Quantitative h-principle in symplectic geometry

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Dedicated to Claude Viterbo, on the occasion of his 60th birthday

Abstract

We prove a quantitative h-principle statement for subcritical isotropic embeddings. As an application, we construct a symplectic homeomorphism that takes a symplectic disc into an isotropic one in dimension at least 6.

1 Introduction

Gromov's h-principle lies at the core of symplectic topology, by reducing many questions on the existence of embeddings or immersions to verifying their compatibility with algebraic topology. Symplectic topology focuses mainly on the other problems, that do not abide by an h-principle : Lagrangian embeddings, existence of symplectic hypersurfaces in specific homology classes etc. In [BO16], we have proved a refined version of h-principle, which in turn yielded applications to $\mathcal{C}^0$-symplectic geometry. For instance, we proved in [BO16] that in dimension at least 6, $\mathcal{C}^0$-close symplectic 2-discs of the same area are isotopic by a small symplectic isotopy, while in dimension 4, this does no longer hold. A similar quantitative h-principle was also used in [BHS18] in order to show that the symplectic rigidity manifested in the Arnold conjecture for the the number of fixed points of a Hamiltonian diffeomorphism completely disappears for Hamiltonian homeomorphisms in dimension at least 4.

The goal of this note is to prove a quantitative h-principle for *isotropic* embeddings and to derive some flexibility statements on symplectic homeomorphisms.

Theorem 1 (Quantitative h-principle for subcritical isotropic embeddings)**.** *Let V be an open subset of $\mathbb{C}^n$, $k<n$, $u_0,u_1 : D^k \hookrightarrow V$ be isotropic embeddings of closed discs. We assume that there exists a homotopy $F : D^k\times[0,1]\to V$ between u_0 and u_1 (so $F(\cdot,0)=u_0$, $F(\cdot,1)=u_1$) of size less than ε ($\mathrm{Diam}\, F(\{z\}\times[0,1])<\varepsilon$ for all $z\in D^k$).*

Then there exists a Hamiltonian isotopy $(\Psi^t)_{t\in[0,1]}$ such that $\Psi^1\circ u_0=u_1$, of size 2ε.

The proof shows that the theorem holds in the relative case, provided u_0,u_1 are symplectically isotopic, relative to the boundary. The method of the proof of theorem 1 follows a very similar track as the quantitative h-principle for symplectic discs that we established in [BO16]. Paralleling the construction of a symplectic homeomorphism whose restriction to a symplectic disc is a contraction in dimension 6, we can deduce from theorem 1 the following statement:

Theorem 2. *There exists a symplectic homeomorphism with compact support in* $\mathbb{C}^3$ *which takes a symplectic disc to an isotropic one.*

Of course, by considering products, we infer that there exists symplectic homeomorphisms that take some codimension 4 symplectic submanifolds to submanifolds which are nowhere symplectic.

The note is organized as follows. We prove theorem 1 in the next section. The construction of a symplectic homeomorphism that takes a symplectic disc to an isotropic one is explained in section 3, where we also explain a relation to relative Eliashberg-Gromov type questions, as posed in [BO16].

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Conventions and Notations We convene the following in the course of this paper:

- All our homotopies and isotopies have parameter space $[0,1]$. For instance (g_t) denotes an isotopy $(g_t)_{t\in[0,1]}$.
- Similarly, by concatenation of homotopies we always mean *reparametrized* concatenation.
- If $F:[0,1]\times X\to Y$ is a homotopy with value in a metric space, $\mathrm{Size}\,(F):=\max\{\mathrm{Diam}\,\big(F([0,1]\times\{x\})\big),\ x\in X\}$.
- For $A\subset B$, $\mathrm{Op}\,(A,B)$ stands for an arbitrarily small neighbourhood of A in B. To keep light notation, we omit B whenever there is no possible ambiguity.
- A homotopy $F:[0,1]\times N\to M$ is said relative to $A\subset N$ if it is constant on A.
- A homotopy $G:[0,1]^2\times N\to M$ between $F_0,F_1:[0,1]\times N\to M$ (that is a continuous map such that $G(i,t,z)=F_i(t,z)$ for $i=0,1$) is said relative to A and $\{0,1\}$ if $G(s,t,z)=F_0(t,z)=F_1(t,z)$ for all $z\in A$ and if $G(s,i,z)=F_0(i,z)$ for all $s\in[0,1]$.

2 Quantitative h-principle for isotropic discs

The aim of this section is to prove theorem 1.

2.1 Standard h-principle for subcritical isotropic embeddings

We recall in this section the main properties of the action of the Hamiltonian group on isotropic embeddings, as described in [Gro86, EM02]. To this purpose, we first fix some notations. In the current note, a disk D^k is always assumed to be closed, unless explicitly stated (hence an embedding of D inside an open set is always compactly embedded). Since we only deal with isotropic embeddings, it is enough to prove theorem 1 for subcritical isotropic embeddings of $[-1,1]^k$ rather than of a closed disc. By abuse of notation, in this section we denote $D^k = [-1,1]^k$. The set of isotropic framings $G^{\mathrm{iso}}(k,n)$ is the space of $(k,2n)$-matrices of rank k whose columns span an isotropic vector space in $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$.

