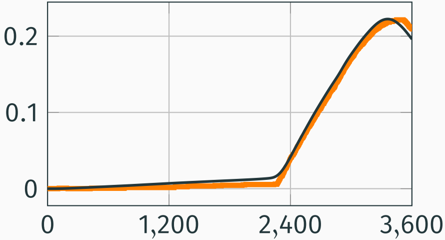
Plots of the time variation
of the water height (in meters).
data in black, simulation in orange



Sensor #2



Sensor #1



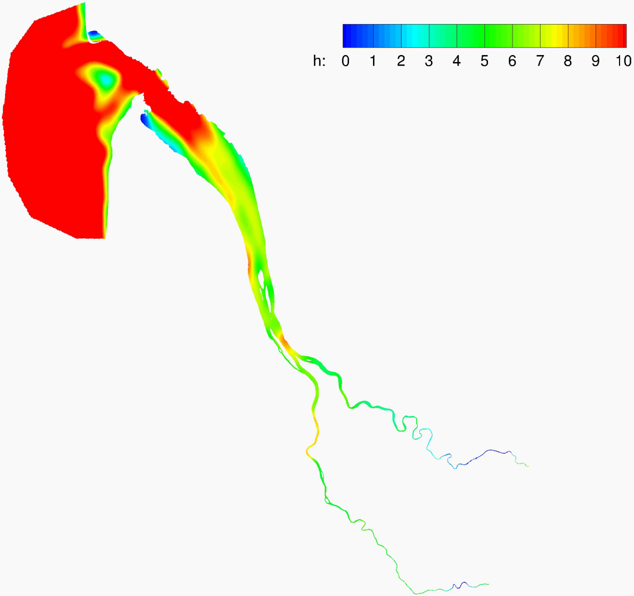
Sensor #3

Estuary modeling

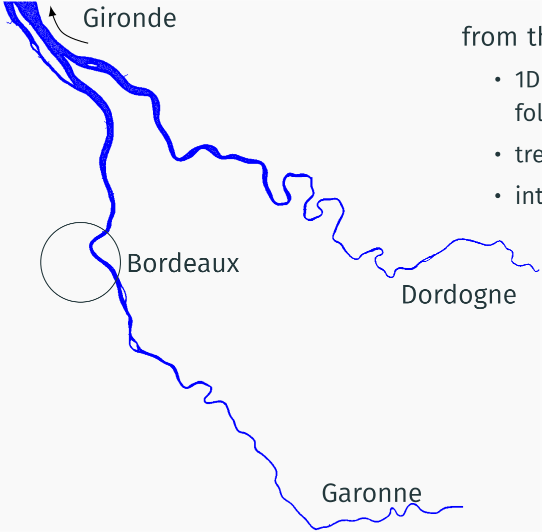
Gironde estuary : satellite picture

Gironde estuary :
2D mesh

Estuary modeling



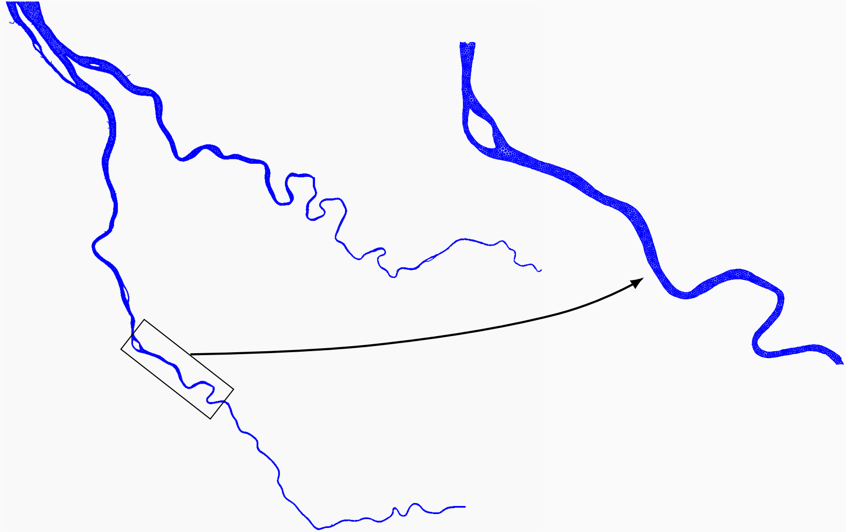
Estuary modeling



from the 2D mesh:

- 1D description of the rivers following their meanders
- treatment of the confluence
- interaction with the tide

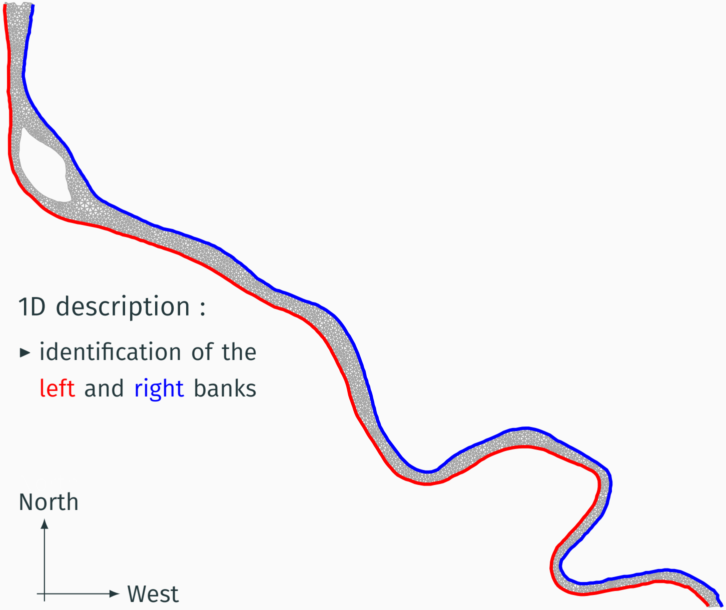
Estuary modeling



Estuary modeling

1D description :

