# LIOUVILLE POLARIZATION AND THE RIGIDITY OF THEIR LAGRANGIAN SKELETA IN DIMENSION $4$

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ABSTRACT. In [3], Biran introduced polarizations of closed symplectic manifolds and showed that their Lagrangian skeleta exhibit remarkable rigidity properties. He in particular found that their complements contain only small balls. In this paper, we introduce so-called Liouville polarizations of certain open 4-dimensional symplectic manifolds. This leads to several symplectic embedding results, that in turn lead to new Lagrangian non-removable intersections and a novel phenomenon of Legendrian barriers.

We show for instance that given any connected symplectic 4-manifold $(M, \omega)$ and a 4-ball of smaller volume, their exists an explicit finite union of Lagrangian discs in the 4-ball such that their complement symplectically embeds into $(M, \omega)$, extending a result by Sackel-Song-Varolgunes-Zhu and Brendel.

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## 1. Introduction and main results

A polarization of a *closed* symplectic manifold $(M, \omega)$ is a decomposition of the symplectic class $[\omega]$ along positive real multiples of the Poincaré-dual classes of codimension 2 symplectic submanifolds. Closed symplectic manifolds always admit polarizations, and these have proven useful to study the flexible side of symplectic embedding problems [6, 25]. In this work, we introduce polarizations of *open* symplectic manifolds and show that this concept offers new opportunities, at least in dimension $4$. Referring to § 2.3 for the definition, we focus in this introduction on the applications.

1.1. **Symplectic embeddings.** As usual, $D(a)$, $B^4(a)$, $C^4(a)$, and $Z^4(a)$ denote the open round disc of area $a$, the open 4-ball of capacity $\pi r^2 = a$, the open "cube" $D(a) \times D(a)$, and the open cylinder $D(a) \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$, all endowed with the standard symplectic form $\omega_{\rm st} = \sum_{j=1}^2 dx_j \wedge dy_j$ on $\mathbb{R}^4$ with primitive $\alpha_{\rm st} = \frac{1}{2} \sum_{j=1}^2 x_j dy_j - y_j dx_j$.

A cornerstone of modern symplectic geometry is Gromov's non-squeezing theorem from [17], stating in dimension four that $B^4(a)$ symplectically embeds into $Z^4(1)$ only for $a \leq 1$. In [29], Sackel, Song, Varolgunes, and Zhu investigated a refinement of this result, asking how big a set (in the sense of the Hausdorff dimension, for instance) one has to remove from $B^4(a)$ such that the complement symplectically embeds into $Z^4(1)$. They first proved that it does not suffice to remove a set of lower Minkowski dimension $< 2$, and then showed optimality of this result by proving that for $a \leq 2$ it suffices to remove a Lagrangian coordinate disc from $B^4(a)$. Later, Brendel in the appendix to [29] showed that for $a < 3$ it suffices to remove three Lagrangian pinwheels (certain simple Lagrangian CW-complexes) and a symplectic torus. In their Question 3, these authors then asked:

**Question 1.1.** *What is the largest $a$ such that the complement of a 2-dimensional subset of $B^4(a)$ symplectically embeds into $Z^4(1)$?*

We denote by $\Gamma_{\frac{1}{k}}$ the union of the $k$ half-lines emanating from the origin in $\mathbb{C}$ and cutting the plane into isometric sectors, as partly shown in (i), (ii), and (iii) of Figure 1.1 on page 5 for $k = 2, 3, 4$, and write

$$\Delta_k := \Gamma_{\frac{1}{k}} \times \Gamma_{\frac{1}{k}} \subset \mathbb{C}^2.$$

Then $\Delta_k$ is the orbit of the Lagrangian quadrant $\mathbb{R}^2_{\geq 0} \subset \mathbb{C}^2$ under the subgroup $G$ of $\mathrm{U}(2)$ generated by $(e^{2\pi i/k}, 1)$ and $(1, e^{2\pi i/k})$. When $k$ is even, then $\Delta_k$ is also the $G$-orbit of the standard Lagrangian plane $\mathbb{R}^2 \subset \mathbb{C}^2$.

A symplectic embedding $\varphi\colon (M, d\alpha) \to (M', d\alpha')$ between exact symplectic manifolds is $(\alpha, \alpha')$**-exact** if $\varphi^*\alpha' = \alpha + df$ for a smooth function $f$ on $M$. Since $B^4(1) \subset C^4(1)$, our first symplectic embedding result give a positive answer to Question 1.1 for every $a$.

**Theorem 1.** *There exists an* $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$*-exact symplectic embedding*

$$C^4(1)\backslash\Delta_k \;\to\; Z^4\big(\tfrac{2}{k}\big).$$

**Remarks 1.2.** (i) There exists a symplectic embedding of $D(\frac{1}{k}) \times D(\frac{1}{k})$ into $C^4(1) \setminus \Delta_k$, so $C^4(1) \setminus \Delta_k$ does not symplectically embed into $Z^4(A)$ for $A < \frac{1}{k}$ by the nonsqueezing theorem. Thus Theorem 1 is sharp up to a possible factor of 2. We do not know the exact size of the thinnest cylinder into which $C^4(1)\backslash\Delta_k$ or $B^4(1)\backslash\Delta_k$ symplectically embed, except for the optimal embedding $B^4(1) \setminus \Delta_2 \to Z^4(\frac{1}{2})$ from [29]. For a different such embedding see Remark 4.7.

(ii) A different positive answer to Question 1.1 was independently given by Haim-Kislev, Hind, and Ostrover in [18], and it is interesting to compare the two constructions, see § 1.5.

Our construction also applies to unbounded domains. For instance, consider the union of Lagrangian planes $\Gamma \times \Gamma$ in $\mathbb{R}^4$, where $\Gamma \subset \mathbb{R}^2$ is the square grid

$$\Gamma = \bigcup_{(n,m)\in\mathbb{Z}^2} \{n\} \times \mathbb{R} \cup \mathbb{R} \times \{m\}.$$

**Theorem 2.** *There exists an* $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$*-exact symplectic embedding*

$$\mathbb{R}^4 \setminus (\Gamma \times \Gamma) \;\to\; Z^4(1).$$

We can construct similar embeddings when the source or the target have topology. If we take the 4-ball as domain, we can fill "anything" after removing sufficiently many Lagrangian planes:

**Theorem 3.** *Let* $(M, \omega)$ *be a connected symplectic 4-manifold of finite volume. Let* $B^4(a)$ *be the ball of the same volume. Then for every* $\varepsilon > 0$ *there exist an even number* $k \in \mathbb{N}$ *such that* $B^4(a-\varepsilon) \setminus \Delta_k$ *symplectically embeds into* $(M, \omega)$.

In fact a much more general statement holds. Any affine part of a closed symplectic 4-manifold (defined as the complement of a polarizing divisor, see Definition 2.2) verifies the same kind of flexibility: After removing a suitable Lagrangian CW-complex, it can be embedded into any other affine part of a closed symplectic 4-manifold of larger volume. We shall give the proof of this result in a separate paper, [27], because in addition to the embedding method developped here the proof relies on a relative version of Giroux's result in [16] and requires specific techniques. In contrast to the results of this paper, the removed CW-complex cannot be explicitly described, however.

1.2. **Rigidity properties of Lagrangian skeleta.** Following Biran [2], we notice that if the complement of a Lagrangian CW-complex $\Delta$ in a domain $U$ symplectically embeds into a cylinder, then any domain $V \subset U$ that does not symplectically embed into that cylinder must intersect $\Delta$. The results of the previous paragraph therefore have implications on non-removable intersections of $\Delta$ with various subsets, and even with Lagrangian submanifolds. We shall remove Lagrangian CW-complexes more general than the $\Delta_k$, namely products of arbitrary connected graphs.

**Definition 1.3.** A **grid** $\Gamma$ in $D(A)$ is the part in $D(A)$ of a connected graph $\Gamma \cup \partial D(A)$ in $\overline{D(A)}$ that has no 1-valent vertex, contains the boundary of $D(A)$, and has smooth edges.

Note that the complement of a grid consists of topological discs. Figure 1.1 shows examples of grids. Note that for two grids $\Gamma$ and $\Gamma'$, the Lagrangian CW-complex $\Gamma \times \Gamma'$ is smooth if and only if both $\Gamma$ and $\Gamma'$ have no vertex, as in (i), (iv), and (vii).

Cieliebak and Mohnke in [9] defined for every closed Lagrangian submanifold $L \subset \mathbb{R}^4$ its **minimal symplectic area** by

$$(1.1)\quad \mathcal{A}_{\min}(L) := \inf \left\{ \int_\sigma \omega_{\rm st} \mid [\sigma] \in \pi_2(\mathbb{R}^4, L),\ \int_\sigma \omega_{\rm st} > 0 \right\} \ \in\ [0, \infty].$$

**Theorem 4.** *Let $\Gamma_{\leq a} \subset D(A)$ and $\Gamma_{\leq b} \subset D(B)$ be any two grids whose complements are a union of topological discs of area $\leq a$ and $\leq b$, respectively. Then a closed Lagrangian submanifold $L$ of $D(A) \times D(B)$ with*

$$\mathcal{A}_{\min}(L) \geq a + b$$

*cannot be mapped to $\big(D(A) \times D(B)\big) \setminus \big(\Gamma_{\leq a} \times \Gamma_{\leq b}\big)$ by a Hamiltonian diffeomorphism of $\mathbb{R}^4$.*

**Example.** For every Markov triple $\mathfrak{m}$ there exists a monotone Lagrangian torus $\widehat{L}(\mathfrak{m})$ in the complex projective plane endowed with the Fubini–Study form integrating to 3 over complex lines. Each such torus gives rise to a Lagrangian torus $L(\mathfrak{m}) \subset B^4(3)$, and $L(\mathfrak{m})$ is Hamiltonian isotopic to $L(\mathfrak{m}')$ in $B^4(3)$ only if $\mathfrak{m} = \mathfrak{m}'$, see [31]. For these Lagrangians, $A_{\min}(L(\mathfrak{m})) = 1$. If $\phi_H$ is a Hamiltonian diffeomorphism of $\mathbb{R}^4$ such that $\phi_H(L(\mathfrak{m})) \subset B^4(A) \setminus \Delta_k$, then $A \geq \frac{k}{2}$. ◇

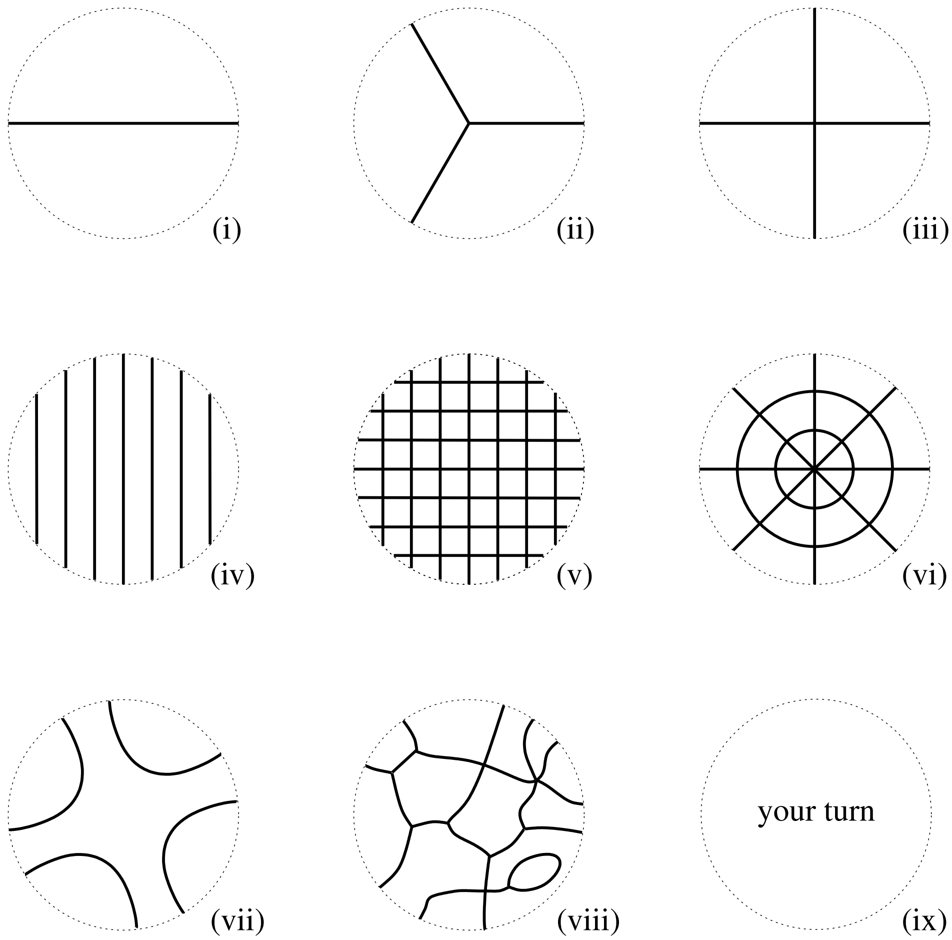


FIGURE 1.1. Examples of grids.

1.3. **Legendrian barriers.** Lagrangian CW-complexes $\Delta$ with symplectically small complement were called Lagrangian barriers in [3] (cf. § 1.5). A further consequence of the phenomenon of Lagrangian barriers, that apparently went unnoticed, is a similar Legendrian barrier phenomenon in contact dynamics. A grid is **radial** if it consists of straight rays emanating from the origin.

**Theorem 5.** *Let $\Gamma_{\leq \delta_1}$ and $\Gamma_{\leq \delta_2}$ be two radial grids that divide $D(1)$ into sectors of area $\leq \delta_1$ and area $\leq \delta_2$, respectively. Let $U \subset C^4(1)$ be a star-shaped domain with smooth boundary $S$, endowed with the usual contact form $\lambda_S = \alpha_{\rm st}|_S$, and consider the connected Legendrian graph*

$$\Lambda_\delta \;=\; \big(\Gamma_{\leq \delta_1} \times \Gamma_{\leq \delta_2}\big) \cap S.$$

*Then any Legendrian knot $\Lambda$ in $S$ has a Reeb chord from $\Lambda$ to $\Lambda \cup \Lambda_\delta$ of length $\leq \delta_1 + \delta_2$ for the contact form $\lambda_S$.*

**Remarks 1.4.** (i) While the proof of Theorem 5 relies on tools from hard symplectic geometry, the case of the round sphere can be shown by elementary geometric arguments, see Section 6.

(ii) The smooth arcs of $\Lambda_\delta$ are indeed Legendrian curves on $S$, because $\iota_{r\partial_r}\omega_{\rm st} = \alpha_{\rm st}$ and $\partial_r$ is tangent to $\Gamma_{\leq \delta_1} \times \Gamma_{\leq \delta_2}$. The theorem says that for any Legendrian knot $\Lambda$, every embedded Reeb cylinder of length $\geq \delta_1 + \delta_2$ over $\Lambda$ must intersect $\Lambda_\delta$, hence the name Legendrian barrier for $\Lambda_\delta$. It follows in particular that for $a > \frac{2}{k}$ there exists no exact contact embedding

$$\big(D(a) \times \mathbb{R}/\mathbb{Z}, \alpha_{\rm st} + dt\big) \hookrightarrow \big(S \backslash \Delta_k, \lambda_S\big).$$

It is then natural to ask whether there is a purely contact topological statement of this kind. For instance, using the Reeb flows one constructs exact contact embeddings

$$\big(D(a) \times \mathbb{R}/\mathbb{Z}, \alpha_{\rm st} + dt\big) \hookrightarrow \big(S^3(1), \lambda_S\big)$$

for all $a \leq 1$. Is it true that for $a \in (\frac{2}{k}, 1]$ the image of these embeddings cannot be displaced from $\Delta_k \cap S^3(1)$ by a (not necessarily exact) contact isotopy?

(iii) In fact, the proof will show that there exist both forwards and backwards Reeb chord from $\Lambda$ to $\Lambda \cup \Lambda_\delta$ of length $\leq \delta_1 + \delta_2$.

(iv) (Improved return time) Theorem 5 states an alternative: There must be short Reeb chords from $\Lambda$ to $\Lambda$ *or* $\Lambda_\delta$. This is a relative version of the solution of the Arnold chord conjecture by Cieliebak–Mohnke in [22, 9]. They proved that for $\Lambda \subset S = \partial U$ as in the theorem, there must be a Reeb chord *from* $\Lambda$ *to* $\Lambda$ of length at most

$$\min\big\{e(U), \tfrac{1}{2}\big\}$$

where $e(U)$ denotes the displacement energy of $U$. Theorem 5 improves this bound on the return time for all those $\Lambda$ that lie in the complement of the Reeb cylinder of $\Lambda_\delta$ of length $\delta_1 + \delta_2$, whenever $\delta_1 + \delta_2 < \min\big\{e(U), \frac{1}{2}\big\}$.

1.4. **A general embedding result.** The upper bounds on the cylindrical capacity of the complement of our Lagrangian CW-complexes all rely on a single embedding result. To state it, we need to introduce the subclass of regular grids.

**Definition.** A grid $\Gamma$ on $D(A)$ (as defined in 1.3) is **regular** if every vertex of $\Gamma \cup \partial D(A)$ has a Darboux chart in which $\Gamma$ consists of radial rays that cut the disc into equal sectors. When the vertex belongs to the boundary, the Darboux chart must be understood with image $D(\varepsilon) \cap \{\operatorname{im} z \geq 0\} \subset \mathbb{H}$.

In Figure 1.1, all grids are regular except for (iv) and (v) – and possibly (ix).

**Theorem 6.** *Let $\Gamma_{\leq a} \subset D(A)$ and $\Gamma_{\leq b} \subset D(B)$ be any two regular grids whose complements are a union of topological discs of area $\leq a$ and $\leq b$, respectively. Then there exists an $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$-exact symplectic embedding*

$$\big(D(A) \times D(B)\big) \setminus \big(\Gamma_{\leq a} \times \Gamma_{\leq b}\big) \hookrightarrow Z^4(a+b).$$

Theorem 1 is a special case of this theorem, and Theorem 2 is a generalization to a non-compact setting. The exactness in this statement is essential to derive Theorems 4 and 5 on Lagrangian rigidity and Legendrian barriers.

## 1.5. **Related results and references.**

**Biran's Lagrangian barriers.** As we have already alluded to, the present paper is in many ways a sequel of Biran's work [3], that is useful to have in mind while reading our paper. We therefore briefly recall it, integrating Giroux's results from [15, 16] that allow to generalize Biran's results from the Kähler to the symplectic setting. Let $(M, \omega)$ be a closed symplectic manifold with integral symplectic class (meaning that $[\omega] \in H^2(M, \mathbb{Z})$). For every sufficiently large $k$, Donaldson [10] produced a symplectic hypersurface $\Sigma$ Poincaré-dual to $k[\omega]$, called a polarization of degree $k$ of $(M, \omega)$. The complement $M \setminus \Sigma$ was proven to be Weinstein by Giroux [16]. Hence there exists an isotropic CW-complex (called the skeleton) $\Delta \subset M \setminus \Sigma$ to which $M \setminus \Sigma$ retracts. Furthermore, any compact subset of $M \setminus \Delta$ embeds into a ruled symplectic manifold, namely a symplectic sphere bundle over $\Sigma$ whose fibers have area $\frac{1}{k}$, see [2, 3]. A fundamental example to have in mind along this discussion is the following:

**Example 1.5.** ([3, § 3.1.3]) In the complex projective space $\mathbb{C}\mathrm{P}^n$ with Fubini–Study form integrating to 1 over complex lines, the hypersurface

$$\Sigma_{2k} = \left\{ \sum_{j=0}^{n} z_j^{2k} = 0 \right\} \tag{1.2}$$

is a symplectic polarization of degree $2k$, and one can arrange the skeleton of $\mathbb{C}\mathrm{P}^n \setminus \Sigma_{2k}$ to be the union $\Delta_{2k}(\mathbb{C}\mathrm{P}^n)$ of $k^n$ equally distributed images of $\mathbb{R}\mathrm{P}^n$ by projectivized unitary maps. The complement of $2k$ fibers of the symplectic disc bundle over $\Sigma_{2k}$ is symplectomorphic to $B^{2n}(1) \setminus \Delta^n_{2k}$, where

$$\Delta^n_{2k} := (\Gamma_{2k})^n$$

is the union of $k^n$ unitary images of $\mathbb{R}^n$. For $n = 2$ this is our set $\Delta^2_{2k} = \Delta_{2k}$.

Ruled symplectic manifolds are known to have many symplectic invariants bounded by the area of their sphere fibres, which allowed Biran to get interesting non-removable intersections with the skeleton. For instance, for a polarization of degree $k$ with skeleton $\Delta$, every symplectic ball of capacity $\geq \frac{1}{k}$ in $M$ must intersect $\Delta$. He therefore called these skeleta Lagrangian barriers. In particular, such balls cannot be displaced from $\Delta$ by a Hamiltonian isotopy. Combined with the work [9] by Cieliebak and Mohnke, Biran's decomposition result also implies non-displaceability of small Lagrangian submanifolds from the skeleton. Namely, applying the neck-stretching in [9] to the set-up in [2, 3], one finds that the general Cieliebak–Mohnke Lagrangian width

$$\mathcal{A}_{\min}(L,M) := \inf \left\{ \int_\sigma \omega \mid [\sigma] \in \pi_2(M,L),\ \int_\sigma \omega > 0 \right\}$$

can be estimated as follows.

**Theorem 7** (Biran–Cieliebak–Mohnke)**.** *Let $\Delta$ be the skeleton of a polarization of degree $k$ of $(M,\omega)$. Then for any closed Lagrangian submanifold $L \subset M \backslash \Delta$ it holds that*

$$\mathcal{A}_{\min}(L,M) < \frac{1}{k}.$$

*In other words, a closed Lagrangian submanifold with $\mathcal{A}_{\min}(L,M) \geq \frac{1}{k}$ cannot be displaced from the skeleton $\Delta$ by a Hamiltonian isotopy.*

For instance, for $x \geq \frac{1}{k}$ the product torus $\mathbb{T}^n(x) \subset B^{2n}(1)$ whose factors are circles enclosing area $x$ cannot be displaced from $\Delta^n_{2k}$ inside $B^{2n}(1)$. Again, Theorem 7 has a Legendrian corollary:

**Theorem 8.** *Let $U \subset B^{2n}(1)$ be a starshaped domain with smooth boundary $S$, endowed with the usual contact form $\lambda_S = \alpha_{\rm st}|_S$, and consider the Legendrian CW-complex $\Lambda_{2k} = \Delta^n_{2k} \cap S$. Then for every Legendrian submanifold $\Lambda \subset S$ there exists a Reeb chord from $\Lambda$ to $\Lambda \cup \Lambda_{2k}$ of length $T \leq \frac{1}{2k}$.*

Biran's decomposition theorem thus implies the same kind of applications that we have drawn from our embedding results, in any dimension and with better constants. We should therefore explain what this paper adds beyond the embedding results from Section 1.1. Here is a first answer. Biran's intersection result and Theorem 7 state that *some* symplectic capacities of $M \backslash \Delta_k$ are smaller than $\frac{1}{k}$, namely Gromov's ball embedding capacity

$$c_B(M) \,:=\, \sup \left\{ a \mid B^{2n}(a) \text{ symplectically embeds into } M \right\},$$

which is the smallest normalized symplectic capacity, and the Cieliebak–Mohnke capacity

$$c_{\rm Lag}(M) \,:=\, \sup \left\{ \mathcal{A}_{\min}(L,M) \mid L \subset M \right\}$$

where the supremum runs over all closed Lagrangian submanifolds. Since the cylindrical capacity

$$c^Z(U) := \inf\{A \mid U \text{ symplectically embeds into } Z^{2n}(A)\}$$

is the largest normalized capacity, Theorem 1 states instead that *all* symplectic capacities of $B^4(1)\backslash\Delta_k$ are small, and our work [27] generalizes this property to the affine part of any closed symplectic manifold. There are many different capacities, whose estimates have different implications. For instance, applied to the Hofer–Zehnder capacity, Theorem 1 says that the flow of any compactly supported Hamiltonian function on $C^4(1)$ that vanishes on a neighbourhood of $\Delta_k$ and whose maximum is at least $\frac{2}{k}$ has a non-trivial closed orbit of period $\leq 1$.