The following statement is a specialization to $\mathbb{C}^n$ of the h-principle for subcritical isotropic embeddings: Recall that the h-principle for subcritical isotropic embeddings provides existence of isotropic embeddings or homotopies whose derivatives realize homotopy classes of maps to $G^{\mathrm{iso}}(k,n)$. In the following, if $A \subset D^k$, a homotopy of $f : D^k \to G^{\mathrm{iso}}(k,n)$ rel $\mathrm{Op}\,(A)$ is a continuous map $F : [0,1] \times D^k \to G^{\mathrm{iso}}(k,n)$ such that $F(t,z) = f(z)$ for all $z \in \mathrm{Op}\,(A)$. A homotopy $G : [0,1]^2 \times D^k \to G^{\mathrm{iso}}(k,n)$ between $F_0, F_1 : [0,1] \times D^k \to G^{\mathrm{iso}}(k,n)$ (that is a continuous map such that $G(i,t,z) = F_i(t,z)$ for $i = 0,1$) is said relative to $\mathrm{Op}\,(A)$ and $\{0,1\}$ if $G(s,t,z) = F_0(t,z) = F_1(t,z)$ for all $z \in \mathrm{Op}\,(A)$ and if $G(s,i,z) = F_0(i,z)$ for all $s \in [0,1]$ and $i \in \{0,1\}$.

Theorem 2.1 (Parametric $\mathcal{C}^0$-dense relative h-principle for isotropic embeddings [EM02])**.** *Let $k < n$:*

a) Let $\rho : D^k \to \mathbb{C}^n$ be a continuous map whose restriction to a neighbourhood of a closed subset $A \subset D^k$ is an isotropic embedding. Assume that $d\rho$ is homotopic to a map $G : D^k \to G^{\mathrm{iso}}(k,n)$ relative to $\mathrm{Op}\,(A)$. Then, for any $\varepsilon > 0$, there exists an isotropic embedding $u : D^k \hookrightarrow \mathbb{C}^n$ which coincides with ρ on $\mathrm{Op}\,(A)$, $d_{\mathcal{C}^0}(\rho, u) < \varepsilon$ and such that $du : D^k \to G^{\mathrm{iso}}(k,n)$ is homotopic to G rel $\mathrm{Op}\,(A)$.

b) Let $u_0, u_1 : D^k \hookrightarrow \mathbb{C}^n$ be isotropic embeddings, which coincide on a neighbourhood of a closed subset $A \subset D^k$. Let $G : [0,1] \times D^k \to G^{\mathrm{iso}}(k,n)$ be a homotopy between du_0, du_1 rel $\mathrm{Op}\,(A)$ and $\rho_t : D^k \to \mathbb{C}^n$ a homotopy between u_0, u_1 rel $\mathrm{Op}\,(A)$. For any $\varepsilon > 0$, there exists an isotropic isotopy $u_t : D^k \hookrightarrow \mathbb{C}^n$ ($t \in [0,1]$) relative to $\mathrm{Op}\,(A)$ such that $d_{\mathcal{C}^0}(\rho_t, u_t) < \varepsilon$ and $\{du_t\}$ is homotopic to G rel $\mathrm{Op}\,(A)$ and $\{0,1\}$.

We now state a related statement in a proper situation, when the disc D^k is *open*, hence not necessarily compactly embedded into V. The proof is a rather straightforward application of theorem 2.1, and goes exactly along the lines of the proof of lemma A.3 b) in [BO16]. We leave the details to the reader.

Proposition 2.2. *Let $V \subset \mathbb{R}^{2n}$ be a bounded open set, $u_0, u_1 : \overset{\circ}{D^k} \hookrightarrow V$ be subcritical isotropic embeddings which coincide on $\mathrm{Op}\,(\partial D^k)$, are homotopic relative to $\mathrm{Op}\,(\partial D^k)$ in V, and whose differentials are homotopic in $G^{\mathrm{iso}}(k,n)$ relative to $\mathrm{Op}\,(\partial D^k)$. We fix such a relative homotopy $G : [0,1] \times D^k \to G^{\mathrm{iso}}(k,n)$ between du_0 and du_1. If $k = 1$, we further assume that for a 1-form λ which is a primitive of ω in V,*

$$\int_{D^1 \times \{0\}} u_0^* \lambda = \int_{D^1 \times \{0\}} u_1^* \lambda.$$

Then there exists a Hamiltonian isotopy (ψ_t) with compact support in V such that $\psi_1 \circ u_0 = u_1$ and for the induced isotropic isotopy $u_t = \psi_t \circ u_0$, $\{du_t\}$ is homotopic to G rel $\mathrm{Op}\,(\partial D^k)$ and $\{0,1\}$.