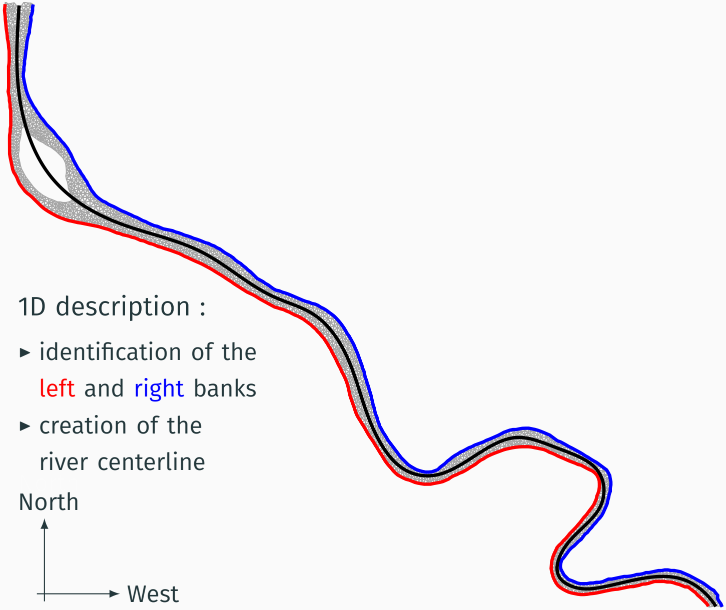
Estuary modeling



1D description :

- identification of the left and right banks

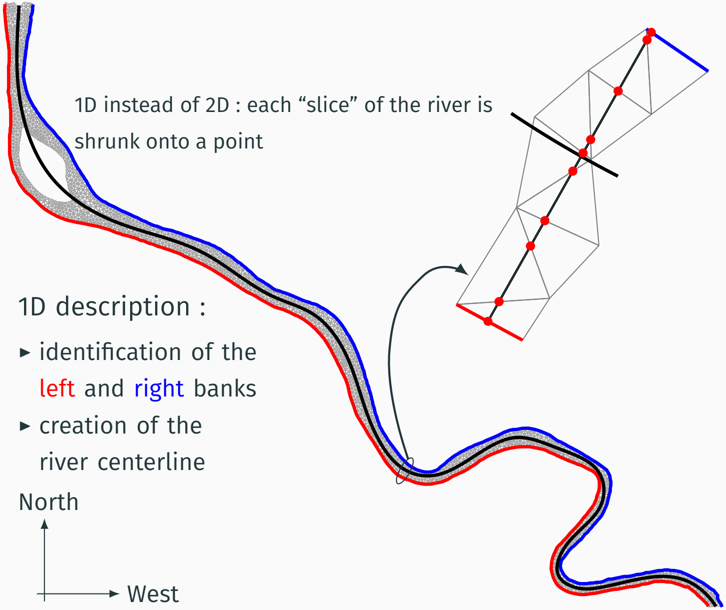
Estuary modeling



1D description :

- identification of the left and right banks
- creation of the river centerline

Estuary modeling



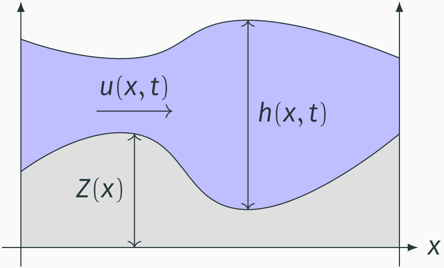
1D instead of 2D : each “slice” of the river is shrunk onto a point

1D description :

- identification of the left and right banks
- creation of the river centerline

The shallow water equations with topography

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh \partial_x Z(x) \end{cases}$$



- $h(x,t)$: water height
- $u(x,t)$: water velocity
- $q = hu$: water discharge
- $Z(x)$: known topography
- g: gravity constant

We will consider solutions of prime importance:

the steady solutions.

For additional details, check the {white, black}board!

Hyperbolic systems of conservation laws

The shallow water equations fall within the broad framework of **hyperbolic systems of conservation laws**.

In one space dimension, they are PDE systems with the following form:

$$\frac{\partial W(t,x)}{\partial t} + \frac{\partial F(W(t,x))}{\partial x} = 0,$$

where:

- $W : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^p$ is the unknown function, which depends on time t and space x,
- $F : \mathbb{R}^p \to \mathbb{R}^p$ is the physical flux function.

Hyperbolic systems of conservation laws

The shallow water equations fall within the broad framework of **hyperbolic systems of conservation laws**.