Another, more important, gain is that our approach allows to study the rigidity properties of certain *singular* polarizations. The role of singular polarizations for symplectic embeddings has been observed earlier, and they were used in several works [6, 7, 25, 26]. But the study of their skeleta was left aside in these works. The question of rigidity of Lagrangian skeleta associated to singular polarizations is more subtle than for smooth polarizations. Indeed, the complement of the skeleton of a singular polarization is not ruled anymore, at if its degree is $k$ the Gromov width may not be $\frac{1}{k}$ but may be arbitrarily large [26, Theorem 1]. The present work opens a way to identify singular polarizations for which the complement of the skeleton has small width. It may also be worth noting that some very natural Lagrangian CW-complexes are the skeleta of singular polarizations and not of smooth ones. For instance, the Clifford torus in $\mathbb{CP}^2$ is the skeleton of a polarization by three lines, but cannot be the skeleton of a smooth polarization, see § 2.2. And even if the skeleton of a smooth polarization is computable (like $\Delta_{2k}$ for $\Sigma_{2k}$), it may be much easier to obtain it from a singular polarization, see e.g. § 3.2.

**Around Theorems 4 and 5.** Traditionally, Lagrangian intersection results were proven for smooth closed monotone Lagrangians. For non-smooth Lagrangians, the Floer machinery does not work directly (see however [13]), and for non-monotone Lagrangians it is more difficult [14]. Our Lagrangian skeleta $\Gamma_{\leq a} \times \Gamma_{\leq b}$ may be smooth or not, are not closed, and they may be very far from monotone.

Theorem 4 says that even rather small Lagrangians must intersect $\Gamma_{\leq a} \times \Gamma_{\leq b}$. Such Lagrangian rigidity at a small scale has been observed recently in [19, 28]. In these papers, the role of our $\Delta$ is played by Lagrangian submanifolds of $S^2 \times S^2$ that are products of one circle in the first factor with a collection of circles in the second factor which, in contrast to our grids, decompose $S^2$ into components that may have arbitrary topology. Another difference is that these Lagrangian submanifolds are secretly monotone (when lifted to a symmetric product), whereas our skeleta do not need to be monotone in any sense: Already each disc $D(A)$ or $D(B)$ may contain several discs bounded by $\Gamma_{\leq a}$ or $\Gamma_{\leq b}$ whose areas are completely independent. Theorem 4 therefore suggests that rigidity at a small scale is not intrinsically related to monotonicity.

Singular Lagrangian were first studied in [12, 13], where it is shown that the product of the 1-skeleton of a fine enough triangulation of the 2-sphere is super-heavy and hence must intersect any heavy Lagrangian. Recall that heavy sets are non-displaceable. Our intersection condition on the minimal action is thus very often much weaker.

**Theorem 6 versus the main result of** [18]**.** A different positive answer to Question 1.1 was given by Haim-Kislev, Hind, and Ostrover afterwards in [18]. They showed that for every $a$ one can remove from the ball $B^4(a)$ a finite number (depending on $a$) of parallel *symplectic* planes such that the complement symplectically embeds into $Z^4(1)$. Their embedding has very different properties from ours: The removed set is symplectic instead of Lagrangian, and the construction is rigid in the sense that one cannot alter the position of the planes, while our grids are rather arbitrary in view of Theorem 6. The key difference is that in contrast to theirs, our embeddings are exact, a property that we crucially need for deriving Theorems 4 and 5.

**Organisation of the paper.** The paper is organized as follows. In Section 2 we recall the results from [24, 25] on symplectic polarisations in dimension four, introduce Liouville polarizations and explain their main relations with symplectic embeddings. In Section 3 we compute some explicit pairs polarizations/skeleta. Combining these tools, we prove our embedding results in Section 4, deduce the Lagrangian rigidities in Section 5 and the Legendrian ones in Section 6.

**Notation and conventions.** Since the paper is long, we list here some of our terminology and conventions, to which the reader may return when necessary.

- A symplectic embedding $\varphi\colon (M, d\alpha) \to (M', d\alpha')$ between exact symplectic manifolds is $(\alpha, \alpha')$**-exact** if $\varphi^*\alpha' = \alpha + df$ for a smooth function $f$ on $M$. Equivalently, $\int_\gamma \alpha = \int_{\varphi(\gamma)} \alpha'$ for all closed curves $\gamma$ in $M$. It is enough to check this equality of actions on a set of closed curves that generate $H_1(M;\mathbb{R})$. If $H^1(M;\mathbb{R}) = 0$, then $(\alpha, \alpha')$-exactness is automatic; and otherwise, the notion of $(\alpha, \alpha')$-exactness may depend on the choice of primitives $\alpha, \alpha'$. We sometimes write $\xhookrightarrow{\alpha_{\mathrm{st}}}$ to abbreviate "there exists an $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$-exact symplectic embedding".

- A **Liouville form** on a symplectic manifold $(M, \omega)$ is just a primitive $\lambda$ of the symplectic form: $d\lambda = \omega$. The associated **Liouville vector field** $X_\lambda$ is defined by $\iota_{X_\lambda} d\lambda = \lambda$. Its flow, the **Liouville flow**, is conformally symplectic: $\left(\phi^t_{X_\lambda}\right)^* \omega = e^t \omega$.
- By a **symplectic curve** in a symplectic 4-manifold we mean a 2-dimensional embedded symplectic submanifold, which is closed if the ambient symplectic manifold is, or properly embedded if it is open. A **normal crossing** between symplectic curves $\Sigma, \Sigma'$ is an intersection point $p \in \Sigma \cap \Sigma'$ such that $T_p\Sigma$ and $T_p\Sigma'$ are $\omega$-orthogonal.
- A **symplectic multi-curve with normal crossings** $\Sigma$ is a union of symplectic curves $\Sigma_i$ (called the components of $\Sigma$) whose pairwise intersections are all normal crossings. The singularities of a multi-curve is the set of intersection points between components and is denoted $\mathrm{Sing}(\Sigma)$. The complement of the singular locus is called the regular part of $\Sigma$.
- Usual objects of differential geometry are generalized to multi-curves by just taking a collection of such objects on each component. For instance, a differential $k$-form $\alpha$ on $\Sigma = \cup \Sigma_i$ is a collection of differential $k$-forms on the $\Sigma_i$. The morphisms between multi-curves $\Sigma, \Sigma'$ are smooth maps from $\Sigma$ to $\Sigma'$ (in the sense that they are smooth when restricted to each components), and send the regular part and singular locus of $\Sigma$ to those of $\Sigma'$. A symplectic embedding between multi-curves is an injective such morphism that pulls back $\omega'|_{\Sigma'}$ to $\omega|_\Sigma$.
- A **weighted** symplectic multi-curve $\mathbf{\Sigma} := \{(\Sigma_i, \mu_i)\}$ with normal crossings is a collection of symplectic curves $\Sigma_i$ weighted by real numbers $\mu_i$, whose total space $\Sigma := \cup \Sigma_i$ is a symplectic multi-curve with normal crossings.
- A **symplectic morphism** from $\mathbf{\Sigma}$ to $\mathbf{\Sigma}'$ is a symplectic embedding of $\Sigma$ into $\Sigma'$ that sends each component to a component of the same weight and sends the regular part and singular locus of $\Sigma$ to those of $\Sigma'$. We write $\mathbf{\Sigma} \to \mathbf{\Sigma}'$.
- Symplectic multi-curves will also be called **symplectic divisors**, or simply divisors.
- Given a subset $S$ of a manifold $M$, we write $\mathrm{Op}(S, M)$ or just $\mathrm{Op}(S)$ instead of "some open neighbourhood of $S$ in $M$".

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## 2. Polarizations of symplectic manifolds

In this section we review those results from [3, 24, 25, 26] that we need in this paper, in a form useful for us. For simplicity, we stick to dimension $4$, although most of the statements can be adapted to higher dimensions.

2.1. **Neighborhoods of symplectic curves in dimension** $4$**.** Let $\pi\colon \mathcal{L} \to (\Sigma, \tau)$ be a complex line bundle with first Chern class $c_1$ over a symplectic curve of area $\mathcal{A} = \int_\Sigma \tau$. The multiplication by $e^{2\pi i\theta}$ defines a vector field $\frac{\partial}{\partial\theta}$ which, together with the complex structure on the fiber, provides a closed 1-form $d\theta$ on the punctured fibers $F \setminus \{0\}$. We now in addition endow $\mathcal{L}$ with a Hermitian metric. This provides a radial coordinate $r$, and throughout the paper we write $R := r^2$. Chern–Weil theory guarantees the existence of a connection 1-form with curvature $\frac{c_1}{\mathcal{A}}\,\tau$. This is a 1-form $\Theta$ on $\mathcal{L} \setminus \Sigma$ that satisfies

$$\Theta|_{F\setminus\{0\}} = d\theta \text{ for each fiber } F \qquad \text{and} \qquad d\Theta = -\frac{c_1}{\mathcal{A}}\,\pi^*\tau \text{ on } \mathcal{L}.$$

The $2$-form

$$\omega_0 \;:=\; \pi^*\tau + d(R\Theta) \;=\; \left(1 - \frac{c_1 R}{\mathcal{A}}\right)\pi^*\tau + dR \wedge \Theta \tag{2.1}$$

is defined on all of $\mathcal{L}$, is closed, and is symplectic on $\left\{R < \frac{\mathcal{A}}{c_1}\right\}$. Furthermore, $\omega_0$ is exact on $\mathcal{L} \setminus \Sigma$ (where we identify $\mathbf{0}_\mathcal{L}$ with $\Sigma$), with Liouville form

$$\lambda_0 := \left(R - \frac{\mathcal{A}}{c_1}\right)\Theta. \tag{2.2}$$

The associated Liouville vector field is

$$\left(R - \frac{\mathcal{A}}{c_1}\right)\frac{\partial}{\partial R}. \tag{2.3}$$

The 1-form $d\theta$ is defined on the punctured fibers $F \setminus \{0\}$, and in a local trivialization of $\mathcal{L}$ over $U \subset \Sigma$ it is defined on $\mathcal{L}|_U \setminus U$, but not on all of $\mathcal{L} \setminus \Sigma$. A transition function between two local trivializations is of the form $(z, w) \mapsto (z, g(z)\, w)$, where $g$ is a smooth $\mathbb{C} \setminus \{0\}$-valued function, and $d\theta$ is pulled back by this map to $d\theta + \operatorname{im} \frac{dg}{g}$. We can therefore define the *angular class* $\{d\theta\}$ as the set of 1-forms on $\mathcal{L} \setminus \Sigma$ that locally are equal to $d\theta$ up to a smooth 1-form on $\mathcal{L}$. The form $\Theta$ is a representative of $\{d\theta\}$. Recall that we have fixed a Hermitian metric on $\mathcal{L}$.

**Definition 2.1.** *The* symplectic disc bundle *over* $(\Sigma, \tau)$ *with Chern class* $c_1$ *is*

$$\mathrm{SDB}(\Sigma, \tau, c_1) \;:=\; \left( \left\{ R < \frac{\mathcal{A}}{c_1} \right\}, \omega_0, \lambda_0 \right) \;\subset\; \mathcal{L}.$$

*It comes with a distinguished class of* 1*-forms on* $\mathrm{SDB}(\Sigma, \tau, c_1) \setminus \Sigma$ *modulo smooth forms on* $\mathrm{SDB}(\Sigma, \tau, c_1)$*, called the angular class. Any connection* 1*-form provides a representative of the angular class. For* $\varepsilon \leq \frac{\mathcal{A}}{c_1}$*, we denote*

$$\mathrm{SDB}_\varepsilon(\Sigma, \tau, c_1) \;:=\; \big( \{R < \varepsilon\}, \omega_0, \lambda_0 \big) \;\subset\; \mathcal{L}.$$

By the symplectic neighbourhood theorem, for any symplectic curve $\Sigma \subset (M^4, \omega)$ and for $\varepsilon > 0$ sufficiently small, an open neighbourhood of $\Sigma$ is symplectomorphic to

$$\mathrm{SDB}_\varepsilon(\Sigma, M) \;:=\; \mathrm{SDB}_\varepsilon\big(\Sigma, \omega|_\Sigma, c_1(T\Sigma^\omega)\big)$$

by a symplectomorphism that lifts any given symplectomorphism between the zero section of $\mathrm{SDB}_\varepsilon(\Sigma, M)$ and $\Sigma \subset M$.

2.2. **Polarizations (closed case).** In [3], Biran introduced the notion of polarization for closed symplectic manifolds with rational symplectic class. This definition was later extended in [25] as follows.

**Definition 2.2.** *A* polarization $\mathbf{\Sigma} := \{(\Sigma_i, \mu_i)\}$ *of a closed* 4*-dimensional symplectic manifold* $(M, \omega)$ *is a finite collection of closed 2-dimensional symplectic submanifolds* $\Sigma_i$ *that intersect* $\omega$*-orthogonally, weighted by real coefficients* $\mu_i > 0$*, such that*

$$[\mathbf{\Sigma}] \;:=\; \sum_i \mu_i \,[\Sigma_i] \;=\; \mathrm{PD}([\omega]) \;\in\; H_2(M; \mathbb{R}). \tag{2.4}$$

*In other terms, a polarization is a weighted symplectic divisor with normal crossings Poincaré-dual to* $[\omega]$ *in the sense of* (2.4)*. We write* $\Sigma := \bigcup_i \Sigma_i$*. We say that the polarization is* smooth *if it consists of a single component. In this case,* $\mu$ *is determined by* $\Sigma$ *and* $[\omega]$*, and we say that* $\Sigma$ *has degree* $d = 1/\mu$*.*

Let $(M, \omega, \mathbf{\Sigma})$ be a polarized closed symplectic manifold. The symplectic form $\omega$ is exact on the complement of $\Sigma$, hence $\omega$ has primitives on $M \setminus \Sigma$. Throughout the paper, we consider only Liouville forms that satisfy a regularity assumption near $\Sigma$, that we discuss now.

When $\mathbf{\Sigma}$ is smooth, a neighbourhood can be modeled on

$$\mathrm{SDB}_\varepsilon(\Sigma, M) := \mathrm{SDB}_\varepsilon\big(\Sigma, \omega|_\Sigma, c_1(T\Sigma^\omega)\big)$$

for $\varepsilon$ small enough, and is therefore endowed with a radial coordinate $R$ and a connection 1-form $\Theta$, as in Section 2.1. When $\mathbf{\Sigma}$ consists of several pairwise $\omega$-orthogonal components $\Sigma_i$, we follow [25, Section 3.1] and make a neighbourhood of $\Sigma$ a symplectic plumbing of the $\varepsilon$-disc bundles over the $\Sigma_i$, which we still denote $\mathrm{SDB}_\varepsilon(\Sigma, M)$: a deleted neighbourhood of each $\Sigma_i$ can be endowed with local coordinates $(R_i, \theta_i)$, $d\theta_i$ belonging to the angular class around $\Sigma_i$, such that on a neighbourhood of an intersection point of $\Sigma_i$ and $\Sigma_j$ we have $\Sigma_i = \{R_j = 0\}$, $\Sigma_j = \{R_i = 0\}$ and $\omega = d(R_i d\theta_i + R_j d\theta_j)$.

**Definition 2.3.** *A* $1$*-form* $\lambda$ *on* $M \setminus \Sigma$ *is* **tame**

- **at a regular point** $p \in \Sigma_i \setminus \bigcup_{j\neq i} \Sigma_j$, *if there exists a real number* $a_i$ *and a bounded* $1$*-form* $\lambda'$ *on* $\mathrm{Op}\,(p) \setminus \Sigma_i$ *such that* $\lambda = a_i\, d\theta_i + \lambda'$ *on* $\mathrm{Op}\,(p) \setminus \Sigma_i$ *(*$\lambda'$ *is then smooth on* $\mathrm{Op}\,(p)\setminus\Sigma_i$*). In other terms,* $\lambda$ *locally represents a multiple of a connection form modulo bounded forms.*
- **at an intersection point** $p \in \Sigma_i \cap \Sigma_j$, *if there exist real numbers* $a_i, a_j$ *such that* $\lambda = (R_i + a_i)\, d\theta_i + (R_j + a_j)\, d\theta_j$ *on* $\mathrm{Op}\,(p) \setminus (\Sigma_i \cup \Sigma_j)$ *in the plumbing coordinates.*
- **along** $\Sigma$*, if it is tame at each point of* $\Sigma$*.*

**Remarks 2.4.** (i) The class of tame forms around $\Sigma$ depends on the choice of the Hermitian metrics on the $\mathcal{L}_i$, and on the identification of our neighbourhood of $\Sigma$ in $M$ with the plumbing of the $\varepsilon$-disc bundles over $\Sigma_i$. Given a polarization, we fix these choices, whether implicitely or after a construction. The notion of tameness will always refer to these choices.

(ii) Liouville forms in the complement of a normal crossing divisor $\Sigma \subset M$ have been used in many works, for instance in [20, 5]. In these works, the main interest was the symplectic homology of $M \setminus \Sigma$, whence it sufficed to understand that the Liouville flow is pointing towards $\Sigma$. Variants of the notion of tameness, that is crucial here, were introduced in [26]. $\diamond$

It is not hard to check that the number $a_i$ depends only on the component $\Sigma_i$ and not on the point $p$ on this component (see [26, Lemma 4.1]). We call $a_i$ the residue of $\lambda$ at $\Sigma_i$ and denote it $\mathrm{Res}\,(\lambda, \Sigma_i)$. By [25, Lemma 3.2] plumbings of symplectic disc bundles have tame Liouville forms with arbitrary residues. When, moreover, the homological polarizing condition (2.4) holds for $\mu_i = -\mathrm{Res}\,(\lambda, \Sigma_i)$, then these tame Liouville forms extend to Liouville forms on $M \setminus \Sigma$, see the proof of [26, Lemma 4.1 (iii)]. For further reference we state:

**Proposition 2.5.** *Let $(M, \omega, \mathbf{\Sigma})$ be a polarized closed symplectic manifold with $\mathbf{\Sigma} = \{(\Sigma_i, \mu_i)\}$. Then*

(i) *There exists a tame Liouville form $\lambda$ on $M \backslash \Sigma$ with residues* $\operatorname{Res}(\lambda, \Sigma_i) = -\mu_i$.

(ii) *Let $U$ be an open subset of $M$. Two Liouville forms on $U$ tame along $\Sigma$ with the same residues differ on $U \backslash \Sigma$ by a bounded smooth closed 1-form $\vartheta$.*

(iii) *Let $\lambda$ be a tame Liouville form on $M \backslash \Sigma$. For every small enough $\varepsilon > 0$ the vector field $X_\lambda$ is pointing inwards along the boundary of $\mathrm{SDB}_\varepsilon(\Sigma, M)$, and for every point $p \in \mathrm{SDB}_\varepsilon(\Sigma, M) \setminus \Sigma$ the flow line of $X_\lambda$ starting at $p$ hits $\Sigma$ in finite positive time. The Liouville field $X_\lambda$ is therefore backward complete, meaning that any trajectory is defined on $\mathbb{R}_{\leq 0}$.*

*Proof.* (i) has been discussed above and (ii) holds by definition of tameness. We prove (iii). Let $\lambda$ be a Liouville form tame along $\Sigma$. Near a regular point of $\Sigma_i$, $\lambda$ is then equal to $-\mu_i \Theta_i$ modulo a bounded one-form on $M \backslash \Sigma_i$, and we have seen that $\Theta_i$ is $\omega$-dual to $\frac{\partial}{\partial R_i}$, so

$$X_\lambda \;=\; -\mu_i \frac{\partial}{\partial R_i} + Z$$

for a bounded vector field $Z$. Since $\frac{\partial}{\partial R_i}$ has norm of order $\frac{1}{r_i}$ and since $\mu_i > 0$, $X_\lambda \cdot R_i < 0$ near $\Sigma_i = \{R_i = 0\}$. A similar argument applies near a singular point. □

**Definition 2.6** (Biran decomposition)**.** *Given a polarized closed symplectic manifold $(M, \omega, \mathbf{\Sigma})$ and a tame Liouville form $\lambda$ on $M \setminus \Sigma$, we denote*

$$\begin{aligned} \mathcal{B}(\mathbf{\Sigma}, \lambda, M) \;&:=\; \Sigma \cup \{p \in M \mid \exists\, t(p) > 0, \;\lim_{t \nearrow t(p)} \phi^t_{X_\lambda}(p) \in \Sigma\}, \\ \mathrm{Skel}(\mathbf{\Sigma}, \lambda, M) \;&:=\; M \setminus \mathcal{B}(M, \mathbf{\Sigma}, \lambda). \end{aligned}$$

*More generally, for any subset $X \subset \Sigma$, we define*

$$\mathcal{B}(X, \lambda, M) := X \cup \{p \in M \mid \exists\, t(p) > 0, \;\lim_{t \nearrow t(p)} \phi^t_{X_\lambda}(p) \in X\}.$$

*There is a natural continuous projection map $\pi \colon \mathcal{B}(X, \lambda, M) \to X$ that associates to $x$ the point on $X$ to which its flow line arrives.*

By (iii) of Proposition 2.5, the set $\mathcal{B}(\mathbf{\Sigma}, \lambda, M)$ is open. It is the **basin of attraction** of $\Sigma$ under the Liouville flow of $\lambda$. The closed set $\mathrm{Skel}(\mathbf{\Sigma}, \lambda, M)$

is the maximal compact subset of $M \setminus \Sigma$ that is invariant under the Liouville flow. It is called the **skeleton** of $(M, \omega)$ with respect to the polarization $\mathbf{\Sigma}$ and the Liouville form $\lambda$. Finally, point (iii) also guarantees that $\overline{\mathcal{B}(X, \lambda, M)} \cap \Sigma = \overline{X}$.

The following examples should be useful for readers unfamiliar with these notions.