The next lemma will be also used in the proof of theorem 1.

Lemma 2.3. *Let A, B be two closed subsets of D^k. Let $u_0, u_1 : D^k \hookrightarrow \mathbb{C}^n$ be subcritical isotropic embeddings that coincide on $\mathrm{Op}\,(A)$. Assume that we are given a homotopy $G_t : D^k \to G^{\mathrm{iso}}(k,n)$ between du_0 and du_1 rel $\mathrm{Op}\,(A)$. Let $v_t : D^k \hookrightarrow \mathbb{C}^n$ be an isotropic isotopy between u_0 and v_1 rel $\mathrm{Op}\,(A)$, such that $v_{1|\mathrm{Op}\,(B)} = u_1$, and such that$\{dv_{t|\mathrm{Op}\,(B)}\}$ is homotopic to $\{G_{t|\mathrm{Op}\,(B)}\}$ relative to $\mathrm{Op}\,(A)$ and $\{0,1\}$*[1]*. Then dv_1 and du_1 are homotopic rel $\mathrm{Op}\,(A \cup B)$ among maps $D^k \to G^{\mathrm{iso}}(k,n)$.*

Remark 2.4. *In the setting of lemma 2.3, since v_1 and u_1 are homotopic rel $\mathrm{Op}\,(A \cup B)$ (just consider the linear homotopy between them), the lemma and theorem 2.1 immediately imply that v_1 is in fact isotropic isotopic to u_1 rel $\mathrm{Op}\,(A \cup B)$.*

Proof of lemma 2.3: Consider the homotopy $K_t := dv_t : D^k \to G^{\mathrm{iso}}(k,n)$ between du_0 and dv_1 relative to $\mathrm{Op}\,(A)$, and the homotopy $G_t : D^k \to G^{\mathrm{iso}}(k,n)$ between du_0 and du_1 rel $\mathrm{Op}\,(A)$, provided by the assumption. Letting $\overline{K}_t := K_{1-t}$, we now consider the concatenation $H_t := \overline{K}_t \star G_t$. Since $\{dv_{t|\mathrm{Op}\,(B)}\}$ is homotopic to $\{G_{t|\mathrm{Op}\,(B)}\}$ relative to $\mathrm{Op}\,(A)$ and $\{0,1\}$ (as assumed by the lemma), there exists a homotopy $H_{s,t}$ ($s \in [0,1]$) between $H_{t|\mathrm{Op}\,(B)}$ and I_t relative to $\mathrm{Op}\,(A)$ and $\{0,1\}$, where $I_t \equiv du_{1|\mathrm{Op}\,(B)} = dv_{1|\mathrm{Op}\,(B)}$ is a constant homotopy. Let $\chi : D^k \to [0,1]$ be a continuous function such that $\chi(x) = 0$ on a complement of a sufficiently small neighborhood of B in D^k, and $\chi(x) = 1$ on a (smaller) neighborhood of B. Now define a homotopy $\tilde{G}_t : D^k \to G^{\mathrm{iso}}(k,n)$ ($t \in [0,1]$) by

$$\tilde{G}_t(z) := \begin{cases} H_{\chi(z),t}(z) & \text{when } z \in Op(B), \\ G_t(z) & \text{otherwise.} \end{cases}$$

Then $\tilde{G}_t$ is a desired homotopy between du_1 and dv_1 rel $\mathrm{Op}\,(A \cup B)$. $\square$

We will also need the following lemma, which allows to achieve general positions by Hamiltonian perturbations.

[1] Recall that this means there exists a continuous map $G : [0,1]^2 \times \mathrm{Op}\,(B) \to G^{\mathrm{iso}}(k,n)$ such that $G(0,t,z) = G_t(z)$ and $G(1,t,z) = dv_t(z)$ $\forall (t,z) \in [0,1] \times \mathrm{Op}\,(B)$, $G(s,t,z) = du_0(z)$ $\forall (s,t,z) \in [0,1]^2 \times \mathrm{Op}\,(A \cap B)$, $G(s,0,z) = G_0(z) = du_0(z)$ and $G(s,1,z) = G_1(z) = dv_1(z)$ $\forall (s,z) \in [0,1] \times \mathrm{Op}\,(B)$).

Lemma 2.5. *Let $V \subset \mathbb{C}^n$ be an open set, Σ_1, Σ_2 be two smooth submanifolds of V, which are transverse in a neighbourhood of ∂V. Then there exists an arbitrarily small Hamiltonian flow $(\varphi^t)_{t\in[0,1]}$ with compact support in V, such that $\varphi^1(\Sigma_1) \pitchfork \Sigma_2$.*

2.2 Proof of theorem 1

Let $k < n$, $D^k := [-1,1]^k, D^k(\mu) := [-1-\mu, 1+\mu]^k$, $u_0, u_1 : D^k \hookrightarrow V \subset \mathbb{C}^n$ be smooth isotropic embeddings, and $F : D^k \times [0,1] \to V$ a homotopy between u_0, u_1 with $\operatorname{Size} F < \varepsilon$. We need to prove that there exists a *Hamiltonian* isotopy of size 2ε, which takes u_0 to u_1 on D^k.

