A fully well-balanced hydrodynamic reconstruction

Christophe Berthon[*], Victor Michel-Dansac[‖]

April 6, 2023

JPV66, Banyuls-sur-Mer

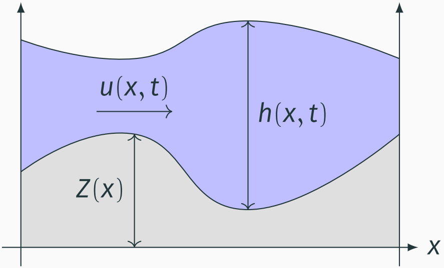
[*]LMJL, Université de Nantes, France
[‖]Université de Strasbourg, CNRS, Inria, IRMA, France

Institut de Recherche Mathématique Avancée

The shallow water equations with topography

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z(x) \end{cases}$$

The equations are written under the form $\partial_t W + \partial_x F(W) = S(W)$.



- $h(x,t)$: water height
- $u(x,t)$: water velocity
- $q = hu$: water discharge
- $Z(x)$: known topography
- g: gravity constant

We will consider solutions of prime importance:

the steady solutions.

Steady solutions and well-balanced schemes

Definition: steady solution

W is a steady solution of $\partial_t W + \partial_x F(W) = S(W)$ if, and only if, $\partial_t W = 0$, i.e. W satisfies the following ODE:

$$\partial_x F(W) = S(W).$$

Example: For the shallow water equations with topography, the ODE governing smooth steady solutions can be simplified.

Shallow water equations: steady solutions

Taking $\partial_t W = 0$ in the shallow water system yields

$$\begin{cases} \partial_x q = 0, \\ \partial_x \left(\frac{q^2}{h} + \frac{1}{2} g h^2 \right) = -gh \partial_x Z, \end{cases} \underset{\text{solution}}{\overset{\text{smooth}}{\Longrightarrow}} \begin{cases} q = \text{cst} = q_0, \\ \partial_x \left(\frac{q_0^2}{2h^2} + g(h+Z) \right) = 0. \end{cases}$$

We summarize the second relation by introducing a function B such that, for a steady solution, $B(h, q_0, Z) = B_0$.

Shallow water equations: steady solutions

Taking $\partial_t W = 0$ in the shallow water system yields

$$\begin{cases} \partial_x q = 0, \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z, \end{cases} \underset{\text{solution}}{\overset{\text{smooth}}{\Longrightarrow}} \begin{cases} q = \text{cst} = q_0, \\ \partial_x \left(\dfrac{q_0^2}{2h^2} + g(h+Z) \right) = 0. \end{cases}$$

We summarize the second relation by introducing a function B such that, for a steady solution, $B(h, q_0, Z) = B_0$.

Two cases are distinguished:

- $\mathbf{q_0 = 0} \rightsquigarrow$ lake at rest
  we get $B(h, q_0, Z) = h + Z = \text{cst}$: linear equation in h
- $\mathbf{q_0 \neq 0} \rightsquigarrow$ moving steady solution
  we get $B(h, q_0, Z) = \dfrac{q_0^2}{2h^2} + g(h+Z) = B_0$: nonlinear equation in h!

Finite volume scheme

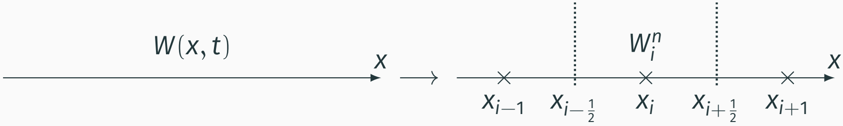
Recall the compact form of the shallow water equations:

$$\partial_t W + \partial_x F(W) = S(W).$$

We take a generic finite volume numerical scheme approximating the shallow water equations:

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} = \frac{1}{\Delta x}\left[\mathcal{F}(W_i^n, W_{i+1}^n) - \mathcal{F}(W_{i-1}^n, W_i^n)\right] + \mathcal{S}(W_{i-1}^n, W_i^n, W_{i+1}^n),$$

with $\mathcal{F}$ a consistent numerical flux, i.e. $\mathcal{F}(W, W) = F(W)$, and $\mathcal{S}$ a consistent numerical source term.



Finite volume scheme

Recall the compact form of the shallow water equations:

$$\partial_t W + \partial_x F(W) = S(W).$$

We take a generic finite volume numerical scheme approximating the shallow water equations:

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} = \frac{1}{\Delta x}\left[\mathcal{F}(W_i^n, W_{i+1}^n) - \mathcal{F}(W_{i-1}^n, W_i^n)\right] + \mathcal{S}(W_{i-1}^n, W_i^n, W_{i+1}^n),$$

with $\mathcal{F}$ a consistent numerical flux, i.e. $\mathcal{F}(W, W) = F(W)$, and $\mathcal{S}$ a consistent numerical source term.

Definition: well-balanced scheme

A numerical method approximating the solution of a balance law is called well-balanced if it exactly preserves the steady solutions.

Question: can we make this generic finite volume scheme well-balanced?

An answer for the lake at rest: the hydrostatic reconstruction

The **hydrostatic reconstruction** was introduced[1] in 2004, as a way to make it possible for any finite volume scheme to capture the **lake at rest** steady solution.

It relies on:

1. providing a relevant expression for $\mathcal{S}$,
2. evaluating the numerical flux at a specific reconstruction of W.

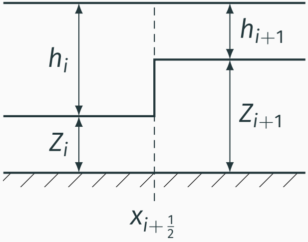
[1] see E. Audusse et al., *SIAM J. Sci. Comput.* (2004)

An answer for the lake at rest: the hydrostatic reconstruction

The **hydrostatic reconstruction** was introduced[1] in 2004, as a way to make it possible for any finite volume scheme to capture the **lake at rest** steady solution.