In one space dimension, they are PDE systems with the following form:

$$\frac{\partial W(t,x)}{\partial t} + \frac{\partial F(W(t,x))}{\partial x} = 0,$$

where:

- $W : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^p$ is the unknown function, which depends on time t and space x,
- $F : \mathbb{R}^p \to \mathbb{R}^p$ is the physical flux function.

Hyperbolic systems have several important (and linked) properties:

1. finite information propagation speed,
2. creation of discontinuities, even from smooth initial data,
3. conservation of the quantity W.

Finite information propagation speed

We assume that the Jacobian matrix of the flux function F has real eigenvalues: this is linked to a **finite information propagation speed**.

Finite information propagation speed

We assume that the Jacobian matrix of the flux function F has real eigenvalues: this is linked to a **finite information propagation speed**.

A typical example is the advection equation

$$\partial_t W + \partial_x (cW) = 0,$$

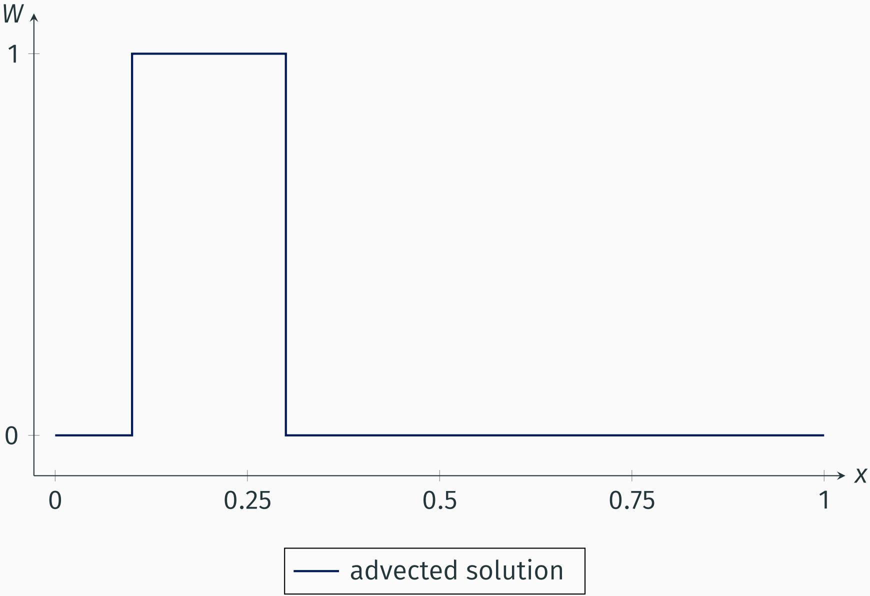
where

- $W(t, x) \in \mathbb{R}$,
- $F : W \mapsto c\,W$ is a linear function, with a fixed $c \neq 0$;
- the derivative of F is $W \mapsto c \in \mathbb{R}^*$.

This equation transports the initial condition $W(0, x)$ with velocity c.

Linear hyperbolic system: the advection equation

Linear hyperbolic system: the advection equation



Creation of discontinuities

Another property of **nonlinear** hyperbolic systems is that continuous initial data can lead to a **discontinuous solution in finite time**.

Creation of discontinuities

Another property of **nonlinear** hyperbolic systems is that continuous initial data can lead to a **discontinuous solution in finite time**.

A typical example is the inviscid Burgers' equation

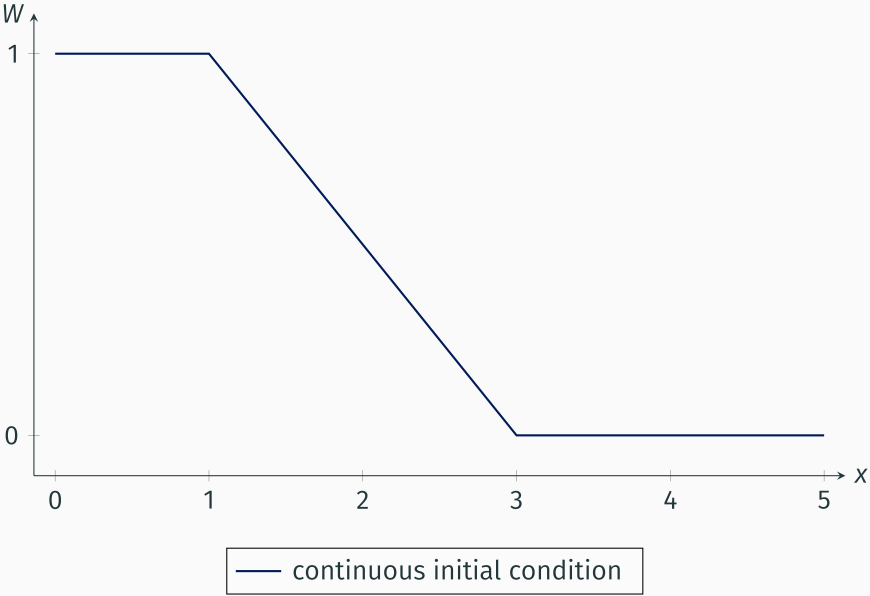
$$\partial_t W + \partial_x \left(\frac{W^2}{2} \right) = 0,$$