**Example 2.7.** If $(\Sigma, \mu)$ is a smooth polarization of $(M, \omega)$, then the basin $\mathcal{B}(\mathbf{\Sigma}, \lambda, M)$ is symplectomorphic to the symplectic disc bundle $\mathrm{SDB}(\Sigma, M) = \mathrm{SDB}_\mu\big(\Sigma, \omega|_\Sigma, c_1(T\Sigma^\omega)\big)$. It therefore depends only very mildly on $M$ or $\lambda$ from a symplectic perspective.

**Examples 2.8.** As before, endow the complex projective plane $\mathbb{C}\mathrm{P}^2$ with the Fubini–Study symplectic form $\omega_{\mathrm{FS}}$ that integrates to $1$ over every projective line. Every smooth algebraic curve in $\mathbb{C}\mathrm{P}^2$ is symplectic and (since $H_2(\mathbb{C}\mathrm{P}^2)$ has rank $1$) is a polarization when weighted by the inverse of its algebraic degree. For instance:

- When $\Sigma$ is a line, then $(\mathbb{C}\mathrm{P}^2 \setminus \Sigma, \omega_{\mathrm{FS}})$ is symplectomorphic to the ball $(B^4(1), \omega_{\mathrm{st}})$. Using the usual toric description of $(\mathbb{C}\mathrm{P}^2, \omega_{\mathrm{FS}})$, one checks that the Liouville form $\lambda_0 = \alpha_{\mathrm{st}} = R_1\, d\theta_1 + R_2\, d\theta_2$ defined on $B^4(1) \cong \mathbb{C}\mathrm{P}^2 \setminus \Sigma$ is tame along $\Sigma$. The skeleton of $\lambda_0$ is a single point, namely the origin of $B^4(1)$.
- When $\Sigma = \Sigma_k := \{z_0^k + z_1^k + z_2^k = 0\} \subset \mathbb{C}\mathrm{P}^2$, then $\Sigma$ is the vanishing locus of a holomorphic section $s_k$ of the Hermitian line bundle $\mathcal{O}(k) \to \mathbb{C}\mathrm{P}^2$ of curvature $-k\,\omega_{\mathrm{FS}}$. The form $-\frac{1}{k} d^c \log \|s_k\|$, defined on $\mathbb{C}\mathrm{P}^2 \setminus \Sigma$, provides a tame Liouville form with residue $-\frac{1}{k}$. For $k$ even, its skeleton is the set $\Delta_k(\mathbb{C}\mathrm{P}^2)$ described in Example 1.5, and for all $k$ the intersection of the skeleton with $B^4(1) = \mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1$ is $\Delta_k$.

For these examples in $\mathbb{C}\mathrm{P}^2$, notice the discrepancy between the "algebraic degree" of a curve and its degree as a polarizartion (the "symplectic degree"). These two degrees coincide when the Fubini–Study form is normalized so that the area of a line is $1$, but they differ for different normalizations. For instance, if $\mathbb{C}\mathrm{P}^2$ is obtained as the compactification of $B^4(2)$, so that the symplectic form becomes $2\,\omega_{\mathrm{FS}}$, a curve of algebraic degree $2$ is a polarization of degree $1$.

**Examples 2.9.**

- $\big(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, a\,\omega_{\mathrm{FS}} \oplus b\,\omega_{\mathrm{FS}}\big)$ is polarized by

$$\mathbf{\Sigma_1} := \Big\{ \big(\mathbb{C}\mathrm{P}^1 \times \{0\}, b\big)\, , \big(\{0\} \times \mathbb{C}\mathrm{P}^1, a\big) \Big\}.$$

For $\lambda = (R_1 - a)\, d\theta_1 + (R_2 - b)\, d\theta_2$ the skeleton is the single point $\{(\infty, \infty)\}$.

- $\big(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, a\omega_{\mathrm{FS}} \oplus b\omega_{\mathrm{FS}}\big)$ is polarized by $\mathbf{\Sigma_2} :=$

$$\Big\{\big(\mathbb{C}\mathrm{P}^1 \times\{0\}, \tfrac{b}{2}\big), \big(\{0\} \times \mathbb{C}\mathrm{P}^1, \tfrac{a}{2}\big), \big(\mathbb{C}\mathrm{P}^1 \times\{\infty\}, \tfrac{b}{2}\big), \big(\{\infty\} \times \mathbb{C}\mathrm{P}^1, \tfrac{a}{2}\big)\Big\}.$$

For a suitable choice of $\lambda$ the skeleton is the Clifford torus, i.e. the product of the two equators.

- $\big(\mathbb{C}\mathrm{P}^2, \omega_{\mathrm{FS}}\big)$ is polarized by

$$\mathbf{\Sigma} := \Big\{\big(\{z_0 = 0\}, \tfrac{1}{3}\big), \big(\{z_1 = 0\}, \tfrac{1}{3}\big), \big(\{z_2 = 0\}, \tfrac{1}{3}\big)\Big\}.$$

For a suitable choice of $\lambda$ the skeleton is the Clifford torus.

We end this section by keeping a promise made in the introduction:

**Proposition 2.10.** *There exists no smooth polarization of* $\mathbb{C}\mathrm{P}^2$ *whose skeleton is the Clifford torus.*

*Proof.* Assume that the Clifford torus $L$ is the skeleton of a smooth polarization of $\mathbb{C}\mathrm{P}^2$ of degree $k$. By [2], the Gromov width of the complement is $c_{\mathrm{G}}(\mathbb{C}\mathrm{P}^2 \setminus L) \leq \frac{1}{k}$. On the other hand, $\mathbb{C}\mathrm{P}^2 \setminus L$ is symplectomorphic to the standard symplectic disc bundle $\mathrm{SDB}_{\frac{1}{k}}(\Sigma, \tau, k)$ over a closed surface of area $k$ with disc fibres of area $\frac{1}{k}$. By [24, Proposition 1.3] this disc bundle contains a symplectic ellipsoid $E(k, \frac{1}{k})$ and hence a ball $B^4(\frac{1}{k})$. It follows that $c_{\mathrm{G}}(\mathbb{C}\mathrm{P}^2 \setminus L) = \frac{1}{k}$. This is in contradiction to the fact that $c_{\mathrm{G}}(\mathbb{C}\mathrm{P}^2 \setminus L) = \frac{2}{3}$, proven in [4, p. 2887]. □

2.3. **Liouville polarizations (open case).** Let $(\Omega, \omega)$ be a connected symplectic manifold (without boundary) and with an exact symplectic form $\omega$. Let $\Sigma \subset \Omega$ be a symplectic divisor with normal crossings. This now means that each of the finitely many components $\Sigma_i$ is a properly embedded symplectic surface and that the singular locus $\mathrm{Sing}(\Sigma)$ is compact.

**Tameness** of a 1-form $\lambda$ on $\Omega \setminus \Sigma$ along $\Sigma$ is defined as in the closed case (Definition 2.3), in terms of Hermitian line bundles $\mathcal{L}_i \to \Sigma_i$ modeled on the symplectic normal bundles, and an identification of a neighbourhood of $\Sigma$ in $\Omega$ with the plumbing of disc bundles over the $\Sigma_i$.

**Definition 2.11.** A **Liouville polarization** $(\mathbf{\Sigma}, \lambda)$ of $(\Omega, \omega)$ consists of $\mathbf{\Sigma} = \{(\Sigma_i, \mu_i)\}$, where $\Sigma = \cup_i \Sigma_i \subset \Omega$ is a symplectic divisor as above and $\mu_i > 0$, together with a Liouville form $\lambda$ on $\Omega \setminus \Sigma$ that is tame along $\Sigma$, with residues $-\mu_i$ at $\Sigma_i$, and such that the Liouville flow $\phi^t_{X_\lambda}$ on $\Omega \setminus \Sigma$ is backward complete and "forward complete up to hitting $\Sigma$": For every point

$p \in \Omega \setminus \Sigma$, the flow line starting at $p$ is defined for $t \in (-\infty, t^+(p))$, where either $t^+(p) = +\infty$ or $\lim_{t \nearrow t^+(p)} \phi^t_{X_\lambda}(p) \in \Sigma$.

The points $p$ with $t^+(p) < \infty$ form the basin of attraction $\mathcal{B}(\mathbf{\Sigma}, \lambda, \Omega)$. The complement of the basin of attraction of $\Sigma$ under the Liouville flow is the **skeleton** of $(\Omega, \omega, \mathbf{\Sigma}, \lambda)$. This is the set of points whose Liouville flow line is defined for all times. If $\mathbf{\Sigma}$ has only one component $\Sigma$, we call the polarization **smooth**, and in analogy with the closed case say that it has **degree** $d = 1/\mu$.

**Remarks 2.12.** (i) In the context of closed manifolds, we have called polarization the sole weighted divisor $\mathbf{\Sigma}$, which could be easily separated from the Liouville form. In the context of open manifolds, the situation is more intricate because the homological condition $[\mathbf{\Sigma}] = \mathrm{PD}([\omega])$ does not make sense anymore. This condition is replaced by the residue and integrability conditions on $\lambda$ and its Liouville flow, making impossible to separate $\mathbf{\Sigma}$ from $\lambda$ in the above definition.

(ii) Whether or not an exact symplectic manifold $(\Omega, \omega)$ has a Liouville polarization does not seem to be easy to decide, in general. We think that if $\Omega$ is the interior of a compact manifold with smooth boudary, this only holds when $(\Omega, \omega)$ is the interior of a very special kind of Liouville domain $(\overline{\Omega}, d\beta)$. Note that for Liouville domains, the Liouville vector field $X_\beta$ is *transverse* to the boundary, pointing outwards. This is in sharp contrast to our Liouville vector fields $X_\lambda$, that in view of the integrability condition on $X_\lambda$ are "*tangent* at infinity".

(iii) The reader may wonder why we insist on assuming $\Omega$ open, and propose a definition that may look a bit artificial. Indeed, if $\Omega$ were assumed compact with smooth boundary, we could as well define our Liouville polarization in terms of a Liouville form on $\Omega \backslash \Sigma$ whose Liouville vector field is tangent to $\partial\Omega$. This makes a perfectly sound definition in this case, that would however discard important examples that we want to consider: open manifolds whose boundaries have corners, like the bidisc and, more generally, symplectic disc bundles over open surfaces as in Example 2.14 below, and $\mathbb{R}^4$, which is not compact.

**Definition 2.13.** A Liouville polarization $(\mathbf{\Sigma}, \lambda)$ of $(\Omega, \omega)$ is **extendable** if there exists an exact symplectic manifold $(\widehat{\Omega}, \widehat{\omega})$ with $\Omega \subset \widehat{\Omega}$ and $\widehat{\omega}|_\Omega = \omega$, a symplectic divisor $\widehat{\Sigma} \subset \widehat{\Omega}$ with $\Sigma \subset \widehat{\Sigma}$ and such that the closure $\overline{\Sigma}$ of $\Sigma$ in $\widehat{\Sigma}$ is a compact surface with boundary and $\widehat{\Sigma}$ is a collar neighbourhood of $\overline{\Sigma}$ and a tame Liouville form $\widehat{\lambda}$ on $\widehat{\Omega} \backslash \widehat{\Sigma}$ with $\widehat{\lambda}|_{\widehat{\Omega}\backslash\widehat{\Sigma}} = \lambda|_{\Omega\backslash\Sigma}$ such that the boundary $\partial\Omega$ of $\Omega$ in $\widehat{\Omega}$ is a smooth manifold to which $\widehat{\Sigma}$ is transverse (cf. Figure 2.1). For an extendable polarization $(\mathbf{\Sigma}, \lambda)$, we denote $\overline{\Sigma}$ the closure of $\Sigma$ in $\widehat{\Sigma}$. ◇

The completeness assumption on $X_\lambda$ implies that $X_{\widehat{\lambda}}$ is tangent to the boundary of $\Omega$ in $\widehat{\Omega}$.

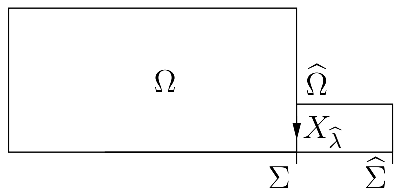


FIGURE 2.1. An extension $(\widehat{\Omega}, \widehat{\Sigma})$ of $(\Omega, \Sigma)$.

In [27] we give a large class of examples of extendable Liouville polarizations. For the moment, we look at the following model example.

**Example 2.14.** Let $(X, \tau)$ be a compact symplectic surface with smooth non-empty boundary, and let $\mu > 0$. Consider an extension $(\Sigma, \tau)$ of $(X, \tau)$ to a closed symplectic surface, whose area $\mathcal{A}_\tau(\Sigma)$ is an integral multiple $c_1$ of $\mu$. Let $\pi \colon \big(\mathrm{SDB}(\Sigma, \tau, c_1), \lambda_0\big) \to \Sigma$ be the symplectic disc bundle defined in Section 2.1, together with its Liouville form of residue $-\mu = -\frac{\mathcal{A}_\tau(\Sigma)}{c_1}$. Then $X_{\lambda_0}$ vanishes on the boundary of $\mathrm{SDB}(\Sigma, \tau, c_1)$ by (2.3) and is tangent to the fibers. Hence $\big((\mathring{X}, \mu), \lambda_0\big)$ is a Liouville polarization of

$$\mathrm{SDB}(\mathring{X}, \tau, \mu) := \pi^{-1}(\mathring{X}) \ \subset \ \mathrm{SDB}(\Sigma, \tau, c_1).$$

The basin of attraction is the whole manifold, and the skeleton is empty. Taking an open collar neighbourhood of $X \subset \Sigma$ we see that this Liouville polarization is extendable.

From the differentiable view point, $\mathrm{SDB}(\mathring{X}, \tau, \mu)$ is just $\mathring{X} \times \mathring{\mathbb{D}}$ with a twisted symplectic form, with the particularity of having a Liouville vector field tangent to the fibers and vanishing at their boundaries. ◇

The above construction does not rule out the possibility that different extensions $(\Sigma, \tau, c_1)$ of $(X, \tau)$ lead to different symplectic manifolds $\mathrm{SDB}(\mathring{X}, \tau, \mu)$. This is not so. While this fact is not crucial for us, its proof is a good warm-up for the main result of this section (Theorem 2.20), and it leads to two lemmas needed later on.

**Lemma 2.15.** *Let $(X, \tau)$ be a compact symplectic surface with boundary, and let $(\widehat{X}, \widehat{\tau})$ be an open collar neighbourhood of $X$. Let $W$ be a neighbourhood of $\widehat{X}$ in $\mathrm{SDB}(\widehat{X}, \hat{\tau}, \mu)$, and let $\vartheta$ be a bounded closed 1-form* *on $W \setminus \widehat{X}$. Let $\lambda$ be a Liouville 1-form for $\mathrm{SDB}(\widehat{X}, \hat{\tau}, \mu)$ as constructed in Example 2.14.*

(i) *There exist neighbourhoods $V, V'$ of $X$ in $W$ and a $C^1$-smooth symplectomorphism $f\colon (V,\omega) \to (V',\omega)$ such that $f|_X = \mathrm{id}$ and $f^*(\lambda+\vartheta) = \lambda$ on $V \setminus \widehat{X}$.*

(ii) *If, moreover, $\vartheta$ vanishes near a union $\partial' X$ of components of the boundary $\partial X$, then we can choose $V, V'$ and $f$ such that $f = \mathrm{id}$ on a neighbourhood of $\pi^{-1}(\partial' X) \cap V$.*

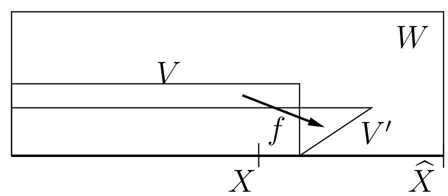


FIGURE 2.2. The symplectomorphism $f\colon V \to V'$ with $f^*(\lambda + \vartheta) = \lambda$.

*Proof.* This is proven in [24, Lemma A.1] in the case that $\widehat{X}$ is a closed surface. The same proof yields Lemma 2.15. □

**Proposition 2.16.** *The symplectic manifold $\mathrm{SDB}(\mathring{X}, \tau, \mu)$ does not depend on the choice of the extension $(\Sigma, \tau, c_1)$.*

*Proof.* For $i = 0, 1$, let $\big(\mathrm{SDB}_i(X, \tau, \mu), \lambda_i\big)$ be defined as above by the inclusion $\iota_i\colon (X, \tau) \to (\Sigma_i, \tau_i)$. We need to show that $\mathrm{SDB}_0(\mathring{X}, \tau, \mu)$ is symplectomorphic to $\mathrm{SDB}_1(\mathring{X}, \tau, \mu)$. Let $\widehat{X}_i \subset \Sigma_i$ be open collar neighbourhoods of $X \subset \Sigma_i$. We can assume that there exists a symplectomorphism $\varphi\colon (\widehat{X}_0, \tau_0) \to (\widehat{X}_1, \tau_1)$ that is the identity on $X$. By the symplectic neighbourhood theorem, there exist open neighbourhoods $U_i$ of $\widehat{X}_i$ in $\mathrm{SDB}(\widehat{X}_i, \tau_i, \mu)$ and a symplectic diffeomorphism $\psi\colon U_0 \to U_1$ that extends $\varphi$.

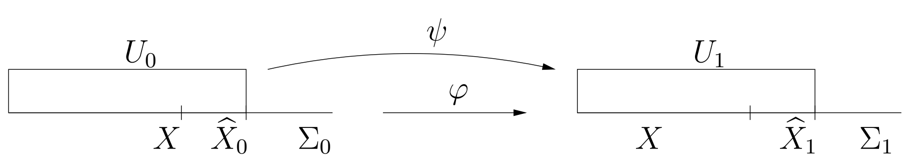


FIGURE 2.3. The extension $\psi\colon U_0 \to U_1$ of $\varphi\colon \widehat{X}_0 \to \widehat{X}_1$.

By Proposition 2.5 (ii) there exists a bounded smooth closed one-form $\vartheta$ on $\operatorname{im}\psi\backslash\widehat{X}_1$ such that $\psi_*\lambda_0 - \lambda_1 = \vartheta$. Let $\widetilde{X}_1$ be a closed collar neighbourhood of $X$ that lies in $\widehat{X}_1$. Applying Lemma 2.15 (i) with $(X, \widehat{X}, W) := (\widetilde{X}_1, \widehat{X}_1, \operatorname{im}\psi)$, we obtain open neighbourhoods $V, V'$ of $X$ in $\operatorname{im}\psi$ and a $C^1$-smooth symplectomorphism $f\colon V \to V'$ that equals the identity on $\widehat{X}_1$ and takes $\lambda_1 + \vartheta$ to $\lambda_1$. Choose $\varepsilon > 0$ so small that $\psi\big(\mathrm{SDB}_{0,\varepsilon}(X, \tau, \mu)\big) \subset \operatorname{Dom} f$. Then the map

$$\varphi := f \circ \psi\colon \mathrm{SDB}_{0,\varepsilon}(X, \tau, \mu) \hookrightarrow \mathrm{SDB}(\Sigma_1, \tau_1, c_1^1)$$

is well-defined, covers $\iota_1 \circ \iota_0^{-1}$, and verifies $\varphi_*\lambda_0 = \lambda_1$. Define

$$\begin{array}{rcccl} \Phi & : & \mathrm{SDB}_0(X, \tau, \mu) & \longrightarrow & \mathrm{SDB}_1(X, \tau, \mu) \\ & & x & \longmapsto & \phi^{-t}_{X_{\lambda_1}} \circ \varphi \circ \phi^t_{X_{\lambda_0}}(x), \end{array}$$

where for $x \in X$ we take $t = 0$ and for $x \notin X$ we take any $t \geq 0$ such that $\phi^t_{X_{\lambda_0}}(x) \in \operatorname{Dom}(\varphi) \setminus X$. Then $\Phi(x)$ is well-defined because $\varphi_*\lambda_0 = \lambda_1$, and a $C^1$-smooth symplectomorphism (see e.g. [24, Section 2.1] if needed). By restriction, we obtain a $C^1$-smooth symplectomorphism $\mathrm{SDB}_0(\mathring{X}, \tau, \mu) \to \mathrm{SDB}_1(\mathring{X}, \tau, \mu)$. The claim in the $C^\infty$-category now follows from the Smoothing Lemma 2.21 relegated to the end of this section. ☐

Almost the same proof yields the following statement.

**Lemma 2.17.** *Let $((\Sigma, \mu), \lambda)$ be a smooth and extendable Liouville polarization of $(\Omega, \omega)$. Then there exists $\varepsilon > 0$ and a $C^1$-smooth symplectic embedding*

$$\Phi\colon\ \mathrm{SDB}_\varepsilon(\Sigma, \omega|_\Sigma, \mu) \to \mathcal{B}(\Sigma, \lambda, \Omega)$$

*that is onto a neighbourhood of $\Sigma$ in $\mathcal{B}(\Sigma, \lambda, \Omega)$ and is such that $\Phi|_\Sigma = \mathrm{id}_\Sigma$ and $\Phi^*\lambda = \lambda_0$.*

*Proof.* The previous proof shows the existence of a symplectic embedding

$$\Phi\colon \mathrm{SDB}_\varepsilon(\Sigma, \omega|_\Sigma, \mu) \to \mathcal{B}(\widehat{\Sigma}, \hat{\lambda}, \widehat{\Omega}),$$

where $(\widehat{\Sigma}, \hat{\lambda}, \widehat{\Omega})$ is an extension of the Liouville polarization $(\Sigma, \lambda)$, with $\Phi|_\Sigma = \mathrm{id}$ and $\Phi^*\widehat{\lambda} = \lambda_0$. The map $\Phi$ is constructed by a dynamical conjugacy procedure that guarantees that $\operatorname{im}\Phi \subset \mathcal{B}(\Sigma, \widehat{\lambda}, \widehat{\Omega})$. Since the Liouville flow associated to $\widehat{\lambda}$ is tangent to $\partial\Omega$ in a neighbourhood of $\widehat{\Sigma} \cap \partial\Omega$, this basin of attraction lies in $\Omega$ and coincides with $\mathcal{B}(\Sigma, \lambda, \Omega)$. ☐

Putting Lemmas 2.15 and 2.17 together, we obtain the following lemma.

**Lemma 2.18.** *Let $(\boldsymbol{\Sigma}, \lambda)$ be an extendable polarization of an exact symplectic manifold $\Omega$. Let $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\lambda})$ be an extension and let $\vartheta$ be a bounded closed 1-form on a deleted neighbourhood of $\widehat{\Sigma}$ in $\widehat{\Omega} \setminus \widehat{\Sigma}$ that vanishes near $\operatorname{Sing}(\Sigma)$. Then there exists a $C^1$-smooth symplectomorphism $\Phi$ between neighbourhoods of $\overline{\Sigma}$ in $\widehat{\Omega}$ such that:*

- *$\Phi$ is the identity on $\overline{\Sigma}$,*
- *$\Phi^*(\widehat{\lambda} + \vartheta) = \widehat{\lambda}$.*

*Proof.* For each $i$ choose small open discs $\coprod_j D_i^j$ around the points in $\Sigma_i \cap \operatorname{Sing}(\Sigma)$ whose closures are disjoint from the support of $\vartheta$, and take the compact surface $X_i := \widetilde{\Sigma}_i \setminus \coprod_j D_i^j$, where $\widetilde{\Sigma}_i$ is a closed collar neighbourhood of $\overline{\Sigma}_i$ in $\widehat{\Sigma}_i$. Then $X := \coprod_i X_i$ is a disjoint union, and so is the basin $\mathcal{B}(X, \widehat{\lambda}) = \sqcup_i \mathcal{B}(X_i, \widehat{\lambda})$.