It relies on:

1. providing a relevant expression for $\mathcal{S}$,
2. evaluating the numerical flux at a specific reconstruction of W.



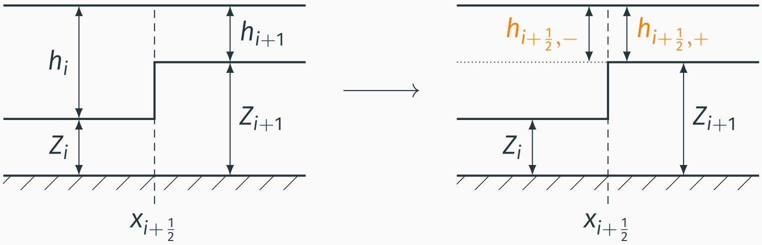
[1]see E. Audusse et al., *SIAM J. Sci. Comput.* (2004)

An answer for the lake at rest: the hydrostatic reconstruction

The **hydrostatic reconstruction** was introduced[1] in 2004, as a way to make it possible for any finite volume scheme to capture the **lake at rest** steady solution.

It relies on:

1. providing a relevant expression for $\mathcal{S}$,
2. evaluating the numerical flux at a specific reconstruction of W.



[1]see E. Audusse et al., *SIAM J. Sci. Comput.* (2004)

The hydrostatic reconstruction

The scheme becomes

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} = \frac{1}{\Delta x}\Big[\mathcal{F}\big(W^n_{i+\frac{1}{2},-}, W^n_{i+\frac{1}{2},+}\big) - \mathcal{F}\big(W^n_{i-\frac{1}{2},-}, W^n_{i-\frac{1}{2},+}\big)\Big] + \mathcal{S}_i^n,$$

where the approximate source term is $\mathcal{S}_i^n = (0, (\mathcal{S}_q)_i^n)^\intercal$, with:

$$(\mathcal{S}_q)_i^n = \frac{g}{2}\Big[\big(h^n_{i+\frac{1}{2},-}\big)^2 - \big(h^n_{i-\frac{1}{2},+}\big)^2\Big],$$

and with the reconstructed values

$$\begin{aligned} Z_{i+\frac{1}{2}} &= \max(Z_i, Z_{i+1}), \\ h^n_{i+\frac{1}{2},-} &= \max(h_i^n + Z_i - Z_{i+\frac{1}{2}}, 0), \quad q^n_{i+\frac{1}{2},-} = h^n_{i+\frac{1}{2},-}\, u_i^n, \\ h^n_{i+\frac{1}{2},+} &= \max(h_{i+1}^n + Z_{i+1} - Z_{i+\frac{1}{2}}, 0), \quad q^n_{i+\frac{1}{2},+} = h^n_{i+\frac{1}{2},+}\, u_{i+1}^n. \end{aligned}$$

Objectives

Main goal of this work: Provide a **linear** reconstruction able to capture the steady solutions with $q \neq 0$.

The objectives of our **hydrodynamic reconstruction** include:

- making sure that the result scheme is consistent,
- ensuring the capture of steady solutions with $q \neq 0$,
- handling dry areas and transitions between wet and dry areas (not presented in this talk),
- a linear and well-balanced high-order extension.

Expression of the hydrodynamic reconstruction

Away from dry areas, the hydrostatic reconstruction reads:

$$h^n_{i+\frac{1}{2},-} = h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right),$$

$$h^n_{i+\frac{1}{2},+} = h^n_{i+1} + \left(Z_{i+1} - Z_{i+\frac{1}{2}}\right).$$

Expression of the hydrodynamic reconstruction

Away from dry areas, the hydrodynamic reconstruction reads:

$$
\begin{aligned}
h^n_{i+\frac{1}{2},-} = h^n_i &+ \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&+ 2\mathrm{Fr}^2(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i)\,\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right), \\
h^n_{i+\frac{1}{2},+} = h^n_{i+1} &+ \left(Z_{i+1} - Z_{i+\frac{1}{2}}\right) \\
&+ 2\mathrm{Fr}^2(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1})\,\mathcal{H}\left(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1}, Z_{i+\frac{1}{2}} - Z_{i+1}\right),
\end{aligned}
$$

with $\mathcal{H}$ a function of h_L, h_R, $\bar{q}$ and $\Delta Z \coloneqq Z_R - Z_L$ and

$$
\mathrm{Fr}^2(h_L, h_R, \bar{q}) = \frac{\bar{q}^2(h_L + h_R)}{2g h_L^2 h_R^2}.
$$

Expression of the hydrodynamic reconstruction

Away from dry areas, the hydrodynamic reconstruction reads:

$$
\begin{aligned}
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i)\,\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right), \\
h^n_{i+\frac{1}{2},+} &= h^n_{i+1} + \left(Z_{i+1} - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1})\,\mathcal{H}\left(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1}, Z_{i+\frac{1}{2}} - Z_{i+1}\right),
\end{aligned}
$$

with $\mathcal{H}$ a function of h_L, h_R, $\bar{q}$ and $\Delta Z \coloneqq Z_R - Z_L$ and

$$
\mathrm{Fr}^2(h_L, h_R, \bar{q}) = \frac{\bar{q}^2(h_L + h_R)}{2g h_L^2 h_R^2}.
$$

The hydrodynamic reconstruction relies on deriving a suitable function $\mathcal{H}$.