Before passing to the proof, we need to modify slightly the framework. First, extend the isotropic embeddings and the homotopy to slightly larger isotropic embeddings: $u_0, u_1 : D^k(\mu) \hookrightarrow V$, $F : D^k(\mu) \times [0,1] \to V$, where $D^k(\mu) = [-\mu, 1+\mu]^k$. By lemma 2.5, we do not lose generality if we assume that the images of u_0 and u_1 are disjoint (since $k < n$), which we do henceforth. Next, the homotopy F can be turned into a more convenient object:

Lemma 2.6 (see [BO16, lemma A.1])**.** *There exists a smooth embedding $\tilde{F} : D^k(\mu) \times [0,1] \hookrightarrow V$, with $\tilde{F}(x,0) = u_0(x)$, $\tilde{F}(x,1) = u_1(x)$, with Diam$(\tilde{F}(\{x\} \times [0,1])) < 2\varepsilon$ for all $x \in D^k(\mu)$. In other words, $\tilde{F}$ has size 2ε when considered as a homotopy between u_0, u_1.*

Now $\tilde{F}$ can be further extended to an embedding, still denoted $\tilde{F}$,

$$\tilde{F} : D^k(\mu) \times [-\mu, 1+\mu] \times [-\mu, \mu]^{2n-k-1} \hookrightarrow V.$$

Consider now a regular grid $\Gamma_0 := \nu\mathbb{Z}^k \cap D^k$ in $D^k \subset D^k(\mu)$, of step $\nu \ll 1$ (to be specified later), where $\nu^{-1} \in \mathbb{N}$. This grid generates a cellular decomposition of D^k, whose l-skeleton Γ_l is the union of the l-faces. The set of k-faces has a natural integer-valued distance, where the distance between k-faces x and x' is the minimal m such that there exists a sequence $x = x_0, x_1, \ldots, x_m = x'$ of k-faces and $x_j \cap x_{j+1} \neq \emptyset$ for each $j \in [0, m-1]$ (note that those intersections are not required to be along full $k-1$-faces). Fix some $\eta < \nu/2$, and for each $x \in \Gamma_0$, let U_x be the η-neighbourhood of $\{x\} \times [0,1] \times \{0\}^{2n-k-1}$ in $\mathbb{C}^n$, and then denote $W_x := \tilde{F}(U_x)$. Similarly, for each k-face x_k, denote by U_{x_k} the η-neighbourhood of $x_k \times [0,1] \times \{0\}^{2n-k-1}$ in $\mathbb{C}^n$, and then put $W_{x_k} := \tilde{F}(U_{x_k})$. For a k-face x and $m \geqslant 0$ we denote $W_x^m := \cup W_{x'}$, where the union is over all the k-faces x' which are at distance at most m from x. Note that $W_x^0 = W_x$, and that W_x^m is a topological ball. Finally, we put $W := \cup_x W_x \subset V$, where the union is over all the k-faces. Hence, $W = \tilde{F}(U)$ where U is the η-neighborhood of $D^k \times [0,1] \times \{0\}^{2n-k-1}$ in $\mathbb{C}^n$.

We will prove theorem 1 by successively isotopying the l-skeleton with a control on each isotopy. Precisely, arguing by induction on l, we prove the following:

Proposition 2.7. *There exist Hamiltonian isotopies (Ψ_l^t), $l \in [0,k]$ with support in W, and modified embeddings $v_0 := \Psi_0^1 \circ u_0$, $v_l := \Psi_l^1 \circ v_{l-1}$, such that*

($\mathcal{I}1$) $v_l \equiv u_1$ on a neighbourhood of the l-skeleton Γ_l, for every $l \in [0,k]$.

($\mathcal{I}2$) $v_l(x) \subset W_x^{3^l-1}$ *for each* k*-face* x *and every* $l \in [0, k-1]$.

($\mathcal{I}3$) $\Psi_l^t(W_x) \subset W_x^{2\cdot 3^{l-1}}$ *for each* k*-face* x *and* $l \in [1, k-1]$*, and* $\Psi_0^t(W_x) \subset W_x$*,* $\Psi_k^t(W_x) \subset W_x^{3^{k(k+1)}}$*, for every* k*-face* x*.*

($\mathcal{I}4$) $v_l(\overset{\circ}{x}_{l+1}) \cap u_1(\overset{\circ}{x}{}'_{l+1}) = \emptyset$ *for every pair of distinct* $(l+1)$*-faces,* $\forall l \in [0, k-1]$*.*

($\mathcal{I}5$) dv_l *and* du_1 *are homotopic rel* $\mathrm{Op}\,(\Gamma_l)$ *among maps* $D^k(\mu) \to G^{\mathrm{iso}}(k,n)$*, for each* $l \in [0, k-1]$*.*