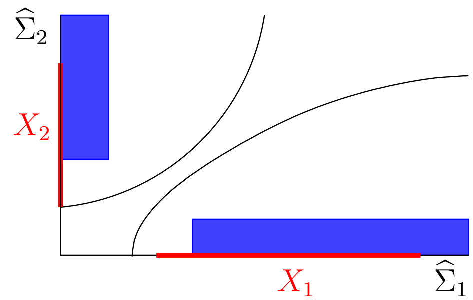


FIGURE 2.4. $X \subset \widehat{\Sigma}$ and $\operatorname{supp} \vartheta$.

By Lemma 2.17 we can identify a neighbourhood of $X_i$ in $\mathcal{B}(X_i, \widehat{\lambda})$ with $\operatorname{SDB}_{\varepsilon_i}(X_i, \widehat{\lambda})$. By Lemma 2.15 (ii) there exists a $C^1$-smooth symplectomorphism $\Phi_i$ between neighbourhoods of $X_i$ in $\widehat{\Omega}$ such that $\Phi_i^*(\widehat{\lambda}+\vartheta) = \widehat{\lambda}$ and $\Phi_i = \mathrm{id}$ on $X_i$ and near the boundaries of the discs $D_i^j$. Extend $\Phi_i$ by the identity to an open neighbourhood of all of $\widetilde{\Sigma}_i$. Then the map $\Phi$ obtained by gluing together the $\Phi_i$ is as required. □

Our main interest in the definition of Liouville polarizations is the next result, that allows to use the techniques set up to study the symplectic embedding problems in [24, 25] in the context of exact symplectic manifolds.

**Definition 2.19.** *A* **symplectic morphism** $\phi\colon \boldsymbol{\Sigma} \to \boldsymbol{\Sigma}'$ *between weighted symplectic divisors* $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}'$ *is an injective continuous map* $\phi\colon \Sigma \to \Sigma'$ *with the following two properties:*

- *For every component* $(\Sigma_i, \mu_i)$ *of* $\mathbf{\Sigma}$ *there exists a component* $(\Sigma'_{i'}, \mu_{i'})$ *of* $\mathbf{\Sigma}'$ *with* $\mu_i = \mu_{i'}$ *such that* $\phi|_{\Sigma_i} : \Sigma_i \to \Sigma'_{i'}$ *is a symplectic embedding.*
- $\phi|_{\Sigma_i} : \Sigma_i \to \Sigma'_{i'}$ *takes smooth points to smooth points, i.e.,*

$$\phi\big(\Sigma_i \setminus \bigcup_{j \neq i} \Sigma_j\big) \subset \Sigma'_{i'} \setminus \bigcup_{j' \neq i'} \Sigma'_{j'}.$$

Recall that an embedding $\varphi$ is $(\alpha, \alpha')$-exact if $\varphi^*\alpha' = \alpha + df$ for a smooth function $f$.

**Theorem 2.20.** *Let* $(\Omega, \omega = d\alpha)$ *and* $(\Omega', \omega' = d\alpha')$ *be exact symplectic manifolds with Liouville polarizations* $(\mathbf{\Sigma}, \lambda)$ *and* $(\mathbf{\Sigma}', \lambda')$*, respectively. Assume that* $(\mathbf{\Sigma}, \lambda)$ *is extendable, with extension* $(\widehat{\mathbf{\Sigma}}, \widehat{\lambda})$*. Assume further that there exists a symplectic morphism* $\phi\colon \widehat{\mathbf{\Sigma}} \to \mathbf{\Sigma}'$ *that is* $(\alpha|_\Sigma, \alpha'|_{\Sigma'})$*-exact. Then there exists an* $(\alpha, \alpha')$*-exact symplectic embedding*

$$\Phi\colon \mathcal{B}(\mathbf{\Sigma}, \lambda) = \Omega \setminus \mathrm{Skel}(\Omega, \mathbf{\Sigma}, \lambda) \to \mathcal{B}(\mathbf{\Sigma}', \lambda') = \Omega' \setminus \mathrm{Skel}(\Omega', \mathbf{\Sigma}', \lambda').$$

*Proof.* Let $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\lambda})$ be the extension of $(\Omega, \Sigma, \lambda)$. Let $\widetilde{\Sigma}$ be a closed collar neighbourhood of $\overline{\Sigma}$ in $\widehat{\Sigma}$. By the symplectic neighbourhood theorem, there exists an open neighbourhood $V$ of $\widetilde{\Sigma}$ in $\widehat{\Omega}$ and a symplectic extension $\psi\colon V \hookrightarrow \mathcal{B}(\mathbf{\Sigma}', \lambda') \subset \Omega'$ of $\phi$ that preserves the symplectic plumbing structure (i.e., the Darboux coordinates $(R_1, \theta_1, R_2, \theta_2)$ near the intersections). Since $\psi^*\lambda'$ and $\widehat{\lambda}$ are tame and have the same residues,

$$\psi^*\lambda' - \widehat{\lambda} = \vartheta$$

where $\vartheta$ is a smooth bounded closed 1-form on $V \setminus \widehat{\Sigma}$. Since $\psi$ preserves the plumbing structure, we also have $\vartheta = 0$ on $\mathrm{Op}\,(\mathrm{Sing}\,\Sigma)$.

By Lemma 2.18 there exists a $C^1$-smooth symplectomorphism $\Phi_1$ between open neighbourhoods of $\overline{\Sigma}$ in $\mathcal{B}(\widetilde{\mathbf{\Sigma}}, \hat{\lambda})$ such that $\Phi_1^*\widehat{\lambda} = \widehat{\lambda} + \vartheta$ and $\Phi_1|_{\widetilde{\Sigma}} = \mathrm{id}$. Then the map $\Phi := \psi \circ \Phi_1^{-1}$ provides a symplectic extension of $\phi$ such that $\Phi^*\lambda' = \widehat{\lambda}$. As in the proof of Proposition 2.16 define

$$\begin{array}{rccc} \Psi_1 & : & \mathcal{B}(\mathbf{\Sigma}, \lambda) & \longrightarrow & \mathcal{B}(\mathbf{\Sigma}', \lambda') \\ & & x & \longmapsto & \phi^{-t}_{X_{\lambda'}} \circ \Phi \circ \phi^t_{X_\lambda}(x), \end{array}$$

where for $x \in \Sigma$ we take $t = 0$ and for $x \notin \Sigma$ we take any $t \geq 0$ such that $\phi^t_{X_\lambda}(x) \in \mathrm{Dom}(\Phi) \setminus \Sigma$. Then $\Psi_1$ is a $C^1$-smooth symplectic embedding. It is $(\alpha, \alpha')$-exact. Indeed, since $\mathcal{B}(\mathbf{\Sigma}, \lambda)$ retracts to $\Sigma$, its first homology is generated by cycles in $\Sigma$, on which $\Psi_1 = \phi$. Since $\phi$ is $(\alpha|_\Sigma, \alpha'|_{\Sigma'})$-exact by assumption, $\Psi_1$ is $(\alpha, \alpha')$-exact. The claim in the smooth category now follows from Lemma 2.21. □

**Lemma 2.21.** *Let $\Psi_1 \colon (M, \omega) \to (M', \omega')$ be a $C^1$-smooth symplectomorphism of symplectic manifolds without boundary. Then there exists a $C^\infty$-smooth symplectomorphism $\Psi \colon (M, \omega) \to (M', \omega')$. Furthermore, if $\Psi_1$ is $(\alpha, \alpha')$-exact, then $\Psi$ can be taken $(\alpha, \alpha')$-exact.*

*Proof.* It is shown in [32, pp. 831–836] that there exists a $C^\infty$-smooth symplectomorphism $\Psi \colon (M, \omega) \to (M', \omega')$ which is $C^1$-close to $\Psi_1$. The proof shows that $\Psi$ is $(\alpha, \alpha')$-exact whenever $\Psi_1$ is $(\alpha, \alpha')$-exact: Take Darboux charts $\varphi_i \colon B^4(a_i) \to U_i \subset M$ such that $\varphi_i\big(B^4(a_i/2)\big) =: U_i'$ form a locally finite covering of $M$. Smoothing a suitable generating function, one replaces $\Psi_1$ by $\Psi_1^1$ that is $C^\infty$-smooth on $U_1'$ and agrees with $\Psi_1$ outside $U_1$. In fact, convex interpolation of the two generating functions yields a smooth symplectic isotopy $\Psi_1^t$ from $\Psi_1$ to $\Psi_1^1$ such that $\Psi_1^{-1} \circ \Psi_1^t$ is supported in $U_1$. Since $U_1 = \varphi_1(B^4(a_1))$ is simply-connected, this isotopy is Hamiltonian, and hence $\Psi_1^{-1} \circ \Psi_1^1$ is $(\alpha, \alpha)$-exact, see e.g. [21, Proposition 9.3.1]. Since $\Psi_1$ is $(\alpha, \alpha')$-exact, we find that $\Psi_1^1$ is also $(\alpha, \alpha')$-exact. In the same way, smoothen $\Psi_1^1$ to $\Psi_1^2$ on $U_2'$, ..., $\Psi_1^k$ to $\Psi_1^{k+1}$ on $U_{k+1}'$, .... The limit map $\Psi$ is then $(\alpha, \alpha')$-exact. □

## 3. Some explicit polarizations and skeleta

We start our computation of explicit polarizations with the simplest case, in dimension 2. This case is then used in § 3.2 to construct Liouville polarizations on bidiscs. In § 3.3 we show how certain singular Liouville polarizations can be surgered to smooth Liouville polarizations.

3.1. **Polarizations in dimension** $2$**.** Let $(S, \omega)$ be a 2-dimensional compact symplectic surface with non-empty boundary $\partial S$. We define a Liouville polarization of $S$ to be a finite set of points $p_i$ in $\operatorname{Int} S$ with weights $\mu_i > 0$ together with a tame Liouville form $\lambda$ defined on $S \setminus \bigcup_i p_i$ such that $X_\lambda$ is tangent to $\partial S$. Tameness now means that for each $i$ there are symplectic polar coordinates $(R, \theta)$ near $p_i$ such that $\lambda = -\mu_i d\theta + \lambda'$ on $\operatorname{Op}(p_i) \setminus \{p_i\}$, where $\lambda'$ is a bounded 1-form on $\operatorname{Op}(p_i) \setminus \{p_i\}$. One easily checks that $\sum_i \mu_i = \mathcal{A}_\omega(S)$. Conversely, any finite set of positively weighted points $\{(p_i, \mu_i)\}$ on $(S, \omega)$ that satisfies $\sum_i \mu_i = \mathcal{A}_\omega(S)$ admits a Liouville polarization, as is easy to see. Because of this very direct link between the residues and the area in dimension 2, we switch notation from $\mu_i$ to $a_i$ throughout this section. Its main purpose is to explain that in dimension 2, the skeleton can be prescribed, which seems much harder in higher dimensions. We recall the notion of regular grid in a slightly more general setting.

**Definition 3.1.** A **regular grid** $\Gamma$ on a compact surface $S$ with boundary is a finite connected graph $\Gamma \subset S$ with smooth edges, whose edges cover $\partial S$

and such that each vertex has a Darboux chart on which $\Gamma$ is a union of radial rays that cut the disc into equal sectors. When the vertex belongs to the interior of $S$, the Darboux chart takes values in $D(\varepsilon) \subset \mathbb{C}$, while it takes values in $D(\varepsilon) \cap \{\operatorname{im} z \geq 0\} \subset \mathbb{H}$ if the vertex belongs to $\partial S$.

**Proposition 3.2.** *Let $(S, \omega)$ be a compact surface with boundary, and let $\Gamma$ be a regular grid that decomposes $S$ into $m$ discs of area $a_i$, in each of which we chose a point $p_i$. Then there exists a Liouville polarization $\big(\{(p_i, a_i)\}, \lambda\big)$ of $S$ whose skeleton is $\Gamma$. This Liouville form can be required to coincide with $(R - a_i)d\theta$ in Darboux coordinates near each $p_i$.*

In the case of a closed disc $\overline{D}(A)$ (which will be the example we will use later on) one may attempt to prove the proposition as follows: Just define a first tame Liouville form with residues $-a_i$ at $p_i$ by $\lambda := \alpha_{\mathrm{st}} - \sum a_i \, d\theta_i$, where $\theta_i$ is the pull-back by the translation of the vector $-p_i$ of the standard angular coordinate in $\mathbb{C} \setminus \{0\}$. Then the periods of $\lambda$ vanish on each loop of $\Gamma$, so we can correct $\lambda$ by adding the differential of a smooth function in order to make $\lambda$ vanish on $\Gamma$. Then the Liouville flow of $\lambda$ fixes $\Gamma$ pointwise, and so $\big(\{(p_i, a_i)\}, \lambda\big)$ provides a Liouville polarization of $\overline{D}(A)$ whose skeleton contains $\Gamma$. Unfortunately, this proof does not constrain the skeleton enough: it could be strictly bigger than $\Gamma$. We therefore proceed in a different manner. Recall that $X_\lambda$ is tangent to $\ker \lambda$, since $\lambda(X_\lambda) = d\lambda(X_\lambda, X_\lambda) = 0$. Our strategy is therefore to first construct a (singular) foliation $\mathcal{F}$ to which we wish $X_\lambda$ to be tangent to. Once this "vanishing foliation" is constructed, the following lemma will readily provide the Liouville form itself.

**Lemma 3.3.** *Let $\gamma_\theta \colon [0,1] \to \mathbb{C}$, $\theta \in [0, \alpha]$, be a family of smooth rays emanating from the origin of $\mathbb{C}$, and set $U := \{\gamma_\theta(t) \mid \theta \in [0, \alpha],\, t \in [0,1]\}$. Fix $a \in \mathbb{R}$. We assume that*

- *$(t, \theta) \mapsto \gamma_\theta(t)$ is smooth and a diffeomorphism except at $t = 0$. Thus the set of curves $\{\operatorname{im} \gamma_\theta\}$ provides a foliation $\mathcal{F}$ of $U \setminus \{0\}$.*
- *There exists $\varepsilon > 0$ such that $\gamma_\theta(t) = te^{i\theta}$ for $t \leq \varepsilon$.*

*Then there exists a unique smooth Liouville form $\lambda_a$ on $(U \setminus \{0\}, \omega_{\mathrm{st}})$ such that*

- *$\ker \lambda_a = T\mathcal{F}$,*
- *$\lambda_a(\gamma_\theta(t)) = (R - a)d\theta$ for $t$ sufficiently small.*

*Proof.* This follows at once from Stokes' theorem. □

*Proof of Proposition 3.2:* We write the proof for the disc $\overline{D}(A)$. The adaptation to a general surface $(S, \omega)$ will be clear. We first set some notation. We write $D$ instead of $\overline{D}(A)$. For each point $p_i$ let $\mathcal{D}_i$ be the closure in $\mathbb{R}^2$

of the connected component of $p_i$ in $D \setminus \Gamma$ (recall that $\Gamma$ contains $\partial D$ by assumption). Write $\mathrm{Sing}(\Gamma)$ for the set of non-smooth points of $\Gamma$, and

$$\mathcal{S}_i := \mathrm{Sing}(\Gamma) \cap \mathcal{D}_i$$

for the set of singularities of $\Gamma$ in the boundary of $\mathcal{D}_i$. For each $p_i$ and a sufficiently small $\varepsilon$ the translation $(D(p_i), p_i) \to (D(\varepsilon), 0)$ is a Darboux chart that induces coordinates $(R, \theta)$ on $D(p_i)$. Similarly, by our assumption on $\Gamma$ we can consider for each $q \in \mathrm{Sing}(\Gamma) \setminus \partial D$ symplectic coordinates $\phi_q = (R, \theta)\colon D(q) \to D(\varepsilon)$ on a disk around $q$ such that

$$\Gamma \cap D(q) \;=\; \Big\{ \theta = \frac{k}{m_q},\; k \in [0, m_q - 1] \Big\},$$

where $m_q$ is the number of branches of $\Gamma$ at $q$. The sectors in $D(q)$ delimited by the rays $\{\theta = \frac{k}{m_q}\}$ are denoted by

$$S_i(q) := \mathcal{D}_i \cap D(q).$$

Also, when $q \in \mathrm{Sing}(\Gamma) \cap \partial D$, we have symplectic coordinates $(R, \theta)$ on a neighbourhood $D^+(q)$ of $q$ in $D$, with values in $\{R < \varepsilon, \theta \in [0, \pi]\}$ such that

$$\Gamma \cap D^+(q) \;=\; \Big\{ \theta = \frac{k}{2m_q},\; k \in [0, m_q - 1] \Big\},$$

where $m_q$ is the number of branches of $\Gamma$ at $q$. The sectors delimited by the rays $\theta = \frac{k}{2m_q}$ are again denoted by

$$S_i(q) := \mathcal{D}_i \cap D^+(q).$$

The radial ray that cuts the sector $S_i(q)$ into two equal sectors is called a local separatrix at $q$ and is denoted $s_i(q)$.

Finally, we fix a symplectic diffeomorphism $f_i\colon \mathcal{D}_i \setminus \mathcal{S}_i \to \overline{D}(a_i) \setminus \mathcal{P}_i$ where $\mathcal{P}_i$ is a finite set of points, and set $\lambda_i^f := f_i^* \alpha_{\mathrm{st}}$. Then $\lambda_i^f$ is positive on $\partial \mathcal{D}_i \setminus \mathcal{S}_i$. The notations are illustrated on the right part of Figure 3.1.

**Step 1: Constructing the vanishing foliation near the $p_i$ and $\mathrm{Sing}(\Gamma)$ (see Figure 3.1)**

In $D(p_i)$, we fix $\mathcal{F}$ to be the radial foliation $\{\theta = \mathrm{const}\}$.

For a point $q \in \mathrm{Sing}(\Gamma) \setminus \partial D$, let $\rho_q\colon D(\varepsilon) \to \mathbb{C}$ be the "ramified covering"

$$\rho_q(R, \theta) \;:=\; \left( \frac{2R}{m_q}, \frac{m_q}{2}\theta \right).$$

Note that $\rho_q$ is smooth and symplectic on $D(\varepsilon) \setminus \{0\}$, but only continuous at $0$. The sectors $S_i(q)$ around $q$ are the preimages under $\rho_q \circ \phi_q$ of $\{y \geq 0\}$ or $\{y \leq 0\}$, and the separatrices $s_i(q)$ are sent by $\rho_q \circ \phi_q$ to $\{x = 0\}$. We define the foliation in $D(q)$ to be the pull-back by $\rho_q \circ \phi_q$ of the vertical foliation:

$$\mathcal{F}|_{D(q)} := (\rho_q \circ \phi_q)^* \{x = c\}.$$

This local foliation has remarkable properties:

(i) The local separatrices emanating from $q$ are leaves of $\mathcal{F}$; they are the pull-back by $\rho_q \circ \phi_q$ of the curves $\{x = 0\}$.

(ii) Since $\rho_q$ sends concentric circles around the origin to concentric circles, and since $\rho_q \circ \phi_q$ is a diffeomorphism between $S_i(q) \setminus \{q\}$ and $\{y \geq 0\} \setminus \{0\}$ or $\{y \leq 0\} \setminus \{0\}$, the leaves $\{x = c\}$ with $c \neq 0$ are pulled back by $\rho_q \circ \phi_q$ to smooth curves that intersect $\Gamma$ orthogonally (in the chart $\phi_q$).

(iii) As a result, the foliation $\mathcal{F}$ can be extended to a smooth foliation on Op $(\Gamma) \setminus \mathrm{Sing}(\Gamma)$ that is transverse to the regular part of $\Gamma$.

Finally, for a point $q \in \mathrm{Sing}(\Gamma) \cap \partial D$, the model is the same. We leave it to the reader to complete the picture, cf. Figure 3.2.

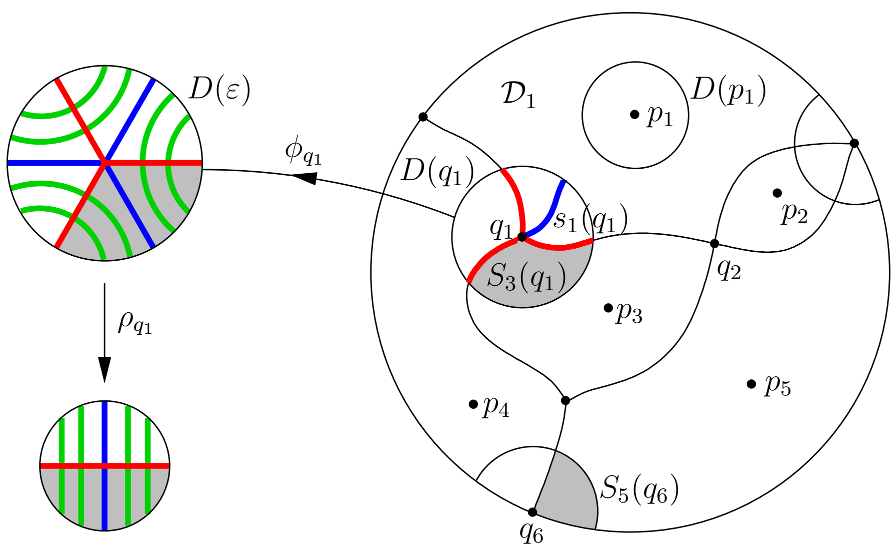


FIGURE 3.1. Notation, and the foliation on $D(\varepsilon)$ for $m_{q_1} = 3$.

**Step 2: Interpolating the separatrices (see Figure 3.2)**

Fix $i \in \{1, \ldots, m\}$. (In Figure 3.2, $i = 1$.) Fix a reference point $q_0 \in \mathcal{S}_i$, denote $q_1, \ldots, q_\ell$ the points of $\mathcal{S}_i$ enumerated by going around $\partial\mathcal{D}_i$ in the anti-clockwise sense from $q_0$, and let $\partial\mathcal{D}_i(q_j \to q_{j+1})$ be the smooth arc in $\Gamma$ joining $q_j$ to $q_{j+1}$ on $\partial\mathcal{D}_i$. Now interpolate between the local separatrix

$s_i(q_0)$ and the ray $\{\theta = 0\}$ in $D(p_i)$ by a smoothly embedded curve. Set

$$\beta_j \;:=\; \frac{1}{a_i} \int_{\partial \mathcal{D}_i(q_j \to q_{j+1})} \lambda_i^f, \qquad j = 0, \dots, |\mathcal{S}_i| - 1,$$

and

$$\theta_j \;:=\; \sum_{k=0}^{j-1} \beta_k.$$

Interpolate now inductively between the local separatrix $s_i(q_j)$ and the ray $\{\theta = \theta_j\}$ by disjoint smoothly embedded curves (for different $j$) in $\mathcal{D}_i$ such that the area enclosed by the three arcs $\partial \mathcal{D}_i(q_j \to q_{j+1})$, $s(p_i \to q_j)$, and $s(p_i \to q_{j+1})$ is $\beta_j\, a_i$. Here, $s(p_i \to q_j)$, $j = 1, \dots, \ell$, denote the global separatrices thus constructed. These interpolations can be found because the total area enclosed by $\partial \mathcal{D}_i$ is $a_i$. Interpolations $s(p_1 \to q_1)$ and $s(p_1 \to q_2)$ are drawn in pink in Figure 3.2. We declare the $s(p_i \to q_j)$ to be leaves of $\mathcal{F}$ in $\mathcal{D}_i$.