Requirements of the hydrodynamic reconstruction

We seek three main properties of the hydrodynamic reconstruction:

1. it should **vanish when the topography is flat**;
2. it should **degenerate towards the hydrostatic reconstruction when the velocity vanishes**;
3. it should be **well-balanced**:

$$\text{steady solution} \implies h^n_{i+\frac{1}{2},-} = h^n_{i+\frac{1}{2}} = h^n_{i+\frac{1}{2},+}.$$

We have defined an interface state in an upwind way:

$$(h^n_{i+\frac{1}{2}}, Z_{i+\frac{1}{2}}) = \begin{cases} (h^n_i, Z_i) & \text{if } Z_i > Z_{i+1}, \\ (h^n_{i+1}, Z_{i+1}) & \text{otherwise.} \end{cases}$$

First property: consistency

$$
\begin{aligned}
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i)\,\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right), \\
h^n_{i+\frac{1}{2},+} &= h^n_{i+1} + \left(Z_{i+1} - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1})\,\mathcal{H}\left(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1}, Z_{i+\frac{1}{2}} - Z_{i+1}\right),
\end{aligned}
$$

For the hydrodynamic reconstruction to vanish when the topography is flat, we impose

$$
\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) \underset{\Delta Z \to 0}{=} \mathcal{O}(\Delta Z).
$$

Second property: lake at rest

$$h^n_{i+\frac{1}{2},-} = h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\ + 2\mathrm{Fr}^2(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i)\,\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right),$$

$$h^n_{i+\frac{1}{2},+} = h^n_{i+1} + \left(Z_{i+1} - Z_{i+\frac{1}{2}}\right) \\ + 2\mathrm{Fr}^2(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1})\,\mathcal{H}\left(h^n_{i+1}, h^n_{i+\frac{1}{2}}, q^n_{i+1}, Z_{i+\frac{1}{2}} - Z_{i+1}\right),$$

Since $\mathrm{Fr}(h_L, h_R, 0) = 0$, the hydrodynamic reconstruction automatically degenerates towards the hydrostatic reconstruction when $q^n_i = q^n_{i+1} = 0$ if **we assume that $\mathcal{H}$ is bounded**.

Third property: general steady solutions

$$
\begin{aligned}
h^n_{i+\frac{1}{2},-} = h^n_i &+ \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&+ 2\mathrm{Fr}^2(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i)\,\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right),
\end{aligned}
$$

We have to prove[2] that $h^n_{i+\frac{1}{2},-} = h^n_{i+\frac{1}{2}}$ when the solution is steady.

Recall that the interface state is defined by

$$
(h^n_{i+\frac{1}{2}}, Z_{i+\frac{1}{2}}) = \begin{cases} (h^n_i, Z_i) & \text{if } Z_i > Z_{i+1}, \\ (h^n_{i+1}, Z_{i+1}) & \text{otherwise.} \end{cases}
$$

Therefore, since the solution is steady, with $\bar{q} = q_i = q_{i+1}$, dropping the time indices for simplicity, we get:

$$
\frac{\bar{q}^2}{2h_i^2} + g(h_i + Z_i) = \frac{\bar{q}^2}{2h_{i+\frac{1}{2}}^2} + g(h_{i+\frac{1}{2}} + Z_{i+\frac{1}{2}}) = \frac{\bar{q}^2}{2h_{i+1}^2} + g(h_{i+1} + Z_{i+1}).
$$

[2] Proving that $h^n_{i+\frac{1}{2},+} = h^n_{i+1}$ leads to the same conclusion.

Third property: general steady solutions

Some algebraic manipulations allow us to write

$$\frac{\bar{q}^2}{2h_i^2} + g(h_i + Z_i) = \frac{\bar{q}^2}{2h_{i+\frac{1}{2}}^2} + g(h_{i+\frac{1}{2}} + Z_{i+\frac{1}{2}})$$

$$\iff$$

$$Z_{i+\frac{1}{2}} - Z_i = -\left(h_{i+\frac{1}{2}} - h_i\right)\left(1 - \mathrm{Fr}^2(h_i, h_{i+\frac{1}{2}}, \bar{q})\right),$$

which is nothing but the usual discrete characterization of smooth steady solutions.

We claim that imposing the following property on $\mathcal{H}$ will be enough to preserve steady solutions:

$$\Delta Z = -(h_R - h_L)(1 - \mathrm{Fr}^2(h_L, h_R, \bar{q})) \implies \mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) = \frac{h_R - h_L}{2}.$$

Third property: general steady solutions

Indeed, we then obtain the following sequence of equalities:

$$h^n_{i+\frac{1}{2},-} = h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) + 2\mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i\right)\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right),$$

Third property: general steady solutions

Indeed, we then obtain the following sequence of equalities:

$$
\begin{aligned}
h^n_{i+\frac12,-} &= h^n_i + \left(Z_i - Z_{i+\frac12}\right) \\
&\quad + 2\mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac12}, q^n_i\right)\mathcal{H}\left(h^n_i, h^n_{i+\frac12}, q^n_i, Z_{i+\frac12} - Z_i\right), \\
h^n_{i+\frac12,-} &= h^n_i + \left(Z_i - Z_{i+\frac12}\right) \\
&\quad + \mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac12}, q^n_i\right)\left(h^n_{i+\frac12} - h^n_i\right),
\end{aligned}
$$

Indeed, we then obtain the following sequence of equalities:

$$
\begin{aligned}
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i\right) \mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right), \\
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + \mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i\right) \left(h^n_{i+\frac{1}{2}} - h^n_i\right), \\
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + \left(Z_{i+\frac{1}{2}} - Z_i\right) + \left(h_{i+\frac{1}{2}} - h_i\right),
\end{aligned}
$$

Third property: general steady solutions

Indeed, we then obtain the following sequence of equalities:

$$
\begin{aligned}
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + 2\mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i\right)\mathcal{H}\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i, Z_{i+\frac{1}{2}} - Z_i\right), \\
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + \mathrm{Fr}^2\left(h^n_i, h^n_{i+\frac{1}{2}}, q^n_i\right)\left(h^n_{i+\frac{1}{2}} - h^n_i\right), \\
h^n_{i+\frac{1}{2},-} &= h^n_i + \left(Z_i - Z_{i+\frac{1}{2}}\right) \\
&\quad + \left(Z_{i+\frac{1}{2}} - Z_i\right) + \left(h_{i+\frac{1}{2}} - h_i\right), \\
h^n_{i+\frac{1}{2},-} &= h^n_{i+\frac{1}{2}},
\end{aligned}
$$

which proves that the scheme is well-balanced.

