Règle de transformation des symboles de Christoffel

Rappel : $D_{\frac{\partial}{\partial x^i}} \dfrac{\partial}{\partial x^j} = \Gamma^k_{ij} \dfrac{\partial}{\partial x^k}$. ds carte $(U, (x^1, \dots, x^n))$

Soit $(V, (y^1, \dots, y^n))$ une autre carte, qui définit des symboles de Christoffel

$$D_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \tilde{\Gamma}^k_{ij} \frac{\partial}{\partial y^k}.$$

Transformation de base : $\boxed{\dfrac{\partial}{\partial y^i} = \dfrac{\partial x^s}{\partial y^i} \dfrac{\partial}{\partial x^s}}$ $\left(\text{Conséquence : } \dfrac{\partial x^s}{\partial y^i} \dfrac{\partial y^t}{\partial x^s} = \delta^t_i\right)$

Alors $D_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j} = D_{\frac{\partial x^s}{\partial y^i} \frac{\partial}{\partial x^s}} \dfrac{\partial x^l}{\partial y^j} \dfrac{\partial}{\partial x^l}$

$$\frac{\partial^2 x^l}{\partial y^i \partial y^j} \frac{\partial}{\partial x^l} = \frac{\partial x^s}{\partial y^i} \frac{\partial^2 x^l}{\partial x^s \partial y^j} \frac{\partial}{\partial x^l} + \frac{\partial x^s}{\partial y^i} \frac{\partial x^l}{\partial y^j} \Gamma^p_{sl} \frac{\partial}{\partial x^p}$$

Par ailleurs $D_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j} = \tilde{\Gamma}^k_{ij} \dfrac{\partial}{\partial y^k} = \tilde{\Gamma}^k_{ij} \dfrac{\partial x^p}{\partial y^k} \dfrac{\partial}{\partial x^p}$

On obtient

$$\tilde{\Gamma}^k_{ij} \frac{\partial x^p}{\partial y^k} = \Gamma^p_{sl} \frac{\partial x^s}{\partial y^i} \frac{\partial x^l}{\partial y^j} + \frac{\partial^2 x^p}{\partial y^i \partial y^j} \qquad \Big| \cdot \frac{\partial y^t}{\partial x^p}$$

$$\Longleftrightarrow \boxed{\tilde{\Gamma}^t_{ij} = \Gamma^p_{sl} \frac{\partial x^s}{\partial y^i} \frac{\partial x^l}{\partial y^j} \frac{\partial y^t}{\partial x^p} + \frac{\partial^2 x^p}{\partial y^i \partial y^j} \frac{\partial y^t}{\partial x^p}} \quad (*) \quad \boxed{\text{M}}$$

<u>Rmq</u> : on retrouve bien la même règle de transformation à partir de la définition

$$(**) \quad \Gamma_{ij}^{l} = \frac{1}{2}\sum_k g^{lk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

((*) est valable pour les coeff. de n'importe quelle connexion linéaire, et ceci montre que (**) définit bien une conn. linéaire).

Ici $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. Ds carte $(y^1, \dots, y^n)$ notons

$$\tilde{g}_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right).$$

<u>Alors :</u> (exercices)

- $\tilde{g}_{ij} = g_{kl}\dfrac{\partial x^k}{\partial y^i}\dfrac{\partial x^l}{\partial y^j}$

  $\tilde{g}^{ij} = g^{kl}\dfrac{\partial y^i}{\partial x^k}\dfrac{\partial y^j}{\partial x^l}$

- $\dfrac{\partial \tilde{g}_{ij}}{\partial y^k} = \dfrac{\partial \tilde{g}_{ij}}{\partial x^s}\dfrac{\partial x^s}{\partial y^k} = \dfrac{\partial g_{kl}}{\partial x^s}\dfrac{\partial x^k}{\partial y^i}\dfrac{\partial x^l}{\partial y^j} + 2$ autres termes

- $\tilde{\Gamma}_{ij}^{l} = \dfrac{1}{2}\tilde{g}^{lk}\left(\dfrac{\partial \tilde{g}_{ik}}{\partial y^j} + \dfrac{\partial \tilde{g}_{jk}}{\partial y^i} - \dfrac{\partial \tilde{g}_{ij}}{\partial y^k}\right)$

$$= \dots\dots = \Gamma_{bc}^{a}\frac{\partial x^b}{\partial y^i}\frac{\partial x^c}{\partial y^j}\frac{\partial y^l}{\partial x^k} + \frac{\partial^2 x^p}{\partial y^i \partial y^j}\frac{\partial y^l}{\partial x^p}. \quad \blacksquare$$

Courbure et relevés horizontaux

__Rappel :__ Les symboles de Christoffel sont définis par la relation $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^l_{ij} \frac{\partial}{\partial x^l}$ (expression dans une carte locale).