Proposition 2.7 readily implies theorem 1. Indeed, denoting by $(\Psi^t)_{t\in[0,1]}$ the (reparametrized) concatenation $\{\Psi_k^t\} \star \cdots \star \{\Psi_1^t\}$ of the flows, from ($\mathcal{I}3$) we conclude that for each k-face x and each t we have $\Psi^t(W_x) \subset W_x^{3^{k^2+k+1}}$ since $\left(\sum_{j=1}^{k-1} 2\cdot 3^j\right) + 3^{k(k+1)} < 3^{k^2+k+1}$. The flow (Ψ^t) is supported in $W = \cup_{x\in\Gamma_k} W_x \subset V$, and if the step ν of the grid is chosen to be sufficiently small, then for each k-face x, the diameter of $W_x^{3^{k^2+k+1}}$ is less than 2ε. Consequently, the size of the flow $(\Psi^t)_{t\in[0,1]}$ is less than 2ε. Moreover, by ($\mathcal{I}1$) we have $\Psi^1 \circ u_0 = v_k = u_1$ on D^k. $\square$

Proof of proposition 2.7: As already explained, the proof goes by induction over the dimension of the skeleton Γ_l. Since $D^k(\mu)$ is contractible, there exists a homotopy $G_t : D^k \to G^{\mathrm{iso}}(k,n)$ between du_0 and du_1.

The 0-skeleton: Let $x \in \Gamma_0$ be a 0-face, $\rho < \eta$, and $D_\rho(x)$ the ρ-neighbourhood of x in $D^k(\mu)$. Then $u_0(D_\rho(x)), u_1(D_\rho(x))$ both lie in W_x, and $\tilde{F}$ provides an isotopy between $u_{0|D_\rho(x)}$ and $u_{1|D_\rho(x)}$ in W_x. By theorem 2.1.b), there exists a Hamiltonian isotopy (ψ_x^t) with support in W_x, such that $\psi_x^1 \circ u_0 = u_1$ on $D_\rho(x)$ and $d\psi_x^t \circ du_0$ is homotopic to G_t rel $\{0,1\}$. Since $W_x \cap W_{x'} = \emptyset$ for different 0-faces x, x', the isotopies ψ_x have pairwise disjoint supports.

The isotopy $\psi_0^t := \circ \psi_x^t$, where the composition runs over all 0-faces x of Γ, verifies ($\mathcal{I}1$) by construction. Moreover, the isotopy satisfies ($\mathcal{I}3$) because it is supported inside the disjoint union $\cup_{x\in\Gamma_0} W_x$, and for every $x \in \Gamma_0$ and a k-face x' we have either $W_x \subset W_{x'}$ or $W_x \cap W_{x'} = \emptyset$. However, $\psi_0^1 \circ u_0$ might not verify ($\mathcal{I}4$). Still, since $\psi_0^1 \circ u_0$ coincides with u_1 on a neighbourhood of Γ_0, there exist closed balls $\overline{B}_{x_0} = \overline{B}(u_1(x_0), r) \subset W_{x_0}$ for each 0-face x_0 of Γ, such that ($\mathcal{I}4$) is verified inside these balls. Therefore the traces of the submanifolds $\psi_0^1 \circ u_0(x_1)$ and $u_1(x_1')$ inside $\underset{x_0\in\Gamma_0}{\cup} \left(W_{x_0} \backslash \overline{B}_{x_0}\right)$ verify the hypothesis of lemma 2.5, for every pair of distinct 1-faces x_1, x_1'. Thus an arbitrarily $\mathcal{C}^1$-small Hamiltonian perturbation $(\tilde{\psi}^t)$ with support in $\underset{x_0\in\Gamma_0}{\cup} \left(W_{x_0} \backslash \overline{B}_{x_0}\right) \subset \underset{x_0\in\Gamma_0}{\cup} W_{x_0}$ achieves $\tilde{\psi}^1 \circ \psi_0^1 \circ u_0(x_1) \pitchfork u_1(x_1')$, for every pair x_1, x_1' of different 1-faces of Γ (hence these intersections are empty). Now $\Psi_0^t := (\tilde{\psi}^t) * (\psi_0^t)$ verifies ($\mathcal{I}4$), and still verifies ($\mathcal{I}1$) and ($\mathcal{I}3$). ($\mathcal{I}2$) follows immediately from ($\mathcal{I}3$), and $v_0 = \Psi_0^1 \circ u_0$ satisfies ($\mathcal{I}5$) by direct application of lemma 2.3.