Summary and source term discretization

To summarize, for the reconstruction to be consistent and well-balanced, we require the **following two properties** on the bounded function $\mathcal{H}$:

1. $\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) \underset{\Delta Z \to 0}{=} \mathcal{O}(\Delta Z)$,
2. $\Delta Z = -(h_R - h_L)(1 - \mathrm{Fr}^2(h_L, h_R, \bar{q})) \implies \mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) = \dfrac{h_R - h_L}{2}$.

Summary and source term discretization

To summarize, for the reconstruction to be consistent and well-balanced, we require the **following two properties** on the bounded function $\mathcal{H}$:

1. $\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) \underset{\Delta Z \to 0}{=} \mathcal{O}(\Delta Z)$,
2. $\Delta Z = -(h_R - h_L)(1 - \mathrm{Fr}^2(h_L, h_R, \bar{q})) \implies \mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) = \dfrac{h_R - h_L}{2}$.

In addition, the whole scheme will also be consistent and well-balanced if the following **numerical source term** is used:

$$\Delta x (\mathcal{S}_q)_i^n = -g \frac{2h^n_{i-\frac{1}{2},+} h^n_{i+\frac{1}{2},-}}{h^n_{i-\frac{1}{2},+} + h^n_{i+\frac{1}{2},-}} \left(Z_{i+\frac{1}{2}} - Z_{i-\frac{1}{2}} \right) + \frac{4g}{h^n_{i-\frac{1}{2},+} + h^n_{i+\frac{1}{2},-}} \mathcal{H}\left(h^n_{i-\frac{1}{2},+}, h^n_{i+\frac{1}{2},-}, q_i, Z_{i+\frac{1}{2}} - Z_{i-\frac{1}{2}} \right)^3.$$

The proof results from algebraic manipulations (not detailed here).

Summary and source term discretization

To summarize, for the reconstruction to be consistent and well-balanced, we require the **following two properties** on the bounded function $\mathcal{H}$:

1. $\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) \underset{\Delta Z \to 0}{=} \mathcal{O}(\Delta Z)$,
2. $\Delta Z = -(h_R - h_L)(1 - \mathrm{Fr}^2(h_L, h_R, \bar{q})) \implies \mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) = \dfrac{h_R - h_L}{2}$.

In addition, the whole scheme will also be consistent and well-balanced if the following **numerical source term** is used:

$$\Delta x (\mathcal{S}_q)_i^n = -g \frac{2 h^n_{i-\frac{1}{2},+} h^n_{i+\frac{1}{2},-}}{h^n_{i-\frac{1}{2},+} + h^n_{i+\frac{1}{2},-}} \left(Z_{i+\frac{1}{2}} - Z_{i-\frac{1}{2}} \right) + \frac{4g}{h^n_{i-\frac{1}{2},+} + h^n_{i+\frac{1}{2},-}} \mathcal{H}\left(h^n_{i-\frac{1}{2},+}, h^n_{i+\frac{1}{2},-}, q_i, Z_{i+\frac{1}{2}} - Z_{i-\frac{1}{2}} \right)^3.$$

The proof results from algebraic manipulations (not detailed here).

Next step: obtain a suitable expression of $\mathcal{H}$.

Satisfying the well-balanced property

Recall that we need

$$\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) = \frac{h_R - h_L}{2}$$

as soon as a steady solution is under consideration.

To obtain an expression of $\mathcal{H}$ satisfying this property, we need to understand how $h_R - h_L$ behaves for discrete steady solutions.

Satisfying the well-balanced property

Recall that we need

$$\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) = \frac{h_R - h_L}{2}$$

as soon as a steady solution is under consideration.

To obtain an expression of $\mathcal{H}$ satisfying this property, we need to understand how $h_R - h_L$ behaves for discrete steady solutions.

We now seek a relation to characterize **the jump of h at the interface**, i.e. an expression of $h_R - h_L$ for steady solutions.

We assume that the solution is steady, and introduce notation

$$\bar{h} = \frac{h_L + h_R}{2} \quad \text{and} \quad \mathcal{H} = \frac{h_R - h_L}{2},$$

so that h_L and h_R satisfy

$$h_L = \bar{h} - \mathcal{H} \quad \text{and} \quad h_R = \bar{h} + \mathcal{H}.$$

The goal is now to rewrite the steady relation in terms of $\bar{h}$ and $\mathcal{H}$.

A local relation to characterize steady solutions

Recall that the steady solutions are governed by

$$B(h, q_0, Z) = \frac{q_0^2}{2h^2} + g(h + Z) = B_0.$$

That is to say, at the interface between states W_L and W_R, the solution is locally steady as soon as $q_L = q_R = \bar{q}$ and

$$B(h_L, \bar{q}, Z_L) = B(h_R, \bar{q}, Z_R) \iff \frac{\bar{q}^2}{2h_L^2} + g(h_L + Z_L) = \frac{\bar{q}^2}{2h_R^2} + g(h_R + Z_R).$$

We set out to rewrite the above relation using $\bar{h}$ and $\mathcal{H}$ instead of h_L and h_R.

A nonlinear relation for the interface jump

$$\frac{\bar{q}^2}{2h_L^2} + g(h_L + Z_L) = \frac{\bar{q}^2}{2h_R^2} + g(h_R + Z_R)$$

$$\iff$$

$$\frac{\bar{q}^2}{2(\bar{h} - \mathcal{H})^2} + g(\bar{h} - \mathcal{H} + Z_L) = \frac{\bar{q}^2}{2(\bar{h} + \mathcal{H})^2} + g(\bar{h} + \mathcal{H} + Z_R)$$

$$\iff$$

$$\dots$$

$$\iff$$

$$2\mathcal{H}\left(g\left(\bar{h}^2 - \mathcal{H}^2\right)^2 - \bar{q}^2\bar{h}\right) = -g(Z_R - Z_L)\left(\bar{h}^2 - \mathcal{H}^2\right)^2.$$

A nonlinear relation for the interface jump: properties

$$2\mathcal{H}\left(g\left(\bar{h}^2 - \mathcal{H}^2\right)^2 - \bar{q}^2\bar{h}\right) = -g\Delta Z\left(\bar{h}^2 - \mathcal{H}^2\right)^2 \qquad (*)$$

Can $\mathcal{H}$, implicitly given by the above expression, satisfy the required consistency and well-balanced properties?