where

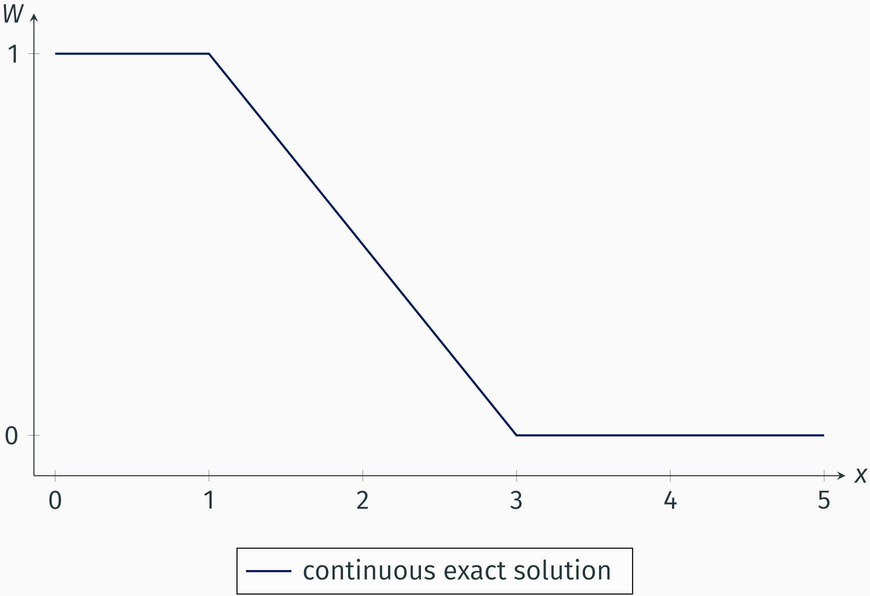
- $W(t,x) \in \mathbb{R}$,
- $F : W \mapsto \frac{1}{2}W^2$ is a nonlinear function;
- the derivative of F is $W \mapsto W \in \mathbb{R}$.

This equation "transports the initial condition $W(0,x)$ with velocity W".

Nonlinear hyperbolic system: Burgers' equation



Nonlinear hyperbolic system: Burgers' equation



Non-hyperbolic systems

To conclude this overview, we give an example of a **non-hyperbolic system**.

The heat equation

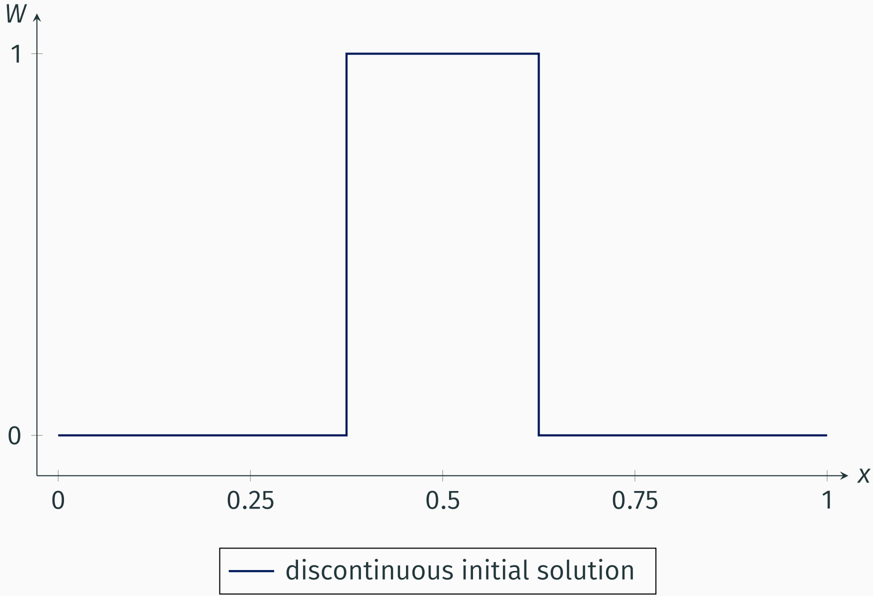
$$\partial_t W + \partial_{xx} W = 0$$

is an example of a *parabolic* system, where[1]

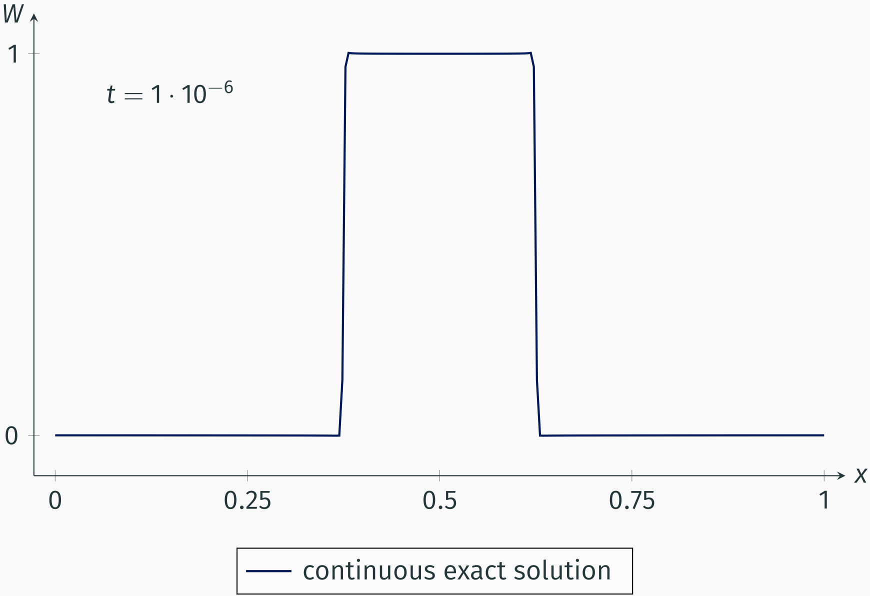
- the information travels at *infinite speed*;
- a *regularizing* effect is applied, rather than a production of discontinuities.