The l-skeleton ($1 \leqslant l < n-1$): We now assume that $\Psi_1, \ldots, \Psi_{l-1}$ have been constructed, and we proceed with the induction step. Recall that $v_{l-1} = \Psi_{l-1}^1 \circ \cdots \circ \Psi_0^1 \circ u_0$ coincides with u_1 on $\mathrm{Op}\,(\Gamma_{l-1})$ and that $v_{l-1}(x_k) \subset W_{x_k}^{3^{l-1}-1}$ for every k-face x_k. Recall also that we have a homotopy $G^l_t : D^k \to G^{\mathrm{iso}}(k,n)$ between dv_{l-1} and du_1 rel $\mathrm{Op}\,(\Gamma_{l-1})$. Our aim is now to find a Hamiltonian flow (Ψ^t_l) which in particular isotopes $v_{l-1|\mathrm{Op}\,(x_l)}$ to $u_{1|\mathrm{Op}\,(x_l)}$, for each l-face x_l.

Step I: Adjusting the actions of the edges (case $l=1$). When $l=1$, beside the formal obstructions, relative isotopies can be performed *via* localized Hamiltonians only when the actions of the edges coincide (see proposition 2.2). In [BO16], we show that there exists a Hamiltonian isotopy $(\psi^t_{\mathcal{A}})$, supported in an arbitrarily small neighborhood $v_0(\Gamma_0) = u_1(\Gamma_0)$, whose flow is the identity on a (smaller) neighbourhood of the Γ_0, such that

$$\mathcal{A}\big(\psi^1_{\mathcal{A}} \circ v_0(x_1)\big) := \int_{\psi^1_{\mathcal{A}}\circ v_0(x_1)} \lambda = \int_{u_1(x_1)} \lambda = \mathcal{A}\big(u_1(x_1)\big) \quad \text{for every 1-face } x_1 \text{ of } \Gamma,$$

and $\psi^1_{\mathcal{A}} \circ v_0(\mathring{x}_1) \cap u_1(\mathring{x}'_1) = \emptyset$ for each pair of distinct 1-faces x_1, x'_1 of Γ. Since $\psi^t_{\mathcal{A}} \equiv \mathrm{Id}$ near Γ_0, $\tilde{\Psi}^t_0 := \psi^t_{\mathcal{A}} \circ \Psi^t_0$ and $v'_0 := \tilde{\Psi}^1_0 \circ u_0$ still verify $(\mathcal{I}1-5)$. In other terms, replacing (Ψ^t_0) by $(\tilde{\Psi}^t_0)$ and v_0 by v'_0, we can freely assume that $\mathcal{A}\big(v_0(x_1)\big) = \mathcal{A}\big(u_1(x_1)\big)$ for each 1-face x_1.

Step II: Isotopying the l-skeleton. Fix an l-face x_l of Γ. By $(\mathcal{I}1)$, there exists a closed box $\hat{x}_l \subset \mathring{x}_l$ such that v_{l-1} and u_1 coincide on $\mathrm{Op}\,(x_l \setminus \mathring{\hat{x}}_l)$. Choose a k-face x_k which contains x_l. Since $u_1(\hat{x}_l)$ and $v_{l-1}(\hat{x}_l)$ both lie in the topological ball $W^{3^{l-1}-1}_{x_k}$ and coincide near their boundary, there exists a homotopy

$$\sigma_{x_l} : \hat{x}_l \times [0,1] \to W^{3^{l-1}-1}_{x_k}$$

such that $\sigma_{x_l}(\cdot,0) = v_{l-1}$, $\sigma_{x_l}(\cdot,1) = u_1$, and $\sigma_{x_l}(z,t) = u_1(z)\ \forall z \in \mathrm{Op}\,(\partial\hat{x}_l), t \in [0,1]$. Since $\hat{x}_l \Subset x_l$ and $l < n$, $(\mathcal{I}4)$ allows to use a general position argument to ensure that moreover $\mathrm{Im}\,\sigma_{x_l}$ admits a regular neighbourhood $\mathcal{V}_{x_l} \subset W^{3^{l-1}-1}_{x_k}$ (a topological ball), such that all these neighbourhoods $\mathcal{V}_{x_l}$ are pairwise disjoint when x_l runs over the l-faces (this is the only point in the proof where we need that $l < n-1$).

By assumption, there exists a homotopy $G^t_l : [0,1] \times D^k \to G^{\mathrm{iso}}(k,n)$ between dv_{l-1} and du_1, with $G^t_{l|\mathrm{Op}\,(\Gamma_{l-1})} = du_1 = dv_{l-1}$. Also, $v_{l-1|\hat{x}_l}$ is clearly homotopic to $u_{1|\hat{x}_l}$ rel $\mathrm{Op}\,(\partial\hat{x}_l)$ in $\mathcal{V}_{x_l}$, and when $l=1$, $\mathcal{A}(v_{l-1} \circ x_l) = \mathcal{A}(u_1 \circ x_l)$. Hence by proposition 2.2, there exist Hamiltonian diffeomorphisms $\psi^t_{x_l}$, where x_l runs over the l-faces, which have support in $\mathcal{V}_{x_l}$, and are such that $\psi^1_{x_l} \circ v_{l-1|\hat{x}_l} = u_1$, and $d(\psi^t_{x_l} \circ v_{l-1})$ are homotopic relative to $\mathrm{Op}\,(\partial\hat{x}_l)$ and $\{0,1\}$ to $G^t_{l|\hat{x}_l}$. Let now $\psi^t_l := \circ\psi^t_{x_l}$ and $\hat{v}_l := \psi^1_l \circ v_{l-1}$. Since the $(\psi^t_{x_l})$ have pairwise disjoint supports, we have $\hat{v}_{l|x_l} = u_{1|x_l}$ for each l-face x_l of Γ. Hence $\hat{v}_l$ and u_1 coincide on a neighbourhood of the l-skeleton of Γ, so $\hat{v}_l$ verifies $(\mathcal{I}1)$. By lemma 2.3, $\hat{v}_l$ verifies $(\mathcal{I}5)$ as well.