A nonlinear relation for the interface jump: properties

$$2\mathcal{H}\Big(g\big(\bar{h}^2 - \mathcal{H}^2\big)^2 - \bar{q}^2\bar{h}\Big) = -g\Delta Z\big(\bar{h}^2 - \mathcal{H}^2\big)^2 \qquad (*)$$

Can $\mathcal{H}$, implicitly given by the above expression, satisfy the required consistency and well-balanced properties?

1. For the consistency, we need $\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) \underset{\Delta Z \to 0}{=} \mathcal{O}(\Delta Z)$: at least one solution to $(*)$ satisfies this property.

A nonlinear relation for the interface jump: properties

$$2\mathcal{H}\left(g\left(\bar{h}^2-\mathcal{H}^2\right)^2-\bar{q}^2\bar{h}\right)=-g\Delta Z\left(\bar{h}^2-\mathcal{H}^2\right)^2 \qquad (*)$$

Can $\mathcal{H}$, implicitly given by the above expression, satisfy the required consistency and well-balanced properties?

1. For the consistency, we need $\mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) \underset{\Delta Z \to 0}{=} \mathcal{O}(\Delta Z)$: at least one solution to $(*)$ satisfies this property.
2. For the well-balanced property, we need

$$\Delta Z = -(h_R - h_L)(1-\mathrm{Fr}^2(h_L, h_R, \bar{q})) \implies \mathcal{H}(h_L, h_R, \bar{q}, \Delta Z) = \frac{h_R - h_L}{2}.$$

This property holds since $(*)$ has been derived so that $2\mathcal{H} = h_R - h_L$ is a solution as soon as the flow is steady.

“Quadratized” relation

$$2\mathcal{H}\left(g\left(\bar{h}^2-\mathcal{H}^2\right)^2-\bar{q}^2\bar{h}\right)=-g\Delta Z\left(\bar{h}^2-\mathcal{H}^2\right)^2 \qquad (*)$$

Equation $(*)$ is nonlinear, and using it would incur considerable computational cost. To avoid this issue, we proceed with linearization-like simplification: for $\mathcal{H}\neq\bar{h}$, we get

$$(*) \iff 2\mathcal{H}\left(1-\frac{\bar{q}^2\bar{h}}{g\left(\bar{h}^2-\mathcal{H}^2\right)^2}\right)=-\Delta Z.$$

We choose a “quadratization” of this expression around $\mathcal{H}=\Delta h/2$:

$$2\mathcal{H}\left(1-\underbrace{\frac{\bar{q}^2(h_L+h_R)}{2gh_L^2h_R^2}}_{\mathrm{Fr}^2}+4\,\mathrm{sgn}(\Delta Z)\sqrt{\frac{|\Delta Z|}{|\Delta h|^3}}(\Delta h-2\mathcal{H})\right)=-\Delta Z.$$

Final expression of $\mathcal{H}$

We are left with $\mathcal{H}$ satisfying a quadratic relation.

Solving this quadratic equation for $\mathcal{H}$ leads to

$$\mathcal{H} = \frac{1}{4}\left(E - \operatorname{sgn}(1-\mathrm{Fr}^2)\operatorname{sgn}(\Delta Z)\sqrt{E^2 + \sqrt{|\Delta Z||\Delta h|^3}}\right),$$

$$\text{with } \; E = \Delta h + \frac{1-\mathrm{Fr}^2}{4}\operatorname{sgn}(\Delta Z)\sqrt{\frac{|\Delta h|^3}{|\Delta Z|}}.$$

We show that, if ΔZ and $1-\mathrm{Fr}^2$ do not simultaneously vanish:

1. this expression of $\mathcal{H}$ is consistent, despite the divisions by ΔZ;
2. this expression of $\mathcal{H}$ is well-balanced.

Well-balanced property

To show the well-balanced property, we take $\Delta Z = -(1-\mathrm{Fr}^2)\Delta h$, to get

$$E = \Delta h + \frac{1-\mathrm{Fr}^2}{4}\mathrm{sgn}(-(1-\mathrm{Fr}^2)\Delta h)\sqrt{\frac{|\Delta h|^3}{|1-\mathrm{Fr}^2||\Delta h|}} = \Delta h\left(1 - \frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}\right),$$

$$E^2 + \sqrt{|\Delta Z||\Delta h|^3} = (\Delta h)^2\left(1 + \frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}\right)^2.$$

Well-balanced property

To show the well-balanced property, we take $\Delta Z = -(1-\mathrm{Fr}^2)\Delta h$, to get

$$E = \Delta h + \frac{1-\mathrm{Fr}^2}{4}\mathrm{sgn}(-(1-\mathrm{Fr}^2)\Delta h)\sqrt{\frac{|\Delta h|^3}{|1-\mathrm{Fr}^2||\Delta h|}} = \Delta h\left(1 - \frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}\right),$$

$$E^2 + \sqrt{|\Delta Z||\Delta h|^3} = (\Delta h)^2\left(1 + \frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}\right)^2.$$

Plugging this in $\mathcal{H}$, we obtain

$$\mathcal{H} = \frac{1}{4}\left(\Delta h\left(1-\frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}\right) + \mathrm{sgn}(\Delta h)\sqrt{(\Delta h)^2\left(1+\frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}\right)^2}\right)$$

$$= \frac{\Delta h}{4}\left(1-\frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}+1+\frac{1}{4}\sqrt{|1-\mathrm{Fr}^2|}\right) = \frac{\Delta h}{2},$$

which proves the well-balanced property.