[1] Note that these two properties are not satisfied by every non-hyperbolic system.

Non-hyperbolic system: the heat equation



Non-hyperbolic system: the heat equation



The finite volume method: discretization

Objective: Approximate the solution $W(t, x)$ of the conservation law.

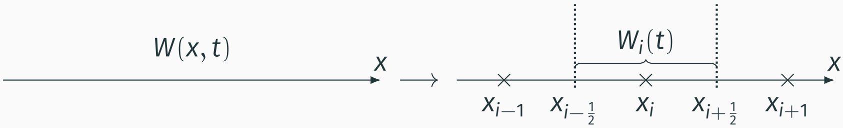
The finite volume method: discretization

Objective: Approximate the solution $W(t,x)$ of the conservation law.

We partition the space domain in **cells** $(\Omega_i)_i$, of length Δx and of evenly spaced centers x_i, and we define:

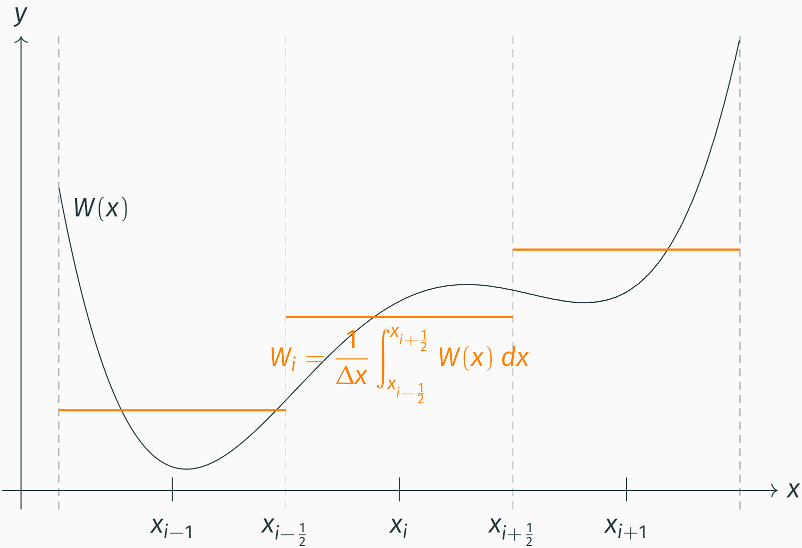
- $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$, the boundaries of cell Ω_i;
- $W_i(t)$, an approximation of $W(t,x)$, defined by

$$W_i(t) = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W(t,x)\, dx.$$



Remark: the approximation $W_i(t)$ of $W(t,x)$ is constant on each cell.

Finite volume space discretization, visualized



The finite volume method: numerical approximation

To derive the **finite volume discretization** of a system of conservation laws, we **average the system in space and time**:

$$\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \int_0^{\Delta t} \partial_t W(t,x)\, dt\, dx + \int_0^{\Delta t} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \partial_x F(W(t,x))\, dx\, dt = 0.$$

We eventually obtain (check the {white, black}board!)

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(W_i^n, W_{i+1}^n) - \mathcal{F}(W_{i-1}^n, W_i^n)\right],$$

where the numerical flux $\mathcal{F}$ is such that $\mathcal{F}(W, W) = F(W)$.

Hyperbolic systems of balance laws

To model complex physical phenomena, we often consider **hyperbolic systems of balance laws**, with the following form:

$$\partial_t W + \partial_x F(W) = S(W, x),$$

where:

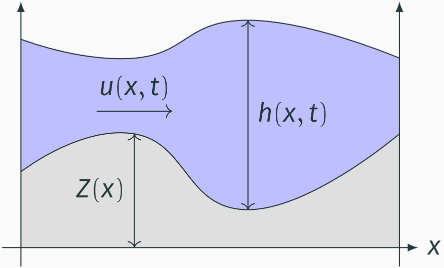
- $W : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^p$ is the unknown function,
- $F : \mathbb{R}^p \to \mathbb{R}^p$ is the physical flux function,
- $S : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ is the source term .

Compared to conservation laws, the presence of the source term disrupts the conservation property.

System of balance laws: the shallow water equations

A typical example of a system of balance laws is the shallow water equations, governed by the following PDE:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z(x). \end{cases}$$



- $h(x,t)$: water height
- $u(x,t)$: water velocity
- $q = hu$: water discharge
- $Z(x)$: known topography
- g: gravity constant

Steady solutions

Moreover, balance laws have an additional kind of solution:
non-trivial **steady solutions**.