**Step 3: Interpolating between the local foliations in $\mathcal{D}_i$**

The foliation $\mathcal{F}_i$ on $\mathcal{D}_i$ is then taken to be any (singular) foliation in $\mathcal{D}_i$ that verifies the following properties:

1. The points $p_i$ and $q_j \in \mathcal{S}_i$ are the only singularities of $\mathcal{F}|_{\mathcal{D}_i}$, so $\mathcal{F}|_{\mathcal{D}_i \setminus \{p_i\}}$ is smooth.
2. The curves $s(p_i \to q_j)$ are leaves.
3. On the $D(p_i)$, $D(q_j)$, $D^+(q_k)$, and Op $(\Gamma) \setminus \operatorname{Sing}(\Gamma)$ the leaves of $\mathcal{F}$ are the ones described in Step 1.
4. All leaves join $p_i$ to $\partial \mathcal{D}_i$ and are transverse to $\partial \mathcal{D}_i \setminus \mathcal{S}_i$.
5. For $c \in (\beta_j, \beta_{j+1})$ the leaf extending the ray $\{\theta = c\} \subset D(p_i)$ joins $p_i$ to the unique point $q \in \partial \mathcal{D}_i(q_j \to q_{j+1})$ such that

$$\int_{\partial \mathcal{D}_i(q_j \to q_{j+1})} \lambda_i^f \;=\; c - \beta_j.$$

Figure 3.2 shows in black seven curves that belong to leaves of $\mathcal{F}_1$ from $p_1$ to $\partial \mathcal{D}_1(q_1 \to q_2)$. By (3) and (4) we can successively construct foliations $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ such that all non-singular leaves are $C^\infty$-smooth.

**Step 4: Joining the foliations $\mathcal{F}_i$**

Since $\mathcal{F}_i$ is a smooth foliation on $\mathcal{D}_i \setminus \big(\{p_i\} \cup \mathcal{S}_i\big)$ and by the smooth fitting of the leaves across the $\partial \mathcal{D}_i(q_j \to q_{j+1})$, the foliation $\mathcal{F}$ on $D \setminus \big(\cup\{p_i\} \bigcup \operatorname{Sing}(\Gamma)\big)$ defined by

$$\mathcal{F}|_{\mathcal{D}_i \setminus (\{p_i\} \cup \mathcal{S}_i)} \;:=\; \mathcal{F}_i$$

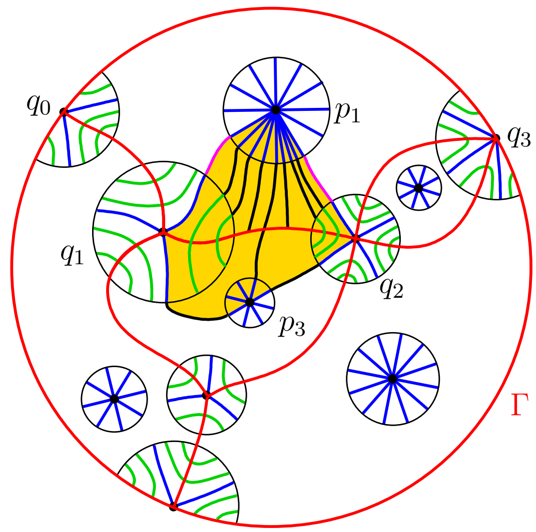


FIGURE 3.2. The foliation $\mathcal{F}$ on the $D(p_i)$, $D(q_j)$, and $D^+(q_k)$ (blue and green), part of the foliation $\mathcal{F}_1$, and $\mathcal{D}(p_1, p_3)$ (yellow).

is $C^\infty$-smooth, and has a unique extension to a singular foliation on the whole disc $D$ whose singularities are $\cup\{p_i\} \bigcup \mathrm{Sing}(\Gamma)$. The different separatrices $s(p_i \to q)$, $q \in \mathcal{S}_i$, are smooth leaves. The leaves of $\mathcal{F}$ that pass through a point in $\Gamma \setminus \mathrm{Sing}(\Gamma)$ that separates between $\mathcal{D}_i$ and $\mathcal{D}_j$ join $p_i$ to $p_j$. Finally, the leaves through a sequence of regular points on $\mathcal{D}_i \cap \mathcal{D}_j$ that converge to a singular point $q \in \mathrm{Sing}(\Gamma)$ break into the two separatrices $s(p_i \to q)$ and $s(p_j \to q)$.

We denote by $\mathcal{D}(p_i, p_j)$ the closure of the union of leaves joining $p_i$ to $p_j$. Figure 3.2 shows $\mathcal{D}(p_1, p_3)$ in yellow.

**Step 5: Fixing the Liouville form itself**

By applying Lemma 3.3 to the curves of the foliation $\mathcal{F}$ that emanate either from $(p_i, a_i)$ or from $(p_j, a_j)$, we obtain on the interior $\mathring{\mathcal{D}}(p_i, p_j)$ two Liouville forms $\lambda_i, \lambda_j$. We claim that they coincide. To see this, we first check that these forms vanish on $\Gamma$. Indeed, let $q \in \Gamma \cap \mathring{\mathcal{D}}(p_i, p_j)$. Since $\lambda_i$ vanishes along $\mathcal{F}$, we already have $\lambda_i(q)v = 0$ for $v \in T_q\gamma$, where $\gamma$ is the leaf through $q$. On the other hand, parametrize the leaves through $q \in \Gamma \cap \mathring{\mathcal{D}}(p_i, p_j)$ by curves $\gamma(\theta, t)$, where $\theta$ represents the angle of the leaf at $p_i$ and the parameter $t$ verifies $\gamma_\theta(1) \in \Gamma$, and put $\frac{\partial}{\partial\theta}(q) := \frac{\partial\gamma}{\partial\theta}(\theta(q), 1) \in T_q\Gamma$. Since $\lambda_i$ vanishes along $\mathcal{F}$, and by Stokes' theorem and the second property of $\lambda_i$ from Lemma 3.3,

$$\lambda_i(q)\,\frac{\partial}{\partial\theta}(q) \;=\; \frac{d\mathcal{A}_i}{d\theta}(\theta(q)) - a_i, \tag{3.1}$$

where $\mathcal{A}_i(\theta)$ is the area of the disc bounded by $\Gamma$, $\gamma(\theta,[0,1])$, and $\gamma(\theta(q),[0,1])$. By the choice of our foliation (point 5 in Step 3), the right hand side of (3.1) vanishes, so $\lambda_i(q)$ vanishes. Now, since $\lambda_i$ and $\lambda_j$ coincide on $\mathcal{F}$ and on $\Gamma$, Stokes' theorem shows that they coincide on $\mathring{\mathcal{D}}(p_i,p_j)$. We therefore define

$$\lambda|_{\mathring{\mathcal{D}}(p_i,p_j)} \;:=\; \lambda_i = \lambda_j.$$

Since $\lambda_i$ is defined and smooth on $\mathring{\mathcal{D}}_i \setminus \{p_i\}$, and not only on $\mathring{\mathcal{D}}(p_i,p_j)$, we can glue the $\lambda_i$ to the smooth Liouville form $\lambda$ on $D \setminus \operatorname{Sing}(\Gamma)$. Notice that for $\gamma(\theta,t)$ as defined above, we have by the same argument

$$\lambda(\gamma(\theta,t))\,\frac{\partial}{\partial\theta} < 0 \qquad \forall\, t<1.$$

Hence on $\mathring{\mathcal{D}}_i \setminus \{p_i\}$ we have $d\lambda(X_\lambda, \frac{\partial}{\partial\theta}) = \lambda(\frac{\partial}{\partial\theta}) < 0$, which shows that the coordinate $t$ is (negative) gradient-like for the Liouville flow $X_\lambda$ of $\lambda$, and so $\mathcal{B}(p_i,\lambda) \supset \mathcal{D}_i$. On the other hand, $\lambda$ vanishes on $\Gamma$, so $\Gamma$ is invariant under the Liouville flow of $\lambda$. Summarizing, we have therefore proved that

- $\lambda$ is a smooth Liouville form on $D \setminus \big(\operatorname{Sing}\Gamma \cup \{p_1,\dots,p_m\}\big)$,
- $\lambda$ is tame at each $p_i$ with residue $-a_i$, in fact $\lambda = (R-a_i)d\theta$ near $p_i$,
- $X_\lambda$ vanishes on $\Gamma \setminus \operatorname{Sing}\Gamma$,
- every trajectory starting in $D \setminus \Gamma$ converges to some $p_i$.

**Step 6: Smoothening $\lambda$ at $\operatorname{Sing}(\Gamma)$**

We only treat the case of $q \in \operatorname{Sing}(\Gamma) \setminus \partial D$, the other case being similar. We recall that a neighbourhood $D(q)$ of $q$, that is identified with $D(\varepsilon)$, covers a neighbourhood of $0 \in \mathbb{C}$ via the map $\rho_q(R,\theta) = (\frac{2R}{m_q}, \frac{m_q}{2}\theta)$, and that in these coordinates, $\mathcal{F}$ is the foliation $\rho_q^*\{x = \text{const}\}$. This foliation is thus also tangent to the kernel of the local Liouville form $\lambda_q := \rho_q^*(-\frac{1}{\pi}\, y\, dx)$. Away from $q$, this form $\lambda_q$ also vanishes on $\rho_q^*\{y=0\} = \Gamma \cap D(q)$. Away from $q$, both forms $\lambda$ and $\lambda_q$ are thus smooth and vanish along $\mathcal{F}$ and on $\Gamma$. Stokes' theorem therefore guarantees that they coincide on $D(q)\setminus\{q\}$. None of the two forms smoothly extends to $q$, however, since at $(x,y) \neq$

$(0,0)$,

$$\begin{aligned}\lambda_q &= \rho_q^*(-\tfrac{1}{\pi}y\,dx)\\ &= \rho_q^*\big(\tfrac{1}{2\pi}(x\,dy-y\,dx)-\tfrac{1}{2\pi}(x\,dy+y\,dx)\big)\\ &= \rho_q^*\big(R\,d\theta-\tfrac{1}{2\pi}d(xy)\big)\\ &= \rho_q^*\big(R\,d\theta-\tfrac{1}{2\pi}d(R\sin(2\pi\theta)\cos(2\pi\theta))\big)\\ &= R\,d\theta-\tfrac{1}{4\pi}\rho_q^*d(R\sin(4\pi\theta))\\ &= R\,d\theta-\tfrac{1}{2\pi m_q}d(R\sin(2\pi m_q\theta))\end{aligned}$$

and the function $R\sin(2\pi m_q\theta)$ is not smooth at the origin for $m_q \geq 2$. We thus alter $\lambda_q$ in $D(q)$ so as to make it smooth at $q$.

The function $r^{m_q}\sin(2\pi m_q\theta)$ is smooth on $\mathbb{R}^2$, since it is the imaginary part of $z \mapsto z^{m_q}$. Let $\chi\colon [0,2\varepsilon] \to \mathbb{R}_{\geq 0}$ be a function that coincides with $r^{m_q}$ near $0$ and with $R$ on $[\varepsilon, 2\varepsilon]$, is positive and smooth except at $0$, and meets $\chi(R) < R$ on $(0,\varepsilon)$. Define the smooth 1-form $\lambda'$ on $D$ by $\lambda' = \lambda$ on $D \setminus \cup_{q\in \operatorname{Sing}\Gamma} D(q)$ and

$$\lambda' = R\,d\theta - \frac{1}{2\pi m_q}d\big(\chi(R)\sin(2\pi m_q\theta)\big) \text{ on } D(q).$$

Then $\lambda'$ is a smooth Liouville form on $D \setminus \{p_1,\dots,p_m\}$, tame at $p_i$ with residue $-a_i$. We claim that the skeleton of $\lambda'$ is $\Gamma$, as required. To see this, notice that $X_{\lambda'} = X_\lambda$ on $D \setminus \cup_{q\in\operatorname{Sing}\Gamma} D(q)$, and

$$X_{\lambda'} = (R-\chi(R)\cos(2\pi m_q\theta))\frac{\partial}{\partial R} + \frac{1}{2\pi m_q}\chi'(R)\sin(2\pi m_q\theta)\frac{\partial}{\partial\theta}$$

on $D(q)$. We first look at the $\frac{\partial}{\partial\theta}$-component of $X_{\lambda'}$. Since

$$\Gamma = \{\sin(\pi m_q\theta)=0\} \subset \{\sin(2\pi m_q\theta)=0\},$$

$X_{\lambda'}$ is radial on $\Gamma \cap D(q)$ (hence its flow preserves $\Gamma$), and the trajectories of the points in $D(q)\setminus\Gamma$ flow away from $\Gamma$. From the $\frac{\partial}{\partial R}$-component, and from $\Gamma \supset \{\cos(2\pi m_q\theta)=1\}$ and $\chi(R) < R$ on $(0,\varepsilon)$, we infer that the trajectories of the points in $D(q)\setminus\Gamma$ all leave $D(q)$, and hence reach the set $\{\lambda=\lambda'\}\setminus\Gamma$ (since $\Gamma$ is invariant). It follows that $\operatorname{Skel}(\lambda') = \Gamma$. □

The following easy result will be needed in the next subsection.

**Lemma 3.4.** *Let $(S,\omega)$ be a compact symplectic surface with or without boundary. Let $\big(\mathcal{P} := \{(p_i,a_i),\ i=1,\dots,n\},\lambda\big)$ with $p_i \in S\setminus\partial S$ be a polarization of $S$ (where $\lambda$ is a tame Liouville form with residues $a_i$ at $p_i$ and with flow tangent to $\partial S$), and set $\Gamma :=$ Skel$(S,\mathcal{P},\lambda)$.*

*For $k \leq n$ and any collection $\{(a_i^1,a_i^2), i=1,\dots,k\}$ of pairs of positive real numbers with $a_i = a_i^1 + a_i^2$ there exist:*

- *a collection of open embedded discs $D_i$ around $p_i$, in any prescribed neighbourhood of $p_i$*
- *for $i \leq k$: points $p_i^1, p_i^2 \in D_i$*
- *and a tame Liouville form $\lambda'$ with residues $a_i^1$ at $p_i^1$ and $a_i^2$ at $p_i^2$ for $i \leq k$ and $a_i$ at $p_i$ for $i \in \{k+1, \dots, n\}$*

*such that*

$$\mathcal{B}(D_i, \lambda') \;:=\; \left\{x \in S \mid \exists\, t > 0 \text{ such that } \phi_{\lambda'}^t(x) \in D_i\right\} \;=\; \mathcal{B}(p_i, \lambda).$$

*Proof.* Since $\lambda$ is tame at $p_i$, its dual vector field points towards $p_i$ on a neighbourhood of $p_i$, so there exist arbitrarily small disjoint open discs $D_i \subset S \setminus (\partial S \cup \Gamma)$ around $p_i$ with

$$\mathcal{B}(D_i, \lambda) = \mathcal{B}(p_i, \lambda).$$

For $i \leq k$ take any pair of points $p_i^j \in D_i$, $j = 1, 2$, and a Hamiltonian diffeomorphisms $\phi_i^j$ with compact support in $D_i$ such that $\phi_i^j(p_i^j) = p_i$. Define

$$\lambda'(x) \;:=\; \begin{cases} \lambda(x) & \text{if } x \in S \setminus \bigcup_{i=1}^k D_i, \\ \frac{a_i^1}{a_i}\, \phi_i^{1*}\lambda(x) + \frac{a_i^2}{a_i}\, \phi_i^{2*}\lambda(x) & \text{if } x \in D_i \text{ for some } i \in \{1, \dots, k\}. \end{cases}$$

Then $\lambda'$ is a Liouville form tame at $p_i^j$ with residue $a_i^j$. Since it coincides with $\lambda$ on $\Gamma \cup \partial S$, its flow preserves both $\Gamma$ and $\partial S$, so $\left(\mathcal{P}' := \left\{(p_i^j, a_i^j)\right\}, \lambda'\right)$ is a Liouville polarization of $S$. Since $\lambda'$ coincides with $\lambda$ outside the discs $D_i$,

$$\mathcal{B}(D_i, \lambda') = \mathcal{B}(D_i, \lambda) = \mathcal{B}(p_i, \lambda).$$

The lemma follows. □

3.2. **Bidiscs.** As before we denote by $\overline{D}(A)$ the closed disc of area $A$, and by $D(A)$ its interior.

**Lemma 3.5.** *Let $\Gamma_1 \subset D(A)$ and $\Gamma_2 \subset D(B)$ be two regular grids containing $\partial D(A), \partial D(B)$ whose complements are a union of discs of area $a_i, b_j$ on which we choose points $p_i, q_j$, respectively. Then*

$$\mathbf{\Sigma} := \left\{\big(p_i \times D(B), a_i\big), \big(D(A) \times q_j, b_j\big)\right\}$$

*is an extendable Liouville polarization of $D(A) \times D(B)$, and there is a tame Liouville form $\lambda$ whose skeleton is $\Gamma_1 \times \Gamma_2$.*

*Proof.* By Proposition 3.2 there exist Liouville forms $\lambda_1, \lambda_2$ on $D(A), D(B)$ that are tame along the $p_i, q_j$, with residues $a_i, b_j$ and skeleton $\Gamma_1, \Gamma_2$, respectively. Then the restriction of the form $\lambda := \pi_1^*\lambda_1 \times \pi_2^*\lambda_2$ to $\mathbf{\Sigma}$ is a Liouville form tame along $\mathbf{\Sigma}$ with the correct residues. Since the associated

Liouville flow is simply the product of the flows in each factor, it clearly preserves

$$\partial\big(D(A)\times D(B)\big) \;=\; \big(\partial D(A)\big)\times D(B)\cup D(A)\times\big(\partial D(B)\big).$$

Moreover, for $x\notin\Gamma_1\times\Gamma_2$, we have $\pi_1(x)\in D(A)\setminus\Gamma_1$ or $\pi_2(x)\in D(B)\setminus\Gamma_2$, so the trajectory of $\pi_1(x)$ or $\pi_2(x)$ under the Liouville flow of $\lambda_1$ or $\lambda_2$ is forward attracted by one of the $p_i$ or $q_j$, which shows that the complement of $\Gamma_1\times\Gamma_2$ in $D(A)\times D(B)$ is forward attracted by $\mathbf{\Sigma}$, whence Skel$(\mathbf{\Sigma},\lambda)\subset\Gamma_1\times\Gamma_2$. Finally, for a point in $\Gamma_1\times\Gamma_2$, both components remain in $\Gamma_1,\Gamma_2$, so $\Gamma_1\times\Gamma_2$ is invariant under the Liouville flow and Skel$(\mathbf{\Sigma},\lambda)\supset\Gamma_1\times\Gamma_2$. The extendability is obvious: take $\widehat\Omega := D(A+\varepsilon)\times D(B+\varepsilon)$ and $\widehat\Sigma := \{p_i\times D(B+\varepsilon), D(A+\varepsilon)\times q_j\}$. □

Combining Lemma 3.4 and Lemma 3.5 we obtain:

**Lemma 3.6.** *Let $\big(\{(p_i,a_i)\}_{i=1,\dots,m},\lambda_A\big)$ and $\big(\{(q_j,b_j)\}_{j=1,\dots,n},\lambda_B\big)$ be Liouville polarizations of $D(A)$ and $D(B)$ (so $\sum a_i = A$ and $\sum b_j = B$) with skeleton $\Gamma_A$ and $\Gamma_B$, respectively. Let $m'\le m$, $n'\le n$, and assume that for $i\le m'$ and $j\le n'$ we are given decompositions*

$$a_i = a_i^1 + a_i^2 \quad \textit{and} \quad b_j = b_j^1 + b_j^2, \qquad \textit{with}\ \ a_i^\ell, b_j^\ell > 0.$$

*Then there exist:*

- *open discs $D_i^A$, $D_j^B$ for $i\le m$, $j\le n$ around $p_i$, $q_j$, which lie in any prescribed neighbourhood of $p_i$, $q_j$*
- *for $i\le m'$, $j\le n'$: points $p_i^1,p_i^2\in D_i^A$ and $q_j^1,q_j^2\in D_j^B$*
- *and a tame Liouville form $\lambda'$ on the complement of*

$$\begin{aligned}\mathbf{\Sigma}' \;:=\; \Big\{&\big(p_i^\ell\times D(B), a_i^\ell\big)_{i\le m'}^{\ell=1,2},\ \big(D(A)\times q_j^\ell, b_j^\ell\big)_{j\le n'}^{\ell=1,2},\\ &\big(p_i\times D(B), a_i\big)_{i>m'},\ \big(D(A)\times q_j, b_j\big)_{j>n'}\Big\}\end{aligned}$$

*that makes $(\mathbf{\Sigma}',\lambda')$ a Liouville polarization of $D(A)\times D(B)$*

*such that*

$$\mathcal{B}\Big(\cup_i D_i^A\times D(B)\bigcup\cup_j D(A)\times D_j^B,\lambda'\Big) \;=\; \big(D(A)\times D(B)\big)\setminus(\Gamma_A\times\Gamma_B).$$

*Proof.* Take small open discs $D_i^A$ around $p_i$, for $i\le m'$ take points $p_i^1,p_i^2\in D_i^A$, and let $\lambda_A'$ be the tame Liouville form on

$$D(A)\setminus\Big(\{p_i^\ell\}_{i\le m'}^{\ell=1,2}\cup\{p_i\}_{i>m'}\Big)$$

provided by Lemma 3.4, with residues $a_i^\ell$ at $p_i^\ell$ for $i\le m'$ and $a_i$ at $p_i$ for $i>m'$, and such that

$$\mathcal{B}\big(\cup_i D_i^A,\lambda_A'\big) \;=\; D(A)\backslash\Gamma_A.$$

Construct similarly $\lambda_B$ on

$$D(B) \setminus \left( \{q_j^\ell\}_{i \le n'}^{\ell=1,2} \cup \{q_j\}_{j>n'} \right).$$

As we have seen in the proof of Lemma 3.5, the 1-form $\lambda' := \lambda'_A \oplus \lambda'_B$ is tame along $\mathbf{\Sigma}'$, and

$$\mathcal{B}\Big( \cup_i D_i^A \times D(B) \bigcup \cup_j D(A) \times D_j^B, \lambda' \Big) \;=\; \big( D(A) \times D(B) \big) \setminus (\Gamma_A \times \Gamma_B).$$

The lemma is proved. □

3.3. **Surgery on Liouville polarizations.** The aim of this section is Proposition 3.8 that allows to glue certain components of a Liouville polarization, to obtain a "smoother" Liouville polarization. We will need the following rather obvious statement.