Well-balanced property

To show the well-balanced property, we take $\Delta Z = -(1 - \mathrm{Fr}^2)\Delta h$, to get

$$E = \Delta h + \frac{1 - \mathrm{Fr}^2}{4}\mathrm{sgn}(-(1 - \mathrm{Fr}^2)\Delta h)\sqrt{\frac{|\Delta h|^3}{|1 - \mathrm{Fr}^2||\Delta h|}} = \Delta h\left(1 - \frac{1}{4}\sqrt{|1 - \mathrm{Fr}^2|}\right),$$

$$E^2 + \sqrt{|\Delta Z||\Delta h|^3} = (\Delta h)^2\left(1 + \frac{1}{4}\sqrt{|1 - \mathrm{Fr}^2|}\right)^2.$$

Plugging this in $\mathcal{H}$, we obtain

$$\mathcal{H} = \frac{1}{4}\left(\Delta h\left(1 - \frac{1}{4}\sqrt{|1 - \mathrm{Fr}^2|}\right) + \mathrm{sgn}(\Delta h)\sqrt{(\Delta h)^2\left(1 + \frac{1}{4}\sqrt{|1 - \mathrm{Fr}^2|}\right)^2}\right)$$
$$= \frac{\Delta h}{4}\left(1 - \frac{1}{4}\sqrt{|1 - \mathrm{Fr}^2|} + 1 + \frac{1}{4}\sqrt{|1 - \mathrm{Fr}^2|}\right) = \frac{\Delta h}{2},$$

which proves the well-balanced property.

Next step: provide a well-balanced high-order extension.

High-order scheme

A high-order (non-well-balanced) finite volume scheme reads:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\Big(\mathcal{F}(\widehat{W}_{i,+}^n, \widehat{W}_{i+1,-}^n) - \mathcal{F}(\widehat{W}_{i-1,+}^n, \widehat{W}_{i,-}^n)\Big) + \Delta t \widehat{\mathcal{S}}_i^n.$$

In each cell, we reconstruct a polynomial of degree d, under the form

$$\widehat{W}_i^n(x) = W_i^n + \sum_{\alpha=1}^{d} R_i^\alpha (x - x_i)^\alpha,$$

where the coefficients R_i^α depend on the neighboring cells.

The evaluations at the interfaces $x_{i\pm\frac{1}{2}}$ are then given by:

$$\widehat{W}_{i,-}^n = W_i^n + \sum_{\alpha=1}^{d} R_i^\alpha \left(-\frac{\Delta x}{2}\right)^\alpha \quad \text{and} \quad \widehat{W}_{i,+}^n = W_i^n + \sum_{\alpha=1}^{d} R_i^\alpha \left(\frac{\Delta x}{2}\right)^\alpha,$$

and the high-order source term is the following approximation:

$$\widehat{\mathcal{S}}_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(\widehat{W}_i^n(x))\, dx + \mathcal{O}(\Delta x^{d+1}).$$

Linear well-balanced correction of the high-order scheme

We introduce a convex combination with parameter $\theta_{i\pm\frac{1}{2}}$ to provide a well-balanced correction to the high-order scheme, such that:

- if $\theta_{i\pm\frac{1}{2}} = 0$, the scheme is well-balanced;
- if $\theta_{i\pm\frac{1}{2}} = 1$, the scheme is high-order accurate.

The new evaluations at the interfaces $x_{i\pm\frac{1}{2}}$ are given by:

$$\widetilde{W}_{i,-}^n = W_i^n + \theta_{i-\frac{1}{2}} \sum_{\alpha=1}^{d} R_i^\alpha \left(-\frac{\Delta x}{2}\right)^\alpha \quad \text{and} \quad \widetilde{W}_{i,+}^n = W_i^n + \theta_{i+\frac{1}{2}} \sum_{\alpha=1}^{d} R_i^\alpha \left(\frac{\Delta x}{2}\right)^\alpha,$$

and the new high-order well-balanced source term reads:

$$\widetilde{\mathcal{S}}_i^n = \left(1 - \frac{\theta_{i-\frac{1}{2}}^n + \theta_{i+\frac{1}{2}}^n}{2}\right) \mathcal{S}_i^n + \frac{\theta_{i-\frac{1}{2}}^n + \theta_{i+\frac{1}{2}}^n}{2} \, \widehat{\mathcal{S}}_i^n.$$

Next step: Provide a suitable choice of the convex combination parameter $\theta_{i\pm\frac{1}{2}}$. We follow the general strategy from [C. Berthon, S. Bulteau, F. Foucher, M. M'Baye and V. M.-D., *SIAM SISC, 2022*].

Steady solution detector

The convex combination parameter $\theta^n_{i+1/2}$ must satisfy the following properties:

- vanish when (W^n_i, W^n_{i+1}) are at equilibrium;
- be an approximation of 1 up to $\mathcal{O}(\Delta x^{d+1})$ otherwise.

Steady solution detector

The convex combination parameter $\theta_{i+1/2}^n$ must satisfy the following properties:

- vanish when (W_i^n, W_{i+1}^n) are at equilibrium;
- be an approximation of 1 up to $\mathcal{O}(\Delta x^{d+1})$ otherwise.