The flow (ψ^t_l) is supported in the disjoint union $\cup_{x_l\in\Gamma_l}\mathcal{V}_{x_l}$. Let x be any k-face, and assume that we have an l-face x_l such that $\mathcal{V}_{x_l} \cap W_x \neq \emptyset$. Let $x_k \supset x_l$ be a k-face as above, so that $\mathcal{V}_{x_l} \subset W^{3^{l-1}-1}_{x_k}$. Then the distance between x and x_k is not larger than 3^{l-1}, and we conclude $\mathcal{V}_{x_l} \subset W^{3^{l-1}-1}_{x_k} \subset W^{2\cdot3^{l-1}-1}_x$. To summarise, for any k-face x, if x_l is an l-face with $\mathcal{V}_{x_l} \cap W_x \neq \emptyset$, then $\mathcal{V}_{x_l} \subset W^{2\cdot3^{l-1}-1}_x$. As a result, we get

$$\psi^t_l(W_x) \subset W^{2\cdot3^{l-1}-1}_x \ . \tag{2.2.1}$$

The embedding $\hat{v}_l$ may fail to satisfy $(\mathcal{I}4)$: there might be two different $l+1$-faces x_{l+1}, x'_{l+1} such that

$$\hat{v}_l(\overset{\circ}{x}_{l+1}) \cap u_1(\overset{\circ}{x}'_{l+1}) \neq \emptyset.$$

Notice however that since $\hat{v}_l$ and u_1 coincide on a neighbourhood of Γ_l, the set $\hat{v}_l(x_{l+1}) \cap u_1(x'_{l+1})$ is compactly contained in $W \backslash u_1(\Gamma_l)$. By lemma 2.5, there exists an arbitrarily small Hamiltonian flow $(\varphi_l^t)_{t\in[0,1]}$, with compact support in $W\backslash\Gamma_l$ such that $v_l := \varphi_l^1 \circ \hat{v}_l$ verifies $(\mathcal{I}4)$. By the smallness of the flow (φ_l^t) and by (2.2.1), the flow $(\Psi_l^t) := (\varphi_l^t) * (\psi_l^t)$ satisfies $\Psi_l^t(W_x) \subset W_x^{2\cdot 3^{l-1}}$ for any k-face x. Hence $(\mathcal{I}3)$ holds for (Ψ_l^t). Since the support of (φ_l^t) is compactly contained in $W\backslash\Gamma_l$, $(\mathcal{I}1)$ and $(\mathcal{I}5)$ still holds for v_l. Finally, $(\mathcal{I}2)$ follows as well: if x is any k-face, then by assumption, $v_{l-1}(x) \subset W_x^{3^{l-1}-1}$, hence by (2.2.1) and $(\mathcal{I}3)$ we get

$$\begin{aligned} v_l(x) &= \Psi_l^1 \circ v_{l-1}(x) \subset \Psi_l^1(W_x^{3^{l-1}-1}) = \bigcup_{d(x,y)\leqslant 3^{l-1}-1} \Psi_l^1(W_y) \subset \\ &\subset \bigcup_{d(x,y)\leqslant 3^{l-1}-1} W_y^{2\cdot 3^{l-1}} = W_x^{3^{l-1}-1+2\cdot 3^{l-1}} = W_x^{3^l-1}. \end{aligned} \tag{2.2.2}$$

The k-skeleton: When $k < n-1$, the procedure described above works perfectly. However, when $k = n-1$, the last step of the induction requires some adjustment. As before, for every k-face x_k, $v_{k-1}(x_k)$ and $u_1(x_k)$ both lie in the topological ball $W_{x_k}^{3^{k-1}-1}$ and coincide near the boundary, hence there exist homotopies

$$\sigma_{x_k} : \hat{x}_k \times [0,1] \to W_{x_k}^{3^{k-1}-1}$$

such that $\sigma_{x_k}(\cdot,0) = v_{k-1|x_k}$, $\sigma_{x_k}(\cdot,1) = u_{1|x_k}$ and $\sigma_{x_k}(z,t) = u_1(z)$ for all $t \in [0,1]$, $z \in \mathrm{Op}\,(\partial x_k)$ (as before, $\hat{x}_k \subset \overset{\circ}{x}_k$ is a closed box such that u_1 and v_{k-1} coincide on $\mathrm{Op}\,(x_k \backslash \overset{\circ}{\hat{x}}_k)$). The difference with the previous steps of the induction is that general position does not make the sets $\mathrm{Im}\,\sigma_{x_k}$ pairwise disjoint. Instead we proceed as follows.