**Lemma 3.7.** *Let $\Sigma \subset B^4(1)$ be a smooth symplectic curve that near the boundary of $B^4(1)$ agrees with a collection of complex lines (1-dimensional complex vector spaces). Let $\mu \in \mathbb{R}$. Then there exists a Liouville form $\lambda$ on $B^4(1) \backslash \Sigma$, tame along $\Sigma$, with residue $\mu$. (We do not claim that $(B^4(1), \Sigma, \lambda)$ is a Liouville polarization.)*

*Proof.* Since $\Sigma$ coincides with a union of complex lines near $\partial B^4(1)$, compactifying $B^4(1)$ into $\mathbb{C}\mathrm{P}^2(1)$ provides a smooth symplectic curve $\Sigma'$ in $\mathbb{C}\mathrm{P}^2$. This is a polarization, so by Proposition 2.5 (i) there exists a Liouville form $\lambda'$ on $\mathbb{C}\mathrm{P}^2 \backslash \Sigma'$ that is tame along $\Sigma'$. Restricting $\lambda'$ to $B^4(1) \subset \mathbb{C}\mathrm{P}^2(1)$ provides a Liouville form $\lambda$ on $B^4(1) \backslash \Sigma$ tame along $\Sigma$. It has a well-defined residue which is a constant $\nu$ because $\Sigma$ is assumed to be smooth. Then the form

$$\lambda_{\mathrm{st}} + \frac{\mu}{\nu}(\lambda - \lambda_{\mathrm{st}})$$

is a Liouville form on $B^4(1) \backslash \Sigma$, tame along $\Sigma$ because it coincides with a multiple of $\lambda$ up to a smooth form, with residue $\mu$. □

**Proposition 3.8.** *Let $(M, \omega, \mathbf{\Sigma}, \lambda)$ be a polarized symplectic manifold, either closed or Liouville polarized. Assume that the weights $\mu_i, \mu_j$ of two components $\Sigma_i, \Sigma_j$ of $\mathbf{\Sigma}$ coincide. Then for any neighbourhood $U$ of $\Sigma_i \cap \Sigma_j$ there exist:*

- *A smooth symplectic curve $\Sigma_{ij}$ that coincides with $\Sigma_i \cup \Sigma_j$ in the complement of $U$. We then define*

$$\mathbf{\Sigma}' := \{ (\Sigma_k, \mu'_k = \mu_k)_{k \ne i,j}, (\Sigma_{ij}, \mu'_{ij} = \mu_i = \mu_j) \}.$$

- *A Liouville form $\lambda'$ on $M\backslash\Sigma'$ tame along $\Sigma'$ with residues $\mu'_k$ that coincides with $\lambda$ on the complement of $U$ and is such that $(M,\omega,\mathbf{\Sigma}',\lambda')$ is again a (Liouville) polarized symplectic manifold.*

*Proof.* We first consider a model in $\mathbb{R}^4$. Let $\Sigma_1 = \{z_2 = 0\}$ and $\Sigma_2 = \{z_1 = 0\}$ be the complex coordinate lines in $\mathbb{C}^2$. For $\varepsilon > 0$ let $\chi\colon [0,2\varepsilon] \to [0,1]$ be a smooth function that is 1 near 0 and vanishes on $[\varepsilon, 2\varepsilon]$. For $\delta > 0$ define the smooth cylinder

$$Z_\delta := \{z_1 z_2 = \delta\chi(R)\}.$$

Note that $Z_\delta = \Sigma_1 \cup \Sigma_2$ outside $B^4(\varepsilon)$, see Figure 3.3. As is well known, $Z_\delta$ is symplectic for $\delta = \delta(\varepsilon)$ small enough.

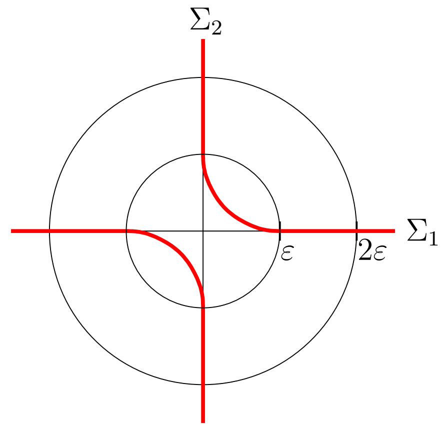


FIGURE 3.3. From $\Sigma_1 \cup \Sigma_2$ to $Z_\delta$, schematically.

Consider the Liouville form

$$\lambda_0 \;=\; \lambda_{\mathrm{st}} - \mu(d\theta_1 + d\theta_2)$$

on $\mathbb{R}^4 \setminus (\Sigma_1 \cup \Sigma_2)$, tame along $\Sigma_1 \cup \Sigma_2$ with residue $-\mu$. Since $Z_\delta = \Sigma_1 \cup \Sigma_2$ on $B^4(2\varepsilon) \setminus B^4(\varepsilon)$, Lemma 3.7 provides a tame Liouville form $\lambda_\delta$ on $B^4(2\varepsilon) \setminus Z_\delta$ with residue $-\mu$. Consider the shell

$$V \;=\; \{R \in (\varepsilon, 2\varepsilon)\} \;=\; B^4(2\varepsilon) \setminus \overline{B}^4(\varepsilon).$$

Then $\lambda_0$ and $\lambda_\delta$ are Liouville forms on $V \setminus Z_\delta$, tame along $Z_\delta$, with equal residue $-\mu$. By Proposition 2.5 (ii) there exist a smooth bounded closed 1-form $\vartheta$ on $V\backslash Z_\delta$ such that

$$\lambda_\delta = \lambda_0 + \vartheta.$$

Since $H_1(V\backslash Z_\delta)$ is generated by two small loops around $Z_\delta$ on which $\vartheta$ integrates to 0, $\vartheta$ is the derivative of a smooth function $f\colon V\backslash Z_\delta \to \mathbb{R}$. Since $\vartheta$ is bounded, $f$ extends to a Lipschitz function on $V$. Let now

$\rho\colon [\varepsilon, 2\varepsilon] \to [0,1]$ be a smooth function that equals 1 near $\varepsilon$ and vanishes near $2\varepsilon$, and define the smooth Liouville form $\lambda'$ on $B^4(2\varepsilon) \setminus Z_\delta$ by

$$\begin{cases} \lambda'|_{\{R\leq\varepsilon\}} & := & \lambda_\delta \\ \lambda'|_{\{\varepsilon\leq R\leq 2\varepsilon\}} & := & \lambda_0 + d(\rho(R)f). \end{cases}$$

Since $\rho(R)f$ is a Lipschitz function, smooth on $V \setminus Z_\delta$, $\lambda'$ is tame on $V \setminus Z_\delta$ with residue $-\mu$, and the same holds on $\{R \leq \varepsilon\}$. Moreover, $\lambda' = \lambda_0$ near the boundary of $B^4(2\varepsilon)$. This finishes the construction of the interpolating model $\big(Z_\delta \cap B^4(2\varepsilon), \lambda'\big)$.

Now take two components $\Sigma_i, \Sigma_j$ of $(\boldsymbol{\Sigma}, \lambda)$ with equal weights $\mu_i = \mu_j$ as in the proposition. Since $\Sigma$ has normal crossings and $\lambda$ is tame along $\Sigma$ by assumption, there exists for each $p \in \Sigma_i \cap \Sigma_j$ a Darboux chart $\phi_p\colon B^4(3\varepsilon_p) \to (M,\omega)$ centered at $p$ such that

$$\phi_p^{-1}(\Sigma) = \phi_p^{-1}(\Sigma_i \cup \Sigma_j) = \{z_1 z_2 = 0\},$$
$$\phi_p^*\lambda = \lambda_{\mathrm{st}} - (\mu_i d\theta_1 + \mu_j d\theta_2) = \lambda_{\mathrm{st}} - \mu_i(d\theta_1 + d\theta_2).$$

Now choose the $\varepsilon_p$ so small that the Darboux balls $\phi_p(B^4(3\varepsilon_p))$ are disjoint, and apply to each ball $B^4(2\varepsilon_p)$ the above model interpolation. When transplanted to $M$, this yields a Liouville form $\lambda'$ on $M \setminus \Sigma'$, tame along $\Sigma'$, with correct residues. Also notice that in the open case the Liouville flow of $\lambda'$ is backward complete, because $\lambda' = \lambda$ outside $\bigcup_p \phi_p(B^4(2\varepsilon))$. □

## 4. SYMPLECTIC EMBEDDINGS

In this section we prove the symplectic embedding results Theorems 1, 2, 3, and 6. The proofs are all based on the same principle, stated in Theorem 2.20, which reduces the proofs to finding convenient polarizations. We first prove Theorem 6. While Theorem 1 is a direct corollary, Theorem 2 deals with an unbounded domain, whence the proof needs some adjustments. We finally prove Theorem 3, using explicit smooth polarizations of the ball.

### 4.1. **The main embedding result (Theorem 6).**

We first prove a special "monotone" case of Theorem 6, and then reduce the general case to this monotone case.

#### 4.1.1. *The monotone case.*

**Proposition 4.1.** *Let $\Gamma_1$ and $\Gamma_2$ be two regular grids in $D(A)$ and $D(B)$, respectively, that cut $D(A)$ into $m$ topological discs of equal area $a$ and $D(B)$ into $n$ topological discs of equal area $b$. Then there exists an $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$-exact symplectic embedding*

$$\big(D(A) \times D(B)\big) \setminus (\Gamma_1 \times \Gamma_2) \;\to\; Z^4(a+b).$$

*Proof.* Let $\mathcal{P} = \{p_1, \ldots, p_m\}$, $\mathcal{Q} = \{q_1, \ldots, q_n\}$,

$$\mathbf{\Sigma} := \left\{ \big(\mathcal{P} \times D(B), a\big), \big(D(A) \times \mathcal{Q}, b\big) \right\},$$

and let $\lambda$ be the Liouville form on $\big(D(A)\times D(B)\big)\setminus\Sigma$ provided by Lemma 3.5 so that $(\mathbf{\Sigma}, \lambda)$ is an extendable Liouville polarization of $D(A) \times D(B)$ with skeleton $\Gamma_1 \times \Gamma_2$.

We now construct a Liouville polarization of $\Omega' := D(a + b) \times D(Kb)$ for $K$ large enough, to be fixed later in the proof. We refer to Figure 4.1 for an illustration of the construction. Take a regular grid (a smooth line) in $D(a + b)$ that divides $D(a + b)$ into topological discs of area $a$ and $b$, and take a point $p'_1$ in the disc of area $a$ and $p'_2$ in the disc of area $b$. For a large integer $K$ (to be chosen later), take a regular grid in $D(Kb)$ that divides $D(Kb)$ into $K$ topological discs of area $b$, and choose a point $q'_j$ in each disc. Lemma 3.5 yields a Liouville polarization

$$\Big(\Big\{ \big(D(a+b) \times \{q'_j\}, b\big), \big(\{p'_1\} \times D(Kb), a\big), \big(\{p'_2\} \times D(Kb), b\big) \Big\}, \lambda' \Big)$$

of $\Omega' = D(a + b) \times D(Kb)$, consisting of $K + 2$ discs.

Define now $\Sigma'_1 := \big\{\{p'_1\} \times D(Kb)\big\}$ and let $\Sigma'_2$ be the symplectic curve obtained by resolving all the intersection points $(p'_2, q'_j)$ of the $K + 1$ remaining discs. We can assume that the smoothening takes place in balls around these points that are disjoint and do not intersect $\partial\Omega'$. Since $\lambda'$ has residue $b$ along each component used to define $\Sigma'_2$, Proposition 3.8 shows that it can be modified in this union of balls to a Liouville form $\lambda''$, tame along $\Sigma'_1 \cup \Sigma'_2$, with residue $a$ along $\Sigma'_1$ and $b$ along $\Sigma'_2$. Since $\lambda''$ coincides with $\lambda'$ near $\partial\Omega'$, its Liouville flow is also tangent to $\partial\Omega'$. In other terms,

$$\big(\mathbf{\Sigma}' := \{(\Sigma'_1, a), (\Sigma'_2, b)\}, \lambda''\big)$$

is a Liouville polarization of $\Omega'$. Now $\Sigma'_1$ has area $Kb$, $\Sigma'_2$ has area $K(a + 2b)$, and $\#(\Sigma'_1 \cap \Sigma'_2) = K$. For $K$ large enough we therefore find $m$ disjoint topological discs $D_1^i$ of area strictly larger than $B$ in $\Sigma'_1$ and $n$ disjoint topological discs $D_2^j$ of area strictly larger than $A$ in $\Sigma'_2$ such that each $D_1^i$ intersects each $D_2^j$ exactly once. Hence there exists a symplectic morphism $\phi\colon \widehat{\mathbf{\Sigma}} \to \mathbf{\Sigma}'$. Since the first cohomology of a disc vanishes, $\phi$ is $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$-exact. Theorem 2.20 now guarantees an exact symplectic embedding of $\big(D(A) \times D(B)\big) \setminus (\Gamma_1 \times \Gamma_2)$ into $D(a + b) \times D(Kb) \subset Z^4(a + b)$. □

4.1.2. *Reduction to the monotone setting.*

**Proposition 4.2.** *Let $\Gamma_1 \subset D(A)$ and $\Gamma_2 \subset D(B)$ be regular grids whose complements are a union of $m$ topological discs of area $a_i \leq a$ and of $n$ topological discs of area $b_j \leq b$, respectively.*

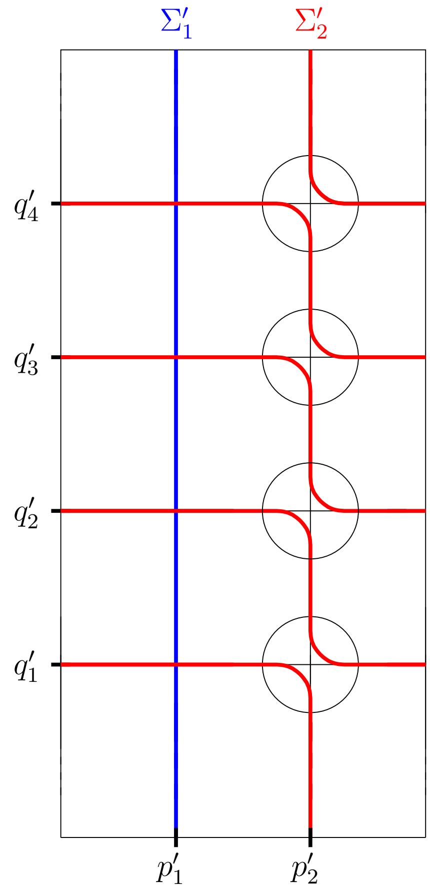


FIGURE 4.1. $\Sigma_1'$ and $\Sigma_2'$ in $\Omega' = D(a+b) \times D(Kb)$ for $K = 4$.

*Then there exists a regular grid* $\Gamma_1' \subset D(ma)$ *that cuts* $D(ma)$ *into* $m$ *topological discs of area* $a$ *and a regular grid* $\Gamma_2' \subset D(nb)$ *that cuts* $D(nb)$ *into* $n$ *topological discs of area* $b$*, and an* $(\alpha_{\rm st}, \alpha_{\rm st})$*-exact symplectic embedding*

$$\big(D(A) \times D(B)\big) \setminus \big(\Gamma_1 \times \Gamma_2\big) \;\to\; \big(D(ma) \times D(nb)\big) \setminus \big(\Gamma_1' \times \Gamma_2'\big).$$

*Proof.* If $a_i = a$ and $b_j = b$ for all $i, j$, then there is nothing to prove. So after switching the factors and re-indexing we can assume that $a_i < a$ for $1 \leq i \leq m' \leq m$ and $a_i = a$ for $i \geq m'+1$. Define $n' \leq n$ for the $b_i$ in the same way, where however $n' = 0$ if $b_i = b$ for all $i$.

Write $\mathcal{P} := \{p_1, \dots, p_m\} \subset D(A)$, $\mathcal{Q} := \{q_1, \dots, q_n\} \subset D(B)$, and let

$$\big(\mathbf{\Sigma} := \big\{(p_i \times D(B), a_i), (D(A) \times q_j, b_j)\big\}, \lambda\big)$$

be the extendable Liouville polarization of $D(A) \times D(B)$ with skeleton $\Gamma_1 \times \Gamma_2$ provided by Lemma 3.5. We recall that in this case, an extension is obtained by taking somewhat larger discs, so $\overline{\Sigma}$ is just the closure of $\Sigma$ in $\mathbb{C}^2$, i.e. the closed discs. Similarly, let $\mathcal{P}' := \{p'_1, \ldots, p'_m\} \subset D(ma)$, $\mathcal{Q}' := \{q'_1, \ldots, q'_n\} \subset D(nb)$, and let

$$\big(\mathbf{\Sigma}' := \big\{(\mathcal{P}' \times D(nb), a), (D(ma) \times \mathcal{Q}', b)\big\}, \lambda'\big)$$

be the Liouville polarization of $D(ma) \times D(nb)$ with skeleton $\Gamma'_1 \times \Gamma'_2$. Since we are free to choose $\Gamma'_1$ and $\Gamma'_2$, and since $D(A) \subset D(ma)$ and $D(B) \subset D(nb)$, we may as well assume that $p'_i = p_i$ and $q'_j = q_j$. This is not an important assumption but will simplify the proof at some point. We will nevertheless keep the primes in order to distinguish the source and the target. Lemma 3.6 provides disjoint open discs $D_i^A$ in $D(ma) \setminus \Gamma_1$, disjoint open discs $D_j^B$ in $D(nb) \setminus \Gamma_2$, points $p_i'^1, p_i'^2 \in D_i^A$ for $i \le m'$, $q_j'^1, q_j'^2 \in D_j^B$ for $j \le n'$, and a Liouville polarization $(\mathbf{\Sigma}'', \lambda'')$ of $D(ma) \times D(nb)$ with

$$\begin{aligned}\mathbf{\Sigma}'' := \Big\{ & \big(p_i'^1 \times D(nb), a_i\big)_{i \le m'}, \big(p_i'^2 \times D(nb), a - a_i\big)_{i \le m'}, \\ & \big(D(ma) \times q_j'^1, b_j\big)_{j \le n'}, \big(D(ma) \times q_j'^2, b - b_j\big)_{j \le n'}, \\ & \big(p'_i \times D(nb), a\big)_{i \ge m'+1}, \big(D(ma) \times q'_j, b\big)_{j \ge n'+1}\Big\}\end{aligned}$$

and such that

$$\begin{aligned}\mathcal{B}\Big(\cup_i D_i^A \times D(nb) \bigcup \cup_j D(ma) \times D_j^B, \lambda''\Big) \\ \subset \big(D(ma) \times D(nb)\big) \setminus (\Gamma'_1 \times \Gamma'_2).\end{aligned} \tag{4.1}$$

As is clear from the proof of Lemma 3.6 we can chose $p_i'^1 = p'_i$ and $q_i'^1 = q'_i$. We can find $\varepsilon > 0$ and two area preserving embeddings

$$\begin{aligned}\sigma &\colon \overline{D(B)} \to \overline{D(nb)} \\ \tau &\colon D(A+\varepsilon) \to D(ma)\end{aligned}$$

such that $\sigma(q_j) = q'_j = q_j$ for all $j \le n$ and the image $\operatorname{im}\sigma$ avoids all the $q_j'^2$, and such that $\tau(p_i) = p'_i$ for all $i \le m$ and $\operatorname{im}\tau$ avoids all the $p_i'^2$. This is obvious for $\tau$ because $A < ma$ by assumption. It is also true for $\sigma$ because if $B < nb$ we can even extend $\sigma$ to $\hat{\sigma}\colon D(B+\varepsilon) \to D(nb)$ with the required properties, and if $B = nb$ then $b_j = b$ for all $j$, so there is no $q_j'^2$ to avoid, and $\sigma$ can be taken to be the identity. The two maps $\sigma$ and $\tau$ induce an embedding $\phi\colon \Sigma \to \Sigma''$ defined by

$$\begin{aligned}\phi|_{p_i \times D(B)} &= p'_i \times \sigma \\ \phi|_{D(A) \times q_j} &= \tau \times q'_j.\end{aligned}$$

There are now two cases.

<u>Case $B < nb$:</u> As already noticed, in this case $\sigma$ can be extended to an area preserving embedding $\hat{\sigma}\colon D(B+\varepsilon) \to D(nb)$, so $\phi$ can be extended to a

smooth area preserving embedding $\widehat{\Sigma} \to \Sigma''$ which is a symplectic morphism between our polarizations. It is $(\alpha_{\rm st}, \alpha_{\rm st})$-exact because all components of $\Sigma$ are discs. Theorem 2.20 now implies that

$$(4.2) \qquad \big(D(A) \times D(B)\big) \setminus (\Gamma_1 \times \Gamma_2) = \mathcal{B}(\mathbf{\Sigma}, \lambda) \xhookrightarrow{\alpha_{\rm st}} \mathcal{B}(\mathbf{\Sigma}'', \lambda'').$$

Moreover, since

$$\mathbf{\Sigma}'' \subset \cup_i D_i^A \times D(nb) \bigcup D(ma) \times \cup_j D_j^B,$$

we obtain together with (4.1) the inclusion

$$(4.3) \qquad \begin{aligned} \mathcal{B}(\mathbf{\Sigma}'', \lambda'') \;&\subset\; \mathcal{B}\Big(\cup_i D_i^A \times D(nb) \bigcup D(ma) \times \cup_j D_j^B\Big) \\ &\subset\; \big(D(ma) \times D(nb)\big) \setminus (\Gamma_1' \times \Gamma_2'). \end{aligned}$$

Composing (4.1) with (4.3) we obtain the asserted exact symplectic embedding.