We propose the following expression:

$$\theta_{i+\frac{1}{2}}^n = \frac{\varepsilon_{i+\frac{1}{2}}^n}{\varepsilon_{i+\frac{1}{2}}^n + C_{i+\frac{1}{2}}^n \Delta x^{d+1}},$$

$$\text{with } \varepsilon_{i+\frac{1}{2}}^n = \left\| \begin{pmatrix} q_{i+1}^n - q_i^n \\ B(h_{i+1}^n, q_{i+1}^n, Z_{i+1}) - B(h_, q_i^n, Z_i) \end{pmatrix} \right\|.$$

Properties of the steady solution detector

$$\theta^n_{i+\frac{1}{2}} = \frac{\varepsilon^n_{i+\frac{1}{2}}}{\varepsilon^n_{i+\frac{1}{2}} + C^n_{i+\frac{1}{2}} \Delta x^{d+1}}, \text{ with } \varepsilon^n_{i+\frac{1}{2}} = \left\| \begin{pmatrix} q^n_{i+1} - q^n_i \\ B(h^n_{i+1}, q^n_{i+1}, Z_{i+1}) - B(h^n_i, q^n_i, Z_i) \end{pmatrix} \right\|$$

(WB) We easily note that $\varepsilon^n_{i+\frac{1}{2}}$ vanishes (and therefore $\theta^n_{i+\frac{1}{2}}$ does too) as soon as W^n_i and W^n_{i+1} are at equilibrium.

(HO) If $\varepsilon^n_{i+\frac{1}{2}} \neq 0$, then

$$\theta^n_{i+\frac{1}{2}} = \frac{1}{1 + \Delta x^{d+1} \frac{C^n_{i+\frac{1}{2}}}{\varepsilon^n_{i+\frac{1}{2}}}} = 1 + \mathcal{O}(\Delta x^{d+1}).$$

$\rightsquigarrow$ The expression of $\theta^n_{i\pm\frac{1}{2}}$ satisfies the required properties.

Next step: perform numerical tests to validate the method.

Setup

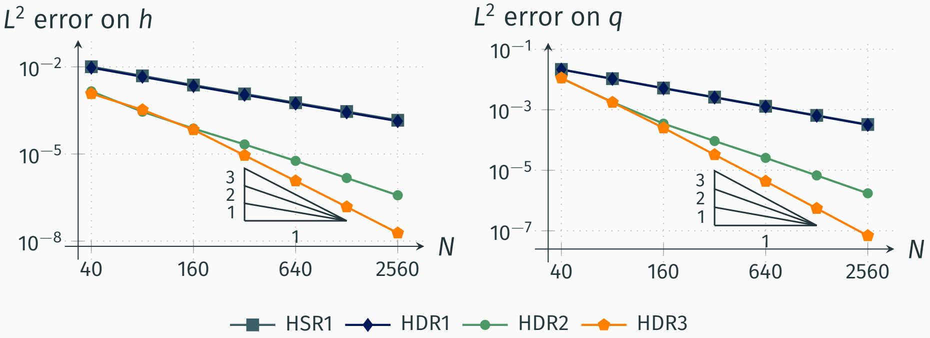
We provide several numerical tests with a finite volume scheme using the HLL flux:

- an order of convergence test,
- three tests of the well-balanced property,
- a dry dam-break.

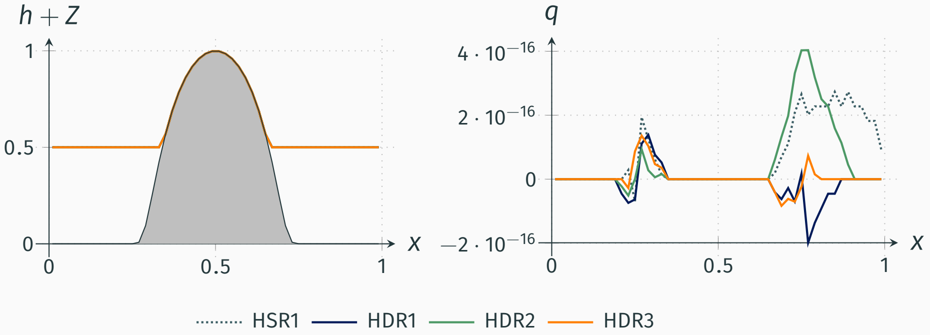
These tests are performed with the **h**ydro**s**tatic **r**econstruction (HSR) and the **h**ydro**d**ynamic **r**econstruction (HDR).

The schemes of order δ are denoted by HSRδ and HDRδ.

Order of convergence

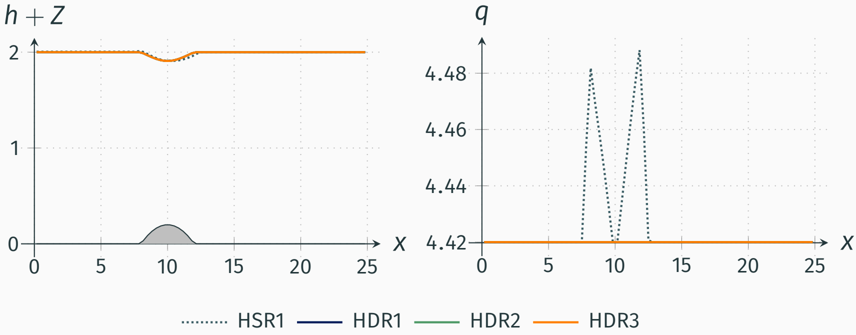


Emerged lake at rest



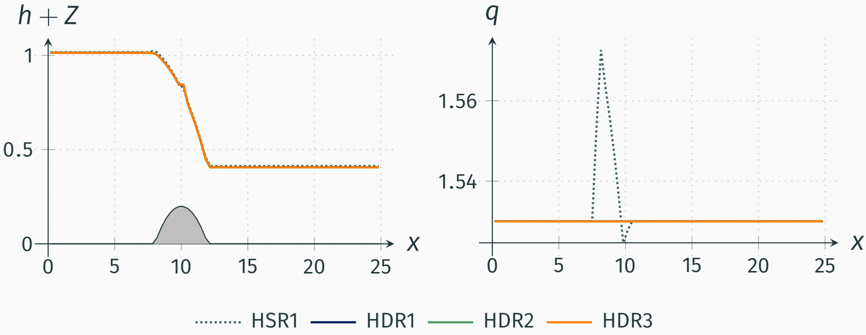
	HSR, $\mathbb{P}_0$	HDR, $\mathbb{P}_0$	HDR, $\mathbb{P}_1$	HDR, $\mathbb{P}_2$
L^2 error on h	$1.85 \cdot 10^{-17}$	$2.75 \cdot 10^{-17}$	$3.07 \cdot 10^{-17}$	$1.32 \cdot 10^{-17}$
L^2 error on q	$1.24 \cdot 10^{-16}$	$5.17 \cdot 10^{-17}$	$1.24 \cdot 10^{-16}$	$3.59 \cdot 10^{-17}$