__Affirmation__ : Soit $(U, (x^1, \dots, x^n))$ carte locale et soient $(y^1, \dots, y^n)$ les coord. associées sur les fibres de TM.

Soit $\boxed{X = X^i \frac{\partial}{\partial x^i}}$ ch. $\vec{v}$ sur U. Son relevé horizontal (pour la connexion de Levi-Civita) en $y = (y^1, \dots, y^n)$ est

$$\boxed{\tilde{X} = X^i \frac{\partial}{\partial x^i} - \Gamma^l_{ij} X^i y^j \frac{\partial}{\partial y^l}}$$

__Preuve :__ Clairement $d\pi \cdot \tilde{X} = X$.

Soit $\gamma(t)$ courbe intégrale pour X et $\tilde{\gamma}(t)$ courbe intégrale pour $\tilde{X}$ par $y = (y^1, \dots, y^n)$. Alors nécessairement

$$\tilde{\gamma}(t) = \left(\gamma(t), f^l(t) \frac{\partial}{\partial y^l}\right), \quad f^l(0) = y$$

Or $D_{\dot\gamma}\left(f^l(t) \frac{\partial}{\partial x^l}\right) = (f^l)'(t) \frac{\partial}{\partial x^l} + f^j X^i \Gamma^l_{ij} \frac{\partial}{\partial x^l}$

$$= -X^i \Gamma^l_{ij} f^j \frac{\partial}{\partial x^l} + f^j X^i \Gamma^l_{ij} \frac{\partial}{\partial x^l} = 0 \quad \blacksquare$$

Conséquence : $\dfrac{\widetilde{\partial}}{\partial x^i} = \dfrac{\partial}{\partial x^i} - \Gamma^l_{ij}\, y^j \dfrac{\partial}{\partial y^l}$

et donc

$$\left[\frac{\widetilde{\partial}}{\partial x^i}, \frac{\widetilde{\partial}}{\partial x^j}\right]_y = \left[\frac{\partial}{\partial x^i} - \Gamma^l_{it}\, y^t \frac{\partial}{\partial y^l}, \frac{\partial}{\partial x^j} - \Gamma^s_{jr}\, y^r \frac{\partial}{\partial y^s}\right]_y$$

$$= -\frac{\partial \Gamma^s_{jr}}{\partial x^i}\, y^r \frac{\partial}{\partial y^s} + \frac{\partial \Gamma^l_{it}}{\partial x^j}\, y^t \frac{\partial}{\partial y^l}$$

$$+ \Gamma^l_{it}\, \Gamma^s_{jr} \underbrace{\left[y^t \frac{\partial}{\partial y^l}, y^r \frac{\partial}{\partial y^s}\right]}_{y^t \delta^r_l \frac{\partial}{\partial y^s} - y^r \delta^t_s \frac{\partial}{\partial y^l}}$$

chang. d'indices

$$= \left(-\frac{\partial \Gamma^l_{jr}}{\partial x^i}\, y^r + \frac{\partial \Gamma^l_{it}}{\partial x^j}\, y^t + \Gamma^s_{it}\, \Gamma^l_{js}\, y^t - \Gamma^s_{jr}\, \Gamma^l_{is}\, y^r\right) \frac{\partial}{\partial y^l}$$

Cas particulier : (identifie $\frac{\partial}{\partial y^l}$ avec $\frac{\partial}{\partial x^l}$)

$$\left[\frac{\widetilde{\partial}}{\partial x^i}, \frac{\widetilde{\partial}}{\partial x^j}\right]_{\frac{\partial}{\partial x^k}} = \left(-\frac{\partial \Gamma^l_{jk}}{\partial x^i} + \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{ik}\, \Gamma^l_{js} - \Gamma^s_{jk}\, \Gamma^l_{is}\right) \frac{\partial}{\partial x^l}$$

Par ailleurs : $\underline{R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)}\frac{\partial}{\partial x^k} = \underline{R^l_{ijk} \frac{\partial}{\partial x^l} = \left[\frac{\widetilde{\partial}}{\partial x^i}, \frac{\widetilde{\partial}}{\partial x^j}\right]}^{\text{vert}}_{\frac{\partial}{\partial x^k}}$

et on obtient $\boxed{R^l_{ijk} = -\dfrac{\partial \Gamma^l_{jk}}{\partial x^i} + \dfrac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{ik}\, \Gamma^l_{js} - \Gamma^s_{jk}\, \Gamma^l_{is}}$