Steady solutions

Moreover, balance laws have an additional kind of solution: non-trivial **steady solutions**.

Definition: steady solution

W is a steady solution of $\partial_t W + \partial_x F(W) = S(W, x)$ if, and only if, $\partial_t W = 0$, i.e.

$$\partial_x F(W) = S(W, x).$$

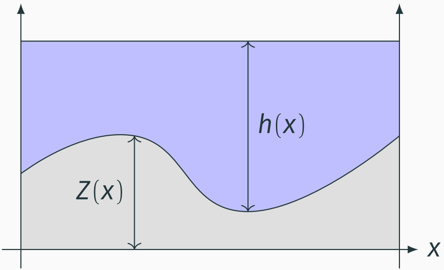
Steady solutions

Moreover, balance laws have an additional kind of solution:
non-trivial **steady solutions**.

Definition: steady solution

W is a steady solution of $\partial_t W + \partial_x F(W) = S(W, x)$ if, and only if, $\partial_t W = 0$, i.e.

$$\partial_x F(W) = S(W, x).$$



For the shallow water equations, if the velocity vanishes, we obtain the lake at rest steady solution:

$$h + Z = \text{cst}.$$

Finite volume scheme: application to balance laws

For a system of balance laws, i.e. with a source term, the finite volume scheme becomes

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(W_i^n, W_{i+1}^n) - \mathcal{F}(W_{i-1}^n, W_i^n)\right] + \Delta t \mathcal{S}(W_{i-1}^n, W_i^n, W_{i+1}^n),$$

where $\mathcal{S}$ is an approximation of the source term.

What about the steady solutions?

Recall that steady solutions are defined by taking $\partial_t W = 0$, which yields the ODE

$$\partial_x F(W) = S(W).$$

The discrete analogue is $W_i^{n+1} = W_i^n$, which is ensured if, and only if,

$$\frac{1}{\Delta x}\left[\mathcal{F}(W_i^n, W_{i+1}^n) - \mathcal{F}(W_{i-1}^n, W_i^n)\right] = \mathcal{S}(W_{i-1}^n, W_i^n, W_{i+1}^n)$$

for all steady solutions. This relation obviously requires *ad hoc* definitions of $\mathcal{F}$ and $\mathcal{S}$, which will depend on the system under consideration, and can be quite involved.

What about the steady solutions?

Recall that steady solutions are defined by taking $\partial_t W = 0$, which yields the ODE

$$\partial_x F(W) = S(W).$$

The discrete analogue is $W_i^{n+1} = W_i^n$, which is ensured if, and only if,

$$\frac{1}{\Delta x}\left[\mathcal{F}(W_i^n, W_{i+1}^n) - \mathcal{F}(W_{i-1}^n, W_i^n)\right] = \mathcal{S}(W_{i-1}^n, W_i^n, W_{i+1}^n)$$

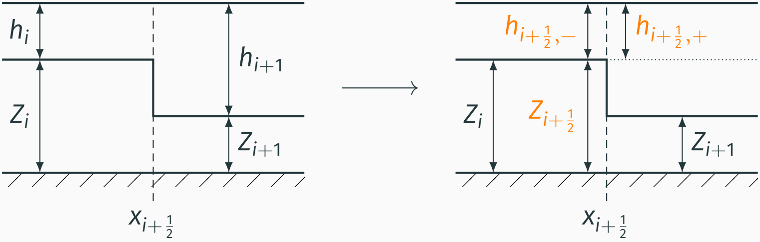
for all steady solutions. This relation obviously requires *ad hoc* definitions of $\mathcal{F}$ and $\mathcal{S}$, which will depend on the system under consideration, and can be quite involved.

Definition: well-balanced scheme

A numerical method approximating the solution of a balance law is called well-balanced if it preserves the steady solutions.

An answer for the lake at rest: the hydrostatic reconstruction

The **hydrostatic reconstruction** was introduced in E. Audusse et al., *SIAM J. Sci. Comput.* (2004), as a way to make it possible for any finite volume scheme to capture the **lake at rest** steady solution.



For additional details, check the {white, black}board!

The hydrodynamic reconstruction

The hydrodynamic reconstruction is an improvement of the hydrostatic reconstruction, introduced in [C. Berthon and V. Michel-Dansac (2023)] to preserve moving steady solutions.

$$
\begin{aligned}
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i\right)\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right), \\
h^n_{i+\frac{1}{2},+} &= h^n_{i+1} + \left(Z_{i+1} - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2\left(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1}\right)\mathcal{H}\left(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1}, Z_{i+\frac{1}{2}} - Z_{i+1}\right),
\end{aligned}
$$

where we have defined the function $\mathcal{H}$ by

$$
\mathcal{H} = \frac{1}{2}\left(E - \mathrm{sgn}(1-\mathrm{Fr}^2)\mathrm{sgn}(\Delta Z)\sqrt{E^2 + \sqrt{\frac{1}{2}|\Delta Z|\|[h]\|^3}}\right),
$$

$$
\text{with} \quad E = [h] + \frac{1-\mathrm{Fr}^2}{2}\mathrm{sgn}(\Delta Z)\sqrt{\frac{\|[h]\|^3}{2|\Delta Z|}}.
$$

High-order accuracy

The Discontinuous Galerkin (DG) method offers a way to increase the **order of accuracy** of traditional finite volume methods.