Subcritical flow



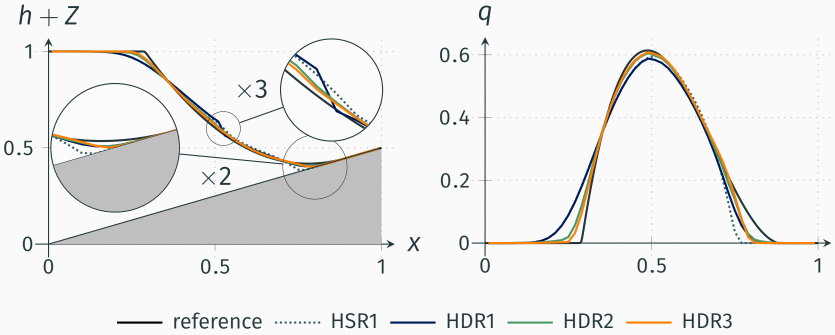
	HSR, $\mathbb{P}_0$	HDR, $\mathbb{P}_0$	HDR, $\mathbb{P}_1$	HDR, $\mathbb{P}_2$
L^2 error on q	$7.73 \cdot 10^{-2}$	$1.06 \cdot 10^{-14}$	$1.31 \cdot 10^{-14}$	$1.30 \cdot 10^{-14}$
L^2 error on B	$1.79 \cdot 10^{-1}$	$2.73 \cdot 10^{-14}$	$3.61 \cdot 10^{-14}$	$2.68 \cdot 10^{-14}$

Transcritical flow



	HSR, $\mathbb{P}_0$	HDR, $\mathbb{P}_0$	HDR, $\mathbb{P}_1$	HDR, $\mathbb{P}_2$
L^2 error on q	$3.74 \cdot 10^{-2}$	$4.73 \cdot 10^{-14}$	$5.15 \cdot 10^{-14}$	$5.21 \cdot 10^{-14}$
L^2 error on B	$1.45 \cdot 10^{-1}$	$4.50 \cdot 10^{-14}$	$5.12 \cdot 10^{-14}$	$5.92 \cdot 10^{-14}$

Dry dam-break



Conclusion

We have developed a linear reconstruction that allows any finite volume scheme to be fully well-balanced for the shallow water system.

This reconstruction has the following properties:

- it leads to a consistent scheme,
- the resulting scheme is well-balanced,
- it can be extended to high-order accuracy with a low computational cost,
- it is able to handle wet/dry transitions,
- it is easy to implement.

Thank you for your attention!

Finite Volumes for Complex Applications 10 (**FVCA10**),

in Strasbourg, 30/10/2023 – 03/11/2023

Limitations of the method

Of course, the method also has a few limitations.

1. It is dependent on a parameter C, which could be different for each experiment.
2. Although the scheme is high-order accurate and well-balanced, there is an issue with high-order well-balanced initialization. Consider an initial condition W_0, steady at interface $x_{i-1/2}$ and unsteady at interface $x_{i+1/2}$; we need the reconstruction $\widetilde{W}_i^0$ to satisfy

$$\widetilde{W}_i^0(x_{i-\frac{1}{2}}) = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W_0(x)dx \quad \text{and} \quad \widetilde{W}_i^0(x_{i+\frac{1}{2}}) = W_0(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^{d+1}).$$

This leads to two conditions in cell i, for one unknown W_i^0...

An expression of $C^n_{i+1/2}$

To implement the scheme, we need to give an expression of $C = C^n_{i+1/2}$. We propose $C^0_{i+1/2} = 1$, and, for $n \geqslant 1$:

$$C^n_{i+\frac{1}{2}} = C_\theta \, \frac{1}{2}\left(\frac{\left\| W^n_{i+1} - W^{n-1}_{i+1} \right\|}{\Delta t} + \frac{\left\| W^n_{i} - W^{n-1}_{i} \right\|}{\Delta t} \right),$$

with C_θ a constant parameter.

Note that

$$\theta^n_{i+\frac{1}{2}} = \frac{\varepsilon^n_{i+\frac{1}{2}}}{\varepsilon^n_{i+\frac{1}{2}} + \left(\frac{\Delta x}{C^n_{i+\frac{1}{2}}} \right)^\delta} = \frac{\varepsilon^n_{i+\frac{1}{2}} (C^n_{i+\frac{1}{2}})^\delta}{\varepsilon^n_{i+\frac{1}{2}} (C^n_{i+\frac{1}{2}})^\delta + \Delta x^\delta} :$$

we get $\theta^n_{i+\frac{1}{2}} = 0$ if $\varepsilon^n_{i+\frac{1}{2}} = 0$ or $C^n_{i+\frac{1}{2}} = 0$. **Why does this make sense?**

An expression of $C^n_{i+1/2}$ – reasoning

$$\theta^n_{i+\frac{1}{2}} = 0 \text{ if } \varepsilon^n_{i+\frac{1}{2}} = 0 \text{ or } C^n_{i+\frac{1}{2}} = 0$$

$\varepsilon^n_{i+\frac{1}{2}} = 0 \implies$ steady state solution for the equations

$\implies \theta^n_{i+\frac{1}{2}}$ must vanish to preserve the steady state solution

$C^n_{i+\frac{1}{2}} = 0 \implies$ vanishing discrete time derivative

$\implies$ steady state solution for the high-order scheme

$\implies$ not a steady state solution for the equations[3]

$\implies \theta^n_{i+\frac{1}{2}}$ must vanish to perturb the solution

[3] Otherwise, the high-order scheme would be well-balanced.