By $(\mathcal{I}4)$, $v_{k-1}(\hat{x}_k) \cap u_1(x'_k) = u_1(\hat{x}_k) \cap u_1(x'_k) = \emptyset$ for every pair of different k-faces x_k, x'_k. By a standard general position argument, since $k < n$, we can therefore assume that $\mathrm{Im}\,\sigma_{x_k} \cap u_1(x'_k) = \emptyset$, and that we have a regular neighbourhood $\mathcal{V}_{x_k} \subset W_{x_k}^{3^{k-1}-1}$ of $\mathrm{Im}\,\sigma_{x_k}$, such that

$$\mathcal{V}_{x_k} \cap u_1(x'_k) = \emptyset \qquad \forall x_k \neq x'_k. \tag{2.2.3}$$

By $(\mathcal{I}5)$, and since $v_{k-1}(\hat{x}_k), u_1(\hat{x}_k)$ are homotopic relative to $\partial\hat{x}_k$ in $\mathcal{V}_{x_k}$, there exists a Hamiltonian isotopy $(\psi_{x_k}^t)$ with support in $\mathcal{V}_{x_k}$ such that $\psi_{x_k}^1 \circ v_{k-1|x_k} = u_1$.

Consider now a partition of the set of the k-faces into $(2\cdot 3^{k-1})^k = 2^k \cdot 3^{k(k-1)}$ subsets F_i ($i = 1, \ldots, 2^k \cdot 3^{k(k-1)}$), such that any two faces $x_k, x'_k \in F_i$ are at distance at least $2\cdot 3^{k-1}$ from each other. Then for any i and any pair $x_k, x'_k \in F_i$ of distinct k-faces, we have $W_{x_k}^{3^{k-1}-1} \cap W_{x'_k}^{3^{k-1}-1} = \emptyset$. Define $(\psi_{k,i}^t) := \underset{x_k\in F_i}{\circ} \psi_{x_k}^t$, which is a composition of Hamiltonian isotopies, compactly supported in the disjoint union $\cup_{x_k\in F_i} W_{x_k}^{3^{k-1}-1}$. For any k-face x, if we have some $x_k \in F_i$ such that $W_x \cap W_{x_k}^{3^{k-1}-1} \neq \emptyset$, then the distance between x and x_k is at most 3^{k-1}, and hence $W_{x_k}^{3^{k-1}-1} \subset W_x^{2\cdot 3^{k-1}-1}$. We conclude that for any k-face x we have $\psi_{k,i}^t(W_x) \subset W_x^{2\cdot 3^{k-1}-1}$.

Now, letting $(\Psi_k^t) := (\psi^t_{k,2^k\cdot 3^{k(k-1)}}) * \cdots * (\psi^t_{k,1})$ and arguing as in (2.2.2), we get for any k-face x

$$\Psi_k^t(W_x) \subset W_x^{N_k} \subset W_x^{3^{k(k+1)}}\ ,$$

where $N_k = 2^k \cdot 3^{k(k-1)} \cdot (2\cdot 3^{k-1} - 1) < 3^{k(k+1)}$. Therefore, $(\mathcal{I}3)$ holds for (Ψ_k^t).

Finally, $\psi^1_{k,i} \circ v_{k-1|\mathrm{Op}\,(x_k)} = u_{1|\mathrm{Op}\,(x_k)}$ for all $x_k \in F_i$, and by (2.2.3), $\psi^1_{x'_k} \circ u_{1|\mathrm{Op}\,(x_k)} = u_{1|\mathrm{Op}\,(x_k)}$ for any pair of k-faces $x'_k \neq x_k$. Thus,

$$\Psi_k^1 \circ v_{k-1|\mathrm{Op}\,(x_k)} = u_1 \text{ for every } k\text{-face } x_k \text{ of } \Gamma,$$

which just means that $\Psi_k^1 \circ v_{k-1|\mathrm{Op}\,(D^k)} \equiv u_{1|\mathrm{Op}\,(D^k)}$. We have verified $(\mathcal{I}1)$ for $v_k := \Psi_k^1 \circ v_{k-1}$. $\square$

3 Action of symplectic homeomorphisms on symplectic submanifolds

3.1 Taking a symplectic disc to an isotropic one

We aim now at proving theorem 2. Although it is completely similar to the proof of the flexibility of the disc area provided in [BO16] once theorem 1 is established, we rewrite below the argument in our situation for the convenience of the reader. Recall that theorem 1 holds for *symplectic* embeddings of discs in $\mathbb{C}