Definition: order of a numerical scheme

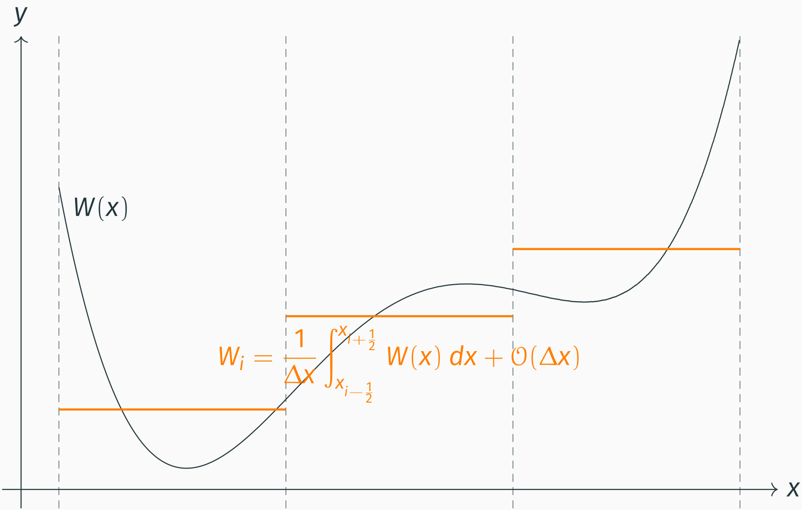
A numerical method is of order p (in space) if the error $e(\Delta x)$ between the approximate and exact solutions behaves as follows:

$$e(\Delta x) = \mathcal{O}(\Delta x^p).$$

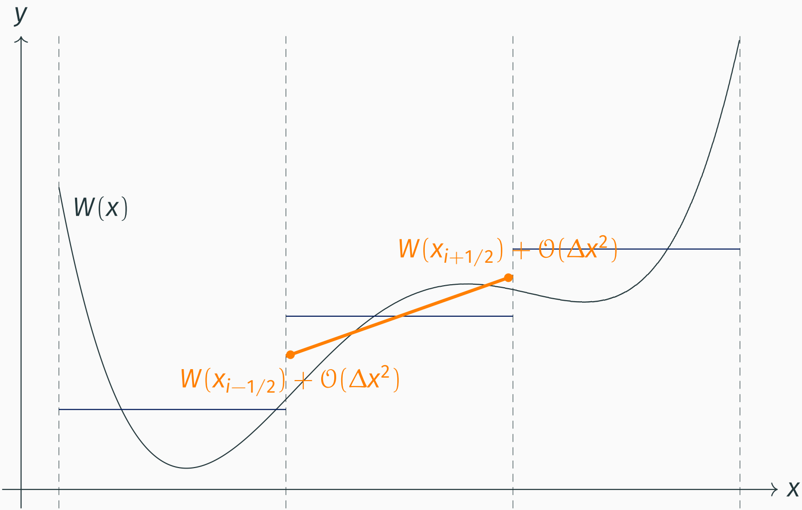
Alternate definition: order of a numerical scheme

A numerical method is of order p (in space) if it is exact on polynomials up to degree $p - 1$.

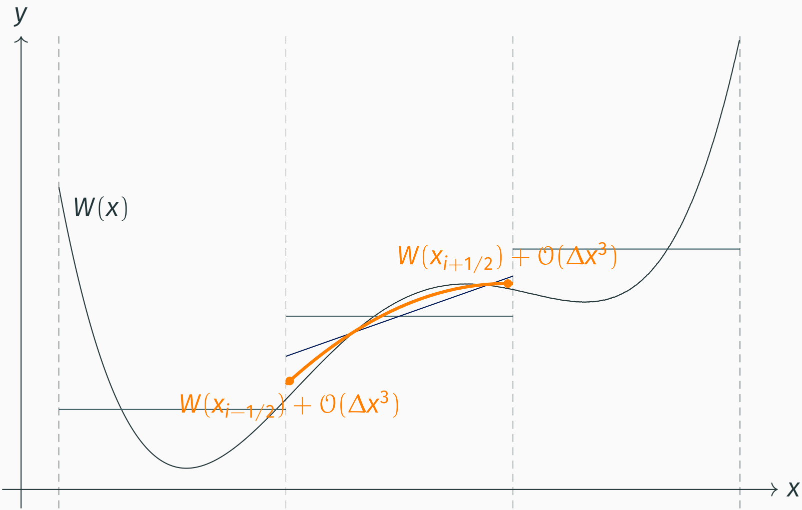
High-order accuracy, visualized



High-order accuracy, visualized



High-order accuracy, visualized



Thank you for your attention!

Conservation

Finally, we present the property of **conservation**.