Case $B = nb$: In this case, $\sigma = \text{id}$ does not extend to a larger disc with image still in $D(nb)$, so formally the above argument does not apply. We solve this difficulty by revisiting the proof of Theorem 2.20 in this particular case: Although $\phi$ does not extend to an extended polarization $\widehat{\Sigma}$ as before, the product map $\psi := \tau \times \text{id}$ extends $\phi$ to a small neighbourhood of $\Sigma$ in $D(A) \times D(b)$, still with image in $D(ma) \times D(nb)$. The forms $\lambda$ on $D(A) \times D(B)$ and $\lambda''$ on $D(ma) \times D(nb)$ are both split:

$$\lambda = \lambda_1 \oplus \lambda_2, \qquad \lambda'' = \lambda_1'' \oplus \lambda_2'',$$

where $\lambda_1, \lambda_2$ are defined on $D(A)\backslash\{p_i\}$, $D(B)\backslash\{q_j\}$ and $\lambda_1'', \lambda_2''$ are defined on $D(ma)\backslash\{p_i', p_i'^2\}$, $D(mb)\backslash\{q_j'\}$. Since $B = mb$ and $q_j = q_j'$ we can take $\lambda_2 = \lambda_2''$, and since $\lambda_1$ and $\lambda_1''$ have the same residues at $p_i$ and $p_i' = p_i$, we can make them coincide on a small neighbourhood of the $p_i$. Since $\tau$ can clearly be taken to be the identity on a neighbourhood of the points $p_i$, the map $\psi = \tau \times \text{id}$ (defined on a neighbourhood of $\Sigma$) then pulls back $\lambda''$ to $\lambda$. Thus the basic conjugacy procedure described in the proof of Theorem 2.20 to embed $\mathcal{B}(\mathbf{\Sigma}, \lambda)$ into $\mathcal{B}(\mathbf{\Sigma}'', \lambda'')$ applies, and now the argument in the previous case applies. □

4.1.3. *Proof of Theorem 6.* Let $\Gamma_1 \subset D(A)$ and $\Gamma_2 \subset D(B)$ be two regular grids that cut the discs into topological discs of area $\leq a$ and $\leq b$, respectively. By Proposition 4.2,

$$\big(D(A) \times D(B)\big) \setminus (\Gamma_1 \times \Gamma_2) \xhookrightarrow{\alpha_{\rm st}} \big(D(ma) \times D(nb)\big) \setminus (\Gamma_a \times \Gamma_b)$$

where $\Gamma_a$ and $\Gamma_b$ cut $D(ma)$ and $D(nb)$ into topological discs of area $a$ and $b$, respectively. By Proposition 4.1, we also have

$$\big(D(ma) \times D(nb)\big) \setminus (\Gamma_a \times \Gamma_b) \xhookrightarrow{\alpha_{\rm st}} Z^4(a+b).$$

Composing these two maps we obtain the searched exact symplectic embedding

$$f\colon \big(D(A) \times D(B)\big) \setminus (\Gamma_1 \times \Gamma_2) \xhookrightarrow{\alpha_{\rm st}} Z^4(a+b).$$

□

4.2. **Proof of Theorem 2.** Let $\Gamma := \Gamma_1 \times \Gamma_1 \subset \mathbb{R}^4$, where $\Gamma_1 \subset \mathbb{R}^2$ is the regular grid

$$\Gamma_1 := \bigcup_{(n,m)\in\mathbb{Z}^2} \{n\} \times \mathbb{R} \cup \mathbb{R} \times \{m\}.$$

Let also

$$\Sigma := \bigcup_{(n,m)\in\mathbb{Z}^2} \big\{(n+\tfrac{1}{2}, m+\tfrac{1}{2})\big\} \times \mathbb{C} \cup \mathbb{C} \times \big\{(n+\tfrac{1}{2}, m+\tfrac{1}{2})\big\}.$$

**Lemma 4.3.** *There exists a Liouville form $\lambda$ on $\mathbb{R}^4\setminus\Sigma$ with the following properties:*

- *$\lambda$ has residue $1$ along $\Sigma$.*
- *For each $p \in \mathbb{C}^2\setminus\Sigma$, the Liouville trajectory $\Phi^t_{X_\lambda}(p)$ is defined for $t \in\, ]-\infty, t^+(p)[$, where $t^+(p) \in\, ]0,+\infty]$. If $t^+(p) < +\infty$, then $\lim_{t\to t^+(p)} \Phi^t_{X_\lambda}(p)$ exists and belong to $\Sigma$.*
- *$\Gamma$ is the skeleton of $(\Sigma, \lambda)$:*
$$\Gamma = \{p \mid t^+(p) = +\infty\} = \mathbb{R}^4\setminus\mathcal{B}(\Sigma,\lambda).$$
- *$X_\lambda$ is tangent to the hyperplanes $\{x_1 \in \mathbb{Z}\}$, $\{x_2 \in \mathbb{Z}\}$, $\{y_1 \in \mathbb{Z}\}$, $\{y_2 \in \mathbb{Z}\}$.*

*Proof.* Again, the form $\lambda$ is a product:

$$\lambda_{(z,w)}(u,v) = \lambda_z(u) + \lambda_w(v),$$

where $\lambda_z = \lambda_w$ is a Liouville form on $\mathbb{R}^2\setminus\big(\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})\big)$, with $\Gamma_1$ as skeleton. We construct this Liouville form exactly as in Section 3.1, where compactness plays no role. Alternatively, we can do the construction in the proof of Proposition 3.2 on $[-2,2]^2$ in a $\mathbb{Z}^2$-periodic way. We then obtain a smooth Liouville form on $[0,1]^2$ that extends to a smooth $\mathbb{Z}^2$-periodic Liouville form on $\mathbb{R}^2\setminus\big(\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})\big)$ with skeleton $\Gamma_1$. □

From Lemma 4.3 we obtain the following two facts:

- For $N \in \mathbb{N}$ define
$$\Sigma_N := \Sigma \cap ]-N,N[^4 \quad \text{and} \quad \Gamma_N := \Gamma \times ]-N,N[^4.$$
Then $\big((\Sigma_N,1),\lambda\big)$ is an extendable Liouville polarization of $]-N,N[^4$ whose skeleton is $\Gamma_N$. Notice that $\Sigma_N \subset \Sigma_{N+1}$.
- Define
$$Z(2) := ]-1,1[\times]0,1[\times\mathbb{R}^2 \quad \text{and} \quad \Sigma_Z := \Sigma \cap Z(2).$$
Then $\big((\Sigma_Z,1),\lambda\big)$ is a Liouville polarization of $Z(2)$.

Now $\Sigma_Z$ is made of two planes $\{(\pm\frac{1}{2},\frac{1}{2})\} \times \mathbb{R}^2$ and infinitely many discs
$$D_{n,m} := ]-1,1[\times]0,1[\times\big\{(n+\tfrac{1}{2},m+\tfrac{1}{2})\big\}, \quad n,m \in \mathbb{Z}.$$
As in the proof of Proposition 4.1, we keep $\Sigma_1' := \big\{(-\frac{1}{2},\frac{1}{2})\big\} \times \mathbb{R}^2$ and glue all other components together to form a symplectic curve $\Sigma_2'$, which is diffeomorphic to a plane with infinitely many punctures arranged on a lattice. Each puncture corresponds to a boundary component of action $2$ (see Figure 4.2). Since $\lambda$ has residue $1$ along each component of $\Sigma_Z$, Proposition 3.8 (applied with $U$ the union of small balls around the singular points of $\Sigma_2'$) yields a Liouville form $\lambda'$ on $Z(2)\backslash(\Sigma_1' \cup \Sigma_2')$ tame along $\Sigma' := \Sigma_1' \cup \Sigma_2'$ with residue $1$ and equal to $\lambda$ near $\partial Z(2)$. In other terms,
$$\big(\{(\Sigma_1',1),(\Sigma_2',1)\},\lambda'\big)$$
is a Liouville polarization of $Z(2)$.

**Lemma 4.4.** *For each $N$ there exists a symplectic morphism $\phi_N \colon \widehat{\Sigma}_N \to \Sigma'$ such that*
$$\phi_{N+1}|_{\widehat{\Sigma}_N} = \phi_N.$$

*Proof.* For describing the construction of $\phi_N$ it is convenient to redescribe the components $\Sigma_1'$ and $\Sigma_2'$ of $\Sigma'$:

- $\Sigma_1' = \{(-\frac{1}{2},\frac{1}{2})\} \times \mathbb{R}^2$ is symplectic with standard area form on the $\mathbb{R}^2$-factor. Its intersections with $\Sigma_2'$ are the points
$$I_{n,m} := (-\tfrac{1}{2},\tfrac{1}{2},n+\tfrac{1}{2},m+\tfrac{1}{2}), \quad (n,m) \in \mathbb{Z}^2.$$
- $\Sigma_2'$ is obtained by gluing $(\frac{1}{2},\frac{1}{2}) \times \mathbb{R}^2$ with the topological discs $D_{n,m}$, $(n,m) \in \mathbb{Z}^2$. This can be done by replacing each round disc $(\frac{1}{2},\frac{1}{2}) \times D\big((n+\frac{1}{2},m+\frac{1}{2}),\varepsilon\big)$ by its gluing with the disc $D_{n,m}$. These gluings can be parameterized by $\mathbb{Z}^2$-translations of a punctured disc $D(0,\varepsilon)\backslash\{(0,\frac{\varepsilon}{2}\}$ with an area form $\omega_\varepsilon$ whose total area is $2+\varepsilon$, and that is standard near the boundary $S^1(\varepsilon)$. Each point $I_{n,m}$ is again parameterized by $(n+\frac{1}{2},m+\frac{1}{2})$.

We also define $\Sigma_1'(N) := \{(-\frac{1}{2},\frac{1}{2})\} \times [-N,N]^2$ and $\Sigma_2'(N) \subset \Sigma_2'$ the part of $\Sigma_2'$ parameterized by $[-N,N]^2 \setminus (\mathbb{Z}^2 + (\frac{1}{2},\frac{1}{2}+\frac{\varepsilon}{2}))$, and notice that they correspond exactly to the intersections of these curves with $Z(2) \cap [-N,N]^4$ (see Figure 4.2).

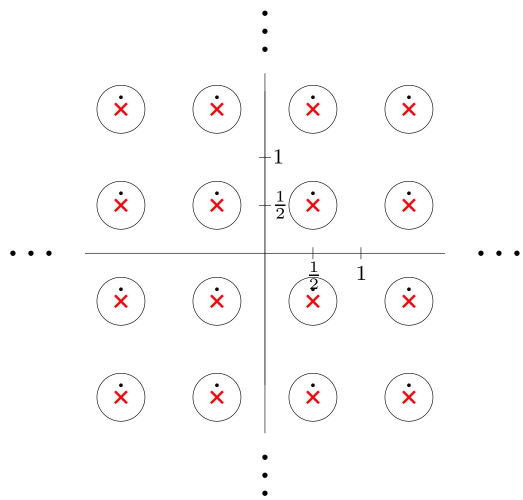


FIGURE 4.2. The parametrization of the curve $\Sigma_2'$. Each depicted disc $D(\varepsilon)$ has area $\varepsilon$ on the picture, but carries the symplectic form $\omega_\varepsilon$ of area $2+\varepsilon$. The black points are the punctures, and the red crosses are the intersection points with $\Sigma_1'$.

Now divide $\Sigma_N$ into its $4N^2$ horizontal components

$$\Sigma_N^H(n,m) := ]-N,N[^2 \times \{(n+\tfrac{1}{2}, m+\tfrac{1}{2})\}, \quad (n,m) \in [-N,N-1]^2$$

and its $4N^2$ vertical components

$$\Sigma_N^V(n,m) := \{(n+\tfrac{1}{2}, m+\tfrac{1}{2})\} \times ]-N,N[^2, \quad (n,m) \in [-N,N-1]^2.$$

Each such component is a disc of area $4N^2$, the horizontal and vertical components do not intersect pairwise, but each horizontal component intersects each vertical component exactly once, thus each component passes through $4N^2$ singular points of $\Sigma_N$.

We construct $\phi_N$ by sending a slightly larger disc around the closure of each $\Sigma_N^V(n,m)$ into $\Sigma_1'(2N^2+1)$, and by sending a slightly larger disc around the closure of each $\Sigma_N^H(n,m)$ into $\Sigma_2'(2N^2+1)$. The areas of these surfaces are

$$\mathcal{A}(\Sigma_1'(2N^2+1)) = (2(2N^2+1))^2 > 16N^4 = \mathcal{A}(\Sigma_N^V)$$
$$\mathcal{A}(\Sigma_2'(2N^2+1)) = (2(2N^2+1))^2 + 2(2N^2+1)^2 > 16N^4 = \mathcal{A}(\Sigma_N^H),$$

where $\Sigma_N^H := \coprod \Sigma_N^H(n,m)$ and $\Sigma_N^V := \coprod \Sigma_N^V(n,m)$, and

$$\#\big(\Sigma_1'(2N^2+1)\cap\Sigma_2'(2N^2+1)\big) = (2(2N^2+1))^2 > 16N^4 = \#(\Sigma_N^V\cap\Sigma_N^H).$$

We can therefore define $\phi_N$ by its restrictions to $\Sigma_1^V$ and $\Sigma_1^H$ as illustrated in Figures 4.3 and 4.4, where $\phi_N$ is explained in blue and $\phi_{N+1}$ in yellow for $N=1$. These are area preserving embeddings that take the points at the red crosses to points at the red crosses. This defines symplectic morphisms $\phi_N \colon \widehat{\Sigma}_N \to \Sigma'$, and we can clearly choose $\phi_{N+1}$ such that $\phi_{N+1}|_{\widehat{\Sigma}_N} = \phi_N$. □

*Proof of Theorem 2.* We can now proceed as in the proof of Theorem 2.20. First we extend each $\phi_N$ from Lemma 4.4 to a symplectic embedding

$$\psi_N \colon V_N \to \operatorname{Op}\big(\Sigma', Z(2)\big)$$

of an open neighbourhood $V_N$ of $\widehat{\Sigma}_N$. Set $U_N := V_N \cap [-N,N]^4$. Since $\phi_{N+1}|_{\widehat{\Sigma}_N} = \phi_N$, Moser's method shows that the maps $\psi_N$ can be chosen such that

$$U_{N+1} \supset U_N \quad \text{and} \quad \psi_{N+1}|_{U_N} = \psi_N|_{U_N}.$$

We can therefore define the $C^\infty$-smooth symplectic embedding

$$\psi\colon \big(U,\Sigma\big) \to \big(Z(2),\Sigma'\big)$$

on $U := \bigcup U_N$ by $\psi|_{U_N} := \psi_N|_{U_N}$.

In the construction of the maps $\phi_N$ of Lemma 4.4 we can assume that their images avoid a whole neighbourhood of the punctures. After choosing $U$ smaller, if necessary, the closure of $\psi(U)$ is then disjoint from the boundary of $Z(2)$. Using Lemma 2.18 as in the proof of Theorem 2.20 we correct $\psi$ to a $C^1$-smooth symplectic embedding

$$\Phi\colon \big(\widehat{U},\Sigma\big) \to \big(Z(2),\Sigma'\big)$$

defined on a neighourhood $\widehat{U}$ of $\Sigma$ such that $\Phi(\widehat{U}) \subset \psi(U)$, whence the closure of $\Phi(\widehat{U})$ is also disjoint from $\partial Z(2)$. Choosing $\widehat{U}$ smaller if necessary, we can also assume that $\widehat{U}$ retracts onto $\Sigma$, whose components are topological discs. We therefore find a Liouville form $\lambda''$ on $Z(2)$ that coincides with $\Phi_*\lambda$ on $\Phi(\widehat{U})$ and with $\lambda'$ near $\partial Z(2)$. Then the map

$$\begin{array}{rccl} \Psi_1 & : & \mathcal{B}(\Sigma,\lambda) & \longrightarrow & Z(2) \\ & & p & \longmapsto & \phi_{X_{\lambda''}}^{-t^+(p)+\varepsilon(p)} \circ \Phi \circ \phi_{X_\lambda}^{t^+(p)-\varepsilon(p)}, \end{array}$$

defined for any small enough positive function $\varepsilon(p)$, is a $C^1$-smooth symplectic embedding.

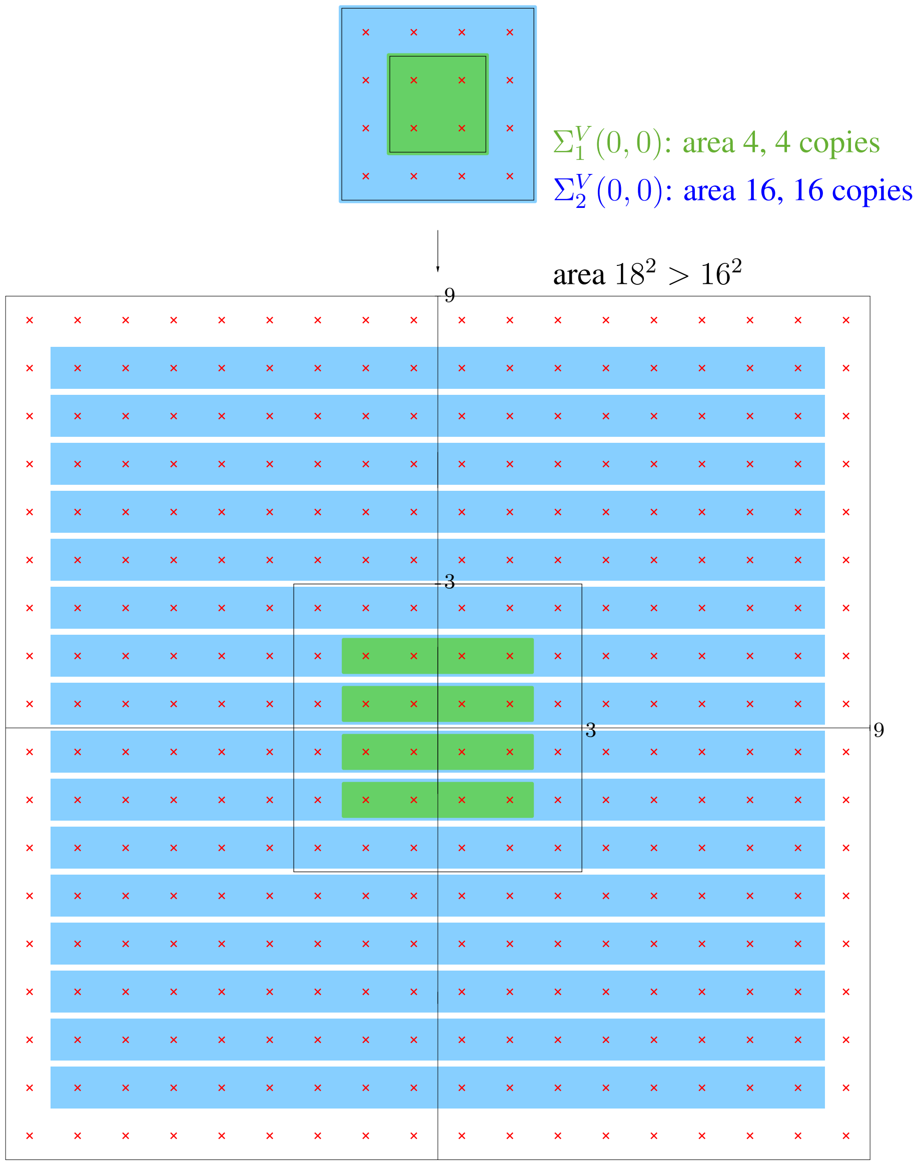


FIGURE 4.3. The restriction of $\phi_1$ to $\Sigma_1^V$ and of $\phi_2$ to $\Sigma_2^V$. Each band has height slightly less than 1, so that we can construct a larger band of height less than 1 in the next step $N+1$.

Since the components of $\Sigma$ are discs, the $\phi_N$ are $(\alpha_{\rm st}|_{\Sigma_N}, \alpha_{\rm st}|_{\Sigma'})$-exact, and so $\Psi_1$ is $(\alpha_{\rm st}, \alpha_{\rm st})$-exact. The asserted $(\alpha_{\rm st}, \alpha_{\rm st})$-exact symplectic embedding $\Psi\colon \mathcal{B}(\Sigma, \lambda) = \mathbb{R}^4 \setminus \Gamma \to Z(2)$ is now obtained from Lemma 2.21.

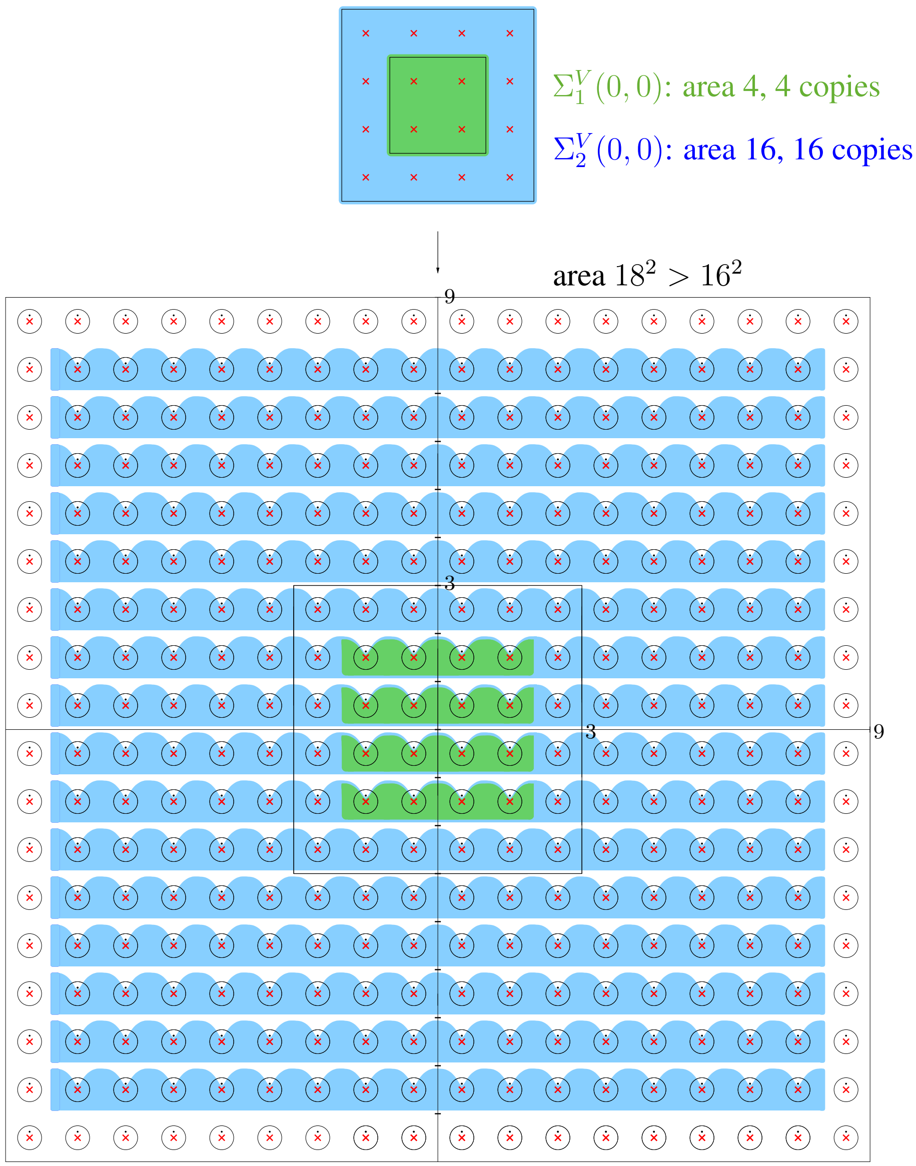


FIGURE 4.4. The restriction of $\phi_1$ to $\Sigma_1^H$ and of $\phi_2$ to $\Sigma_2^H$. These are almost the same embeddings as in the previous figure, except that they avoid the punctures. Note that the area $2+\varepsilon$ of the discs is not even used since for $\varepsilon > 0$ small enough the area of $\Sigma_2'(2N^2+1)$ without the discs is already larger than the area of $\Sigma_N^H$.

$\square$

4.3. **Proof of Theorem 3.** In this subsection we first give another proof of Theorem 6 for the ball as domain, that does not use a singular polarization but Biran's smooth polarization. We then use this result to prove Theorem 3. We start with a lemma that is useful if the polarization of the source is smooth.

**Lemma 4.5.** *Let* $(\Sigma, d\alpha)$ *and* $(\Sigma', d\alpha')$ *be compact connected symplectic surfaces of genus* $g$ *and* $g'$*, respectively. Let* $b$ *be the number of boundary components of* $\Sigma$*. Assume that*

$$\operatorname{area}\Sigma < \operatorname{area}\Sigma' \quad \text{and} \quad g + b - 1 \leq g'.$$

*Then there exists an* $(\alpha, \alpha')$*-exact symplectic embedding* $\varphi\colon \Sigma \to \mathring{\Sigma}'$.

*Proof.* A basis of $H_1(\Sigma, \mathbb{Z})$ is given by oriented closed curves $\gamma_1, \dots, \gamma_{2g}$ around the $g$ holes and $b-1$ oriented boundary components $\gamma_{2g+1}, \dots, \gamma_{2g+b-1}$, see the red curves in Figure 4.5. Since $g + b - 1 \leq g'$, there exists an embedding $f\colon \Sigma \to \mathring{\Sigma}'$ as shown in Figure 4.5.

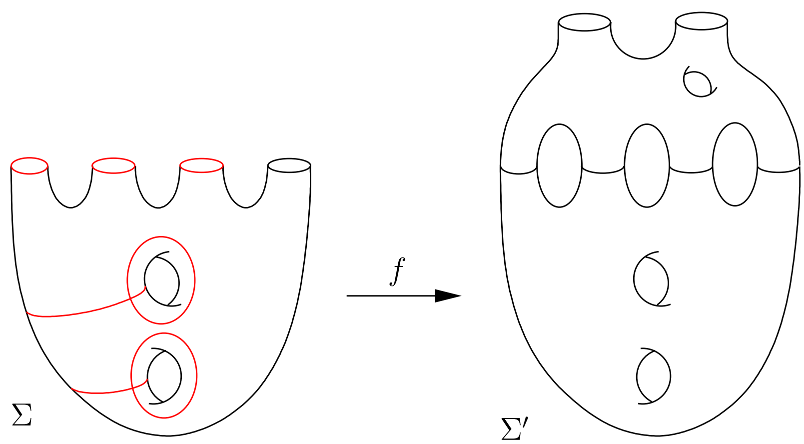


FIGURE 4.5. The embedding $f\colon \Sigma \to \mathring{\Sigma}'$ (in the figure it is the inclusion).

Notice that $f_*\colon H_1(\Sigma) \to H_1(\Sigma', \partial\Sigma')$ is injective, over $\mathbb{Z}$ and hence over $\mathbb{R}$. In our context of oriented surfaces, the de Rham isomorphism gives $H_1(\Sigma; \mathbb{R})^* = H^1(\Sigma)$ and $H_1(\Sigma', \partial\Sigma'; \mathbb{R})^* = H^1_c(\mathring{\Sigma}')$, where both spaces on the right are the de Rham cohomology groups. Thus, $f^*\colon H^1_c(\mathring{\Sigma}') \to H^1(\Sigma)$ is surjective.

Since the area of $\Sigma'$ is larger than the area of $\Sigma$, Moser's theorem allows to further assume that $f$ is area-preserving: $f^*d\alpha' = d\alpha$. Write $\gamma_i' = f(\gamma_i)$. Since the classes $[\gamma_i]$ are linearly independent in $H_1(\Sigma)$ and since $f^*$ is surjective, there exists a compactly supported closed 1-form $\vartheta$ on $\Sigma'$ with periods

$$\int_{\gamma_i'} \vartheta \;=\; \int_{\gamma_i} \alpha - \int_{\gamma_i'} \alpha', \qquad i = 1, \dots, 2g+b-1.$$

Let $\phi_\vartheta^t$ be the flow associated to the vector field $X_\vartheta$ defined by $d\alpha'(X_\vartheta, \cdot) = \vartheta$. This flow is complete because $\vartheta$ is compactly supported, and it is symplectic because $\vartheta$ is closed. A classical computation for the flux shows that

$$\int_{\phi_\vartheta^t(\gamma_i')} \alpha' \;=\; \int_{\gamma_i'} \alpha' + t \int_{\gamma_i'} \vartheta, \qquad i = 1, \dots, 2g+b-1,$$

see e.g. [21, § 10.2]. Thus $\phi_\vartheta^1 \circ f$ is a $(\alpha, \alpha')$-exact symplectic embedding of $\Sigma$ into $\Sigma'$. The proof of the proposition is complete. □

**Theorem 4.6.** $B^4(1) \backslash \Delta_k \overset{\alpha_{\rm st}}{\hookrightarrow} Z^4\big(\frac{2}{k}\big)$.

*Proof.* The polarization of $\mathbb{C}\mathrm{P}^2$ of degree $k$ in Example 2.8 restricts to a tame Liouville polarization $(\Sigma_k, \lambda_k)$ of the ball whose skeleton is $\Delta_k \cap B^4(1)$, so

$$B^4(1) \backslash \Delta_k = \mathcal{B}\big(\Sigma_k, \lambda_k\big).$$

One readily checks that this Liouville polarization is extendable. Moreover, the area, genus, and number of punctures of $\Sigma_k$ are well-known and given by

$$\mathcal{A}_\omega(\Sigma_k) = k, \quad g(\Sigma_k) = \frac{(k-1)(k-2)}{2}, \quad b(\Sigma_k) = k.$$

On the other side, proceeding as in the proof of Proposition 4.1, we take for any $A \in \mathbb{N}$ two points $p_i$ in $D(\frac{2}{k})$ and $kA$ points $q_j$ in $D(A)$, all with weight $\frac{1}{k}$, and consider the union of the discs $D(\frac{2}{k}) \times q_j$ and $p_i \times D(A)$. Using Lemma 3.5 and resolving all the $2kA$ intersections as in Proposition 3.8, we obtain a smooth Liouville polarization $\Sigma_k'$ of $D(\frac{2}{k}) \times D(A)$ with residue $\frac{1}{k}$. Then

$$\mathcal{A}_\omega(\Sigma_k') = 4A \quad \text{and} \quad g(\Sigma_k') = kA - 1.$$

When $A \geq \frac{k}{2}$, we have $\mathcal{A}_\omega(\Sigma_k') > \mathcal{A}_\omega(\Sigma_k)$ and $g(\Sigma_k') \geq g(\Sigma_k) + b(\Sigma_k) - 1$ since $k \geq 2$, so by Lemma 4.5 there exists an $(\alpha_{\rm st}|_{\Sigma_k}, \alpha_{\rm st}|_{\Sigma_k'})$-exact symplectic embedding of $\widehat{\Sigma}_k$ into $\Sigma_k'$, and by Theorem 2.20 there exists an $(\alpha_{\rm st}, \alpha_{\rm st})$-exact symplectic embedding of $\mathcal{B}\big(\Sigma_k, \lambda_k\big) = B^4(1) \backslash \Delta_k$ into $Z^4(\frac{2}{k})$. □

**Remark 4.7.** When $k=2$ we can get a better estimate and symplectically embed $B^4(1)\backslash\Delta_2 = B^4(1)\backslash\mathbb{R}^2$ into the ellipsoid $E(\frac{1}{2}, 2+\varepsilon)$, like [29]. This is because $\Sigma_2$ is a sphere with two punctures, hence has no genus. Thus a cylindrical extension of $\Sigma_2$ symplectically embeds into a disc of area $2+\varepsilon$. Since by Example 2.14, $(D(2+\varepsilon), \frac{1}{2})$ is a polarization of $\mathrm{SDB}(D(2+\varepsilon), \frac{1}{2})$, Theorem 2.20 shows that $B^4(1)\backslash\Delta_2$ embeds into $\mathrm{SDB}(2+\varepsilon, \frac{1}{2})$. And by [23, Lemma 2.1], $\mathrm{SDB}(D(2+\varepsilon), \frac{1}{2})$ is symplectomorphic to $E(2+\varepsilon, \frac{1}{2})$. $\diamond$

Recall that the open ellipsoid $E(a,b)$ is defined as

$$E(a,b) \;=\; \left\{ (z,w) \in \mathbb{C}^2 \;\middle|\; \frac{|z|^2}{a} + \frac{|w|^2}{b} < 1 \right\}.$$

**Proposition 4.8.** *For $d, N \in \mathbb{N}$ coprime there exists for every $m \in \mathbb{N}$ with $m \geq d$ and $(m,d,N) \neq (2,2,1)$ a symplectic embedding*

$$B^4(1) \setminus \Delta_{mNd} \xhookrightarrow{\alpha_{\mathrm{st}}} E\big(\tfrac{1}{d}, d + \tfrac{1}{N}\big).$$

*Proof.* As above, consider the extendable Liouville polarization $(\Sigma_{mNd}, \lambda_{mNd})$ of $B^4(1)$ of degree $mNd$, whose skeleton is $\Delta_{mNd}$. We have that $\mathcal{A}_\omega(\Sigma_{mNd}) = mNd$,

$$g(\Sigma_{mNd}) = \frac{1}{2}(mNd-1)(mNd-2) \quad \text{and} \quad b(\Sigma_{mNd}) = mNd.$$

In order to produce the required Liouville polarization of the target ellipsoid, consider the symplectic ramified covering

$$\begin{array}{rcccl} \Phi & : & B^4(N(Nd^2+d)) & \longrightarrow & E(N, Nd^2+d) \\ & & (R_1, \theta_1, R_2, \theta_2) & \longmapsto & \big(\frac{R_1}{Nd^2+d}, (Nd^2+d)\theta_1, \frac{R_2}{N}, N\theta_2\big). \end{array}$$

One checks without difficulty that the Liouville polarization $(\Sigma_m, \lambda_m)$ of $B^4(N(Nd^2+d))$ of degree $m$ projects under $\Phi$ to a smooth Liouville polarization of degree $m$ of $E(N, Nd^2+d)$ (smoothness requires that $N$ and $Nd^2+d$ are coprime, which is equivalent to our assumption that $N$ and $d$ are coprime). This polarization can therefore be seen as a Liouville polarization $(\Sigma', \lambda')$ of degree $mNd$ of $E(\frac{1}{d}, d+\frac{1}{N})$. The polarizing curve $\Sigma'$ has area

$$\mathcal{A}_\omega(\Sigma') \;=\; m(Nd+1) \;>\; \mathcal{A}_\omega(\Sigma_{mNd}),$$

and by the Riemann–Hurwitz formula its genus is

$$g(\Sigma') \;=\; \frac{1}{2}\big((mN-1)(mNd^2+md-1) - m + 1\big)$$

which is $\geq g\big(\Sigma_{mNd}\big) + b\big(\Sigma_{mNd}\big) - 1$ by our assumption on $(m,d,N)$. By Lemma 4.5, there exists an $(\alpha_{\mathrm{st}}|_{\Sigma_{mNd}}, \alpha_{\mathrm{st}}|_{\Sigma'})$-exact symplectic embedding of $\widehat{\Sigma}_{mNd}$ into $\Sigma'$, and by Theorem 2.20 an $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$-exact symplectic embedding of $\mathcal{B}(\Sigma_{mNd}, \lambda_{mNd}) = B^4(1)\backslash\Delta_{mNd}$ into $E(\frac{1}{d}, d+\frac{1}{N})$. $\square$

**Proof of Theorem 3**. It is shown in [30, § 6.2] that for every $\varepsilon > 0$ there exists $s_0 \in \mathbb{R}$ such that for every $s \geq s_0$ the ellipsoid $\lambda_s E(\frac{1}{s}, s)$ of volume $B^4(a - \frac{\varepsilon}{2})$ symplectically embeds into $(M, \omega)$. The claim now follows by precomposing this embedding with a scaling of a suitable embedding from Proposition 4.8. $\square$

The above proof gives no upper bound for $k$. However, if $(M, \omega)$ is a rational symplectic manifold, or an affine part $M \setminus \Sigma$ therein, then by [23] there exists an explicit volume filling ellipsoid in $M$ that can be chosen to lie in the complement of $\Sigma$. Together with Proposition 4.8 one obtains an upper bound for $k$.

## 5. LAGRANGIAN RIGIDITY

**Proof of Theorem 4.** Let $\iota \colon L \hookrightarrow D(A) \times D(B)$ be a Lagrangian embedding of a closed surface. Assume that there exists a Hamiltonian diffeomorphism $\phi$ of $\mathbb{R}^4$ such that

$$\phi(L) \subset \big(D(A) \times D(B)\big) \setminus \big(\Gamma_{\leq a} \times \Gamma_{\leq b}\big)$$

Since $\phi(L)$ is compact and disjoint from $\big(\Gamma_{\leq a} \cup \partial D(A)\big) \times \big(\Gamma_{\leq b} \cup \partial D(B)\big)$, we can modify $\Gamma_{\leq a}$ and $\Gamma_{\leq b}$ near the vertices of their closure to regular grids $\Gamma'_{\leq a}$ and $\Gamma'_{\leq b}$ that still divide $D(A)$ and $D(B)$ into topological discs of area $\leq a$ and $\leq b$, and such that still

$$\phi(L) \subset \big(D(A) \times D(B)\big) \setminus \big(\Gamma'_{\leq a} \times \Gamma'_{\leq b}\big).$$

By Theorem 6 there exists an exact symplectic embedding

$$\psi \colon \big(D(A) \times D(B)\big) \setminus \big(\Gamma'_{\leq a} \times \Gamma'_{\leq b}\big) \;\to\; Z^4(a + b).$$

Then $\psi \circ \phi \circ \iota \colon L \to L' \subset Z^4(a+b)$ is a Lagragian embedding. Since $\phi$ and $\psi$ are exact, the action classes $\iota^*[\alpha_{\rm st}]$ and $(\psi \circ \phi \circ \iota)^*[\alpha_{\rm st}]$ in $H^1(L; \mathbb{R})$ coincide. Since by Stokes' theorem the minimal area (1.1) is also the minimal action

$$A_{\min}(L) \;=\; \inf \left\{ \int_\gamma \alpha_{\rm st} \mid [\gamma] \in \pi_1(L),\, \int_\gamma \alpha_{\rm st} > 0 \right\},$$

we obtain that

$$A_{\min}(L') = A_{\min}(L).$$

And since $L' \subset Z^4(a+b)$, its displacement energy $e(L)$ in $\mathbb{C}^2$ is $< a + b$, so by Chekanov's result from [8], $A_{\min}(L') \leq e(L) < a + b$. Altogether, $A_{\min}(L) < a + b$, as we wished to prove. $\square$

## 6. LEGENDRIAN BARRIERS

Theorem 5 on the existence of short Reeb chords between Legendrian curves is obtained from the Lagrangian rigidity result of the previous paragraph via a construction of Mohnke [22], that associates to the Reeb trajectory of a Legendrian knot a Lagrangian torus. We first review this construction.

**Lemma 6.1.** *Let $(M^3, \xi, \alpha)$ be a contact manifold, $\big(SM = M \times \mathbb{R}_{>0}, d(R\alpha)\big)$ its symplectization, $\Lambda \subset M$ a Legendrian knot, and $X \subset M$ any subset. Assume that there is no Reeb chord of length $\leq T$ from $\Lambda$ to $\Lambda \cup X$.*

*Then there exists a Lagrangian torus $L$ in $(M \backslash X) \times (0,1] \subset SM$ with $\mathcal{A}_{\min}(L, SM) = T$.*

*Proof.* Consider the map

$$\iota\colon \Lambda \times (0,1] \times [0,\infty) \;\to\; SM, \quad (p,\tau,t) \mapsto \big(\Phi^t_\alpha(p), \tau\big)\,,$$

where $\Phi^t_\alpha$ is the Reeb-flow on $(M,\alpha)$. The restriction of $\iota$ to every band $\{p\} \times (0,1] \times [0,\infty)$, $p \in \Lambda$, is symplectic for the the forms $d\tau \wedge dt$ and $d(R\alpha)$ on the domain and the target, respectively. Since there is no Reeb chord of length $\leq T$ from $\Lambda$ to $\Lambda \cup X$, we find $\varepsilon > 0$ such that $\iota$ is an embedding of $\Lambda \times (0,1] \times [0, T+\varepsilon)$ into

$$V \;:=\; (M \backslash X) \times (0,1] \;\subset\; SM.$$

Take a closed disc $D_\gamma$ in $(0,1] \times [0, T+\varepsilon]$ of area $T$ and smooth oriented boundary $\gamma$. Then $\iota(\Lambda \times \gamma)$ is an embedded Lagrangian torus in $V$. The actions of the generators $[\Lambda]$ and $[\gamma]$ of $\pi_1(L)$ are $\int_\Lambda \alpha = 0$ and $\int_\gamma \alpha = T$, so $\mathcal{A}_{\min}(L, SM) = T$. Moreover, the symplectic disc $\iota(D_\gamma)$ lies in $V$, has boundary on $L$, and has area $T$. □

Returning to the setting of Theorem 5, we notice that when $S$ is the smooth boundary of a starshaped domain $U \subset \mathbb{R}^4$, with contact form $\lambda_S = \alpha_{\rm st}|_S$, then the exact symplectomorphism

$$\begin{array}{lcl} \big(S \times \mathbb{R}_{>0}, R\lambda_S\big) & \longrightarrow & \big(\mathbb{R}^4 \backslash \{0\}, \alpha_{\rm st}\big) \\ (s,R) & \longmapsto & R\,s \end{array}$$

identifies $S \times (0,1]$ with $U \backslash \{0\}$ and $X \times (0,1]$ with the part in $U \backslash \{0\}$ of the cone over $X$ centered at the origin of $\mathbb{R}^4$.

*Proof of Theorem 5:* Let $S$ be the smooth boundary of a starshaped domain $U \subset C^4(1) \subset \mathbb{R}^4$. Arguing by contradiction, assume that $\Lambda$ is a Legendrian knot in $S$ with no Reeb chord of length $\leq \delta_1 + \delta_2$ from $\Lambda$ to $\Lambda \cup \Lambda_\delta$, where $\Lambda_\delta = (\Gamma_{\delta_1} \times \Gamma_{\delta_2}) \cap S$. Since $\Gamma_{\delta_1}$ and $\Gamma_{\delta_2}$ are radial, $\Gamma_{\delta_1} \times \Gamma_{\delta_2}$ lies in the cone over $\Lambda_\delta$. Hence Lemma 6.1 provides a Lagrangian torus

$$L \subset U \setminus (\Gamma_{\delta_1} \times \Gamma_{\delta_2}) \subset C^4(1) \setminus (\Gamma_{\delta_1} \times \Gamma_{\delta_2})$$

with $\mathcal{A}_{\min}(L) = \delta_1 + \delta_2$. This is a contradiction to Theorem 4. $\Box$

**An illustration.** To get a feeling for the phenomenon of Legendrian barriers, we explictely work out the case where $S$ is the round sphere $S^3(1) = \partial B^4(1)$ and $\delta_1 = \delta_2 = \frac{1}{k}$. Recall that

$$\Delta_k = \bigcup_{0 \le i,j \le k-1} \xi^i \mathbb{R}_{\ge 0} \times \xi^j \mathbb{R}_{\ge 0}$$

where $\xi$ is the $k$th root of unity $e^{2\pi i/k}$. Hence the Legendrian graph $\Lambda_k := \Delta_k \cap S^3(1)$ is the union of the $k^2$ Legendrian quarter-circles

$$Q_{i,j} := \left(\xi^i \mathbb{R}_{\ge 0} \times \xi^j \mathbb{R}_{\ge 0}\right) \cap S^3(1).$$

Note that $Q_{i,j}$ and $Q_{i',j'}$ are disjoint if $i \neq i'$ and $j \neq j'$, and intersect at one end-point if either $i \neq i'$ or $j \neq j'$. We group these $k^2$ quarter-circles in $k$ sets

$$Q_j := \coprod_{i=0}^{k-1} Q_{i,i+j}.$$

Since the Reeb flow on $S^3(1)$ is the Hopf flow

$$\Phi_R^t(z_1, z_2) = e^{2\pi i t}(z_1, z_2) = \left(e^{2\pi i t} z_1, e^{2\pi i t} z_2\right),$$

we have $\Phi_R^{\frac{i}{k}}(Q_{0,j}) = \xi^i\, Q_{0,j} = Q_{i,i+j}$, that is, $\Phi_R^{\frac{1}{k}}$ cyclically acts on the components of $Q_j$.

The full sweep out of $Q_{0,j}$ under the (backward) Reeb flow is the Lagrangian surface

$$L_j := \bigcup_{t \in \mathbb{R}} e^{-2\pi i t}\, Q_{0,j} = \bigcup_{t \in [0,1]} e^{-2\pi i t}\, Q_{0,j} = \bigcup_{t \in [0,\frac{1}{k}]} e^{-2\pi i t}\, Q_j.$$

Therefore, if $\pi \colon S^3 \to S^2$ denotes the Hopf fibration (whose fibers are the Reeb trajectories), then

$$L_j = \pi^{-1}(h_j), \quad \text{where } h_j = \pi(Q_{0,j}) = \pi(Q_j).$$

A computation shows that $h_j$ is a half-great circle joining the north pole to the south pole of $S^2$. Since the area of the reduced space $S^2$ is $1$, it follows that the pairs $(h_j, h_{j+1})$ each bound an open disc $D_j$ of area $\frac{1}{k}$. It is also not hard to see that there exists a diffeomorphism $\pi^{-1}(D_j) \to D(\frac{1}{k}) \times S^1$ that takes $\lambda_{S^3(1)}$ to $\alpha_{\text{st}} \oplus dt$. Altogether, we see that

$$L := \bigcup_j L_j = \bigcup_{t \in [0,\frac{1}{k}]} \Phi_R^{-t}(\Lambda_k)$$

cuts $S^3(1)$ into $k$ connected components, each of which can be identified with $\big(D(\frac{1}{k}) \times S^1, \alpha_{\mathrm{st}} + dt\big)$, where $S^1 = \mathbb{R}/\mathbb{Z}$.

Now take a Legendrian knot $\Lambda$ in $S^3(1)$ such that there is no Reeb chord of length $\leq \frac{1}{k}$ from $\Lambda$ to $\Lambda_k$. Equivalently, $\Lambda$ is disjoint from $\bigcup_{t\in[0,\frac{1}{k}]} \Phi_R^{-t}(\Lambda_k)$, which is $L$. So $\Lambda$ lies in one of the connected components of $S^3(1)\setminus L$ and can be seen as a Legendrian knot in $\big(D(\frac{1}{k}) \times S^1, \alpha_{\mathrm{st}} + dt\big)$. But a very classical and elementary argument (see e.g. [1, p. 192]) shows that such a knot has a Reeb chord of length $\leq \frac{1}{k}$ (in fact, of length $< \frac{1}{2k}$).

Summing up this discussion, in the case of the round sphere $S = S^3(1)$, where the Reeb flow is explicit, the barrier property of $\Lambda_k$ directly follows from the fact that the $\frac{1}{k}$-negative Reeb chords starting at $\Lambda_k$ disconnect $S$ into pieces all of whose Legendrian knots have small Reeb chords. There is no reason, however, that this disconnectedness property remain true when the Reeb flow is modified by taking the contact form on an arbitrary star-shaped domain in $C^4(1)$. It is therefore remarkable that the Legendrian rigidity result holds true.

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