To that end, we integrate the conservation law on the space domain $(a, b) \subset \mathbb{R}$.

$$\int_a^b \partial_t W \, dx + \int_a^b \partial_x F(W) \, dx = 0$$
$$\implies \quad \partial_t \left(\int_a^b W \, dx \right) + \big(F(W(t, b)) - F(W(t, a)) \big) = 0.$$

If $F(W(t, a)) = F(W(t, b))$, meaning a balance between what enters and leaves the space domain, then we obtain the **conservation** of W:

$$\partial_t \left(\int_a^b W \, dx \right) = 0.$$

Link to the conservation property

The space domain is $[a, b] = \bigcup_i [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, and so we get

$$\int_a^b W\,dx = \sum_i \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W\,dx = \Delta x \sum_i W_i(t).$$

Therefore, the discrete analogue of the conservation property is

$$\sum_i W_i^{n+1} = \sum_i W_i^n.$$

Link to the conservation property

The space domain is $[a, b] = \bigcup_i [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, and so we get

$$\int_a^b W\,dx = \sum_i \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W\,dx = \Delta x \sum_i W_i(t).$$

Therefore, the discrete analogue of the conservation property is

$$\sum_i W_i^{n+1} = \sum_i W_i^n.$$

Using the definition of the numerical scheme, we obtain

$$\sum_i W_i^{n+1} = \sum_i W_i^n - \frac{\Delta t}{\Delta x} \sum_i \left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n\right)$$

$$\sum_i W_i^{n+1} = \sum_i W_i^n - \frac{\Delta t}{\Delta x}\left(F(W(t^n, b)) - F(W(t^n, a))\right)$$

$$\sum_i W_i^{n+1} = \sum_i W_i^n,$$

and the discrete conservation property is recovered by the finite volume method.

The finite volume method: numerical approximation

To derive the **space discretization** of a system of conservation laws, we fix some $t \geqslant 0$ and **average the system on each cell**:

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_t W(t,x)\,dx + \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_x F(W(t,x))\,dx = 0$$

The finite volume method: numerical approximation

To derive the **space discretization** of a system of conservation laws, we fix some $t \geqslant 0$ and **average the system on each cell**:

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_t W(t,x)\,dx + \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_x F(W(t,x))\,dx = 0$$

$$\implies \partial_t\left(\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W(t,x)\,dx\right) + \frac{1}{\Delta x}\big(F(W(t,x_{i+\frac{1}{2}})) - F(W(t,x_{i-\frac{1}{2}}))\big) = 0$$

The finite volume method: numerical approximation

To derive the **space discretization** of a system of conservation laws, we fix some $t \geqslant 0$ and **average the system on each cell**:

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_t W(t,x)\,dx + \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_x F(W(t,x))\,dx = 0$$

$$\implies \quad \partial_t \left(\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W(t,x)\,dx \right) + \frac{1}{\Delta x}\big(F(W(t,x_{i+\frac{1}{2}})) - F(W(t,x_{i-\frac{1}{2}}))\big) = 0$$

$$\implies \quad \partial_t W_i(t) + \frac{1}{\Delta x}\big(F(W(t,x_{i+\frac{1}{2}})) - F(W(t,x_{i-\frac{1}{2}}))\big) = 0.$$

The finite volume method: numerical approximation

To derive the **space discretization** of a system of conservation laws, we fix some $t \geqslant 0$ and **average the system on each cell**:

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_t W(t,x)\,dx + \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_x F(W(t,x))\,dx = 0$$

$$\implies \partial_t \left(\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W(t,x)\,dx \right) + \frac{1}{\Delta x}\big(F(W(t,x_{i+\frac{1}{2}})) - F(W(t,x_{i-\frac{1}{2}}))\big) = 0$$

$$\implies \partial_t W_i(t) + \frac{1}{\Delta x}\big(F(W(t,x_{i+\frac{1}{2}})) - F(W(t,x_{i-\frac{1}{2}}))\big) = 0.$$

Given an approximation $\mathcal{F}_{i+\frac{1}{2}}(t) \simeq F(W(t,x_{i+\frac{1}{2}}))$, we obtain the **semi-discrete scheme in time**

$$\partial_t W_i(t) + \frac{1}{\Delta x}\big(\mathcal{F}_{i+\frac{1}{2}}(t) - \mathcal{F}_{i-\frac{1}{2}}(t)\big) = 0.$$

Time discretization

Note that the semi-discrete scheme in time

$$\partial_t W_i(t) + \frac{1}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}}(t) - \mathcal{F}_{i-\frac{1}{2}}(t)\right) = 0$$

is nothing but an ordinary differential equation. To approximate its solution, we partition the time domain $(0, T)$ in intervals (t^n, t^{n+1}) of size Δt.

Time discretization

Note that the semi-discrete scheme in time

$$\partial_t W_i(t) + \frac{1}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}}(t) - \mathcal{F}_{i-\frac{1}{2}}(t)\right) = 0$$

is nothing but an ordinary differential equation. To approximate its solution, we partition the time domain $(0, T)$ in intervals (t^n, t^{n+1}) of size Δt.

Using the usual **explicit Euler time discretization**, we obtain the fully discrete scheme:

$$\frac{W_i(t^{n+1}) - W_i(t^n)}{\Delta t} + \frac{1}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}}(t^n) - \mathcal{F}_{i-\frac{1}{2}}(t^n)\right) = 0.$$

Introducing notation $W_i^n = W_i(t^n)$, we obtain the final form of the finite volume scheme:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}}\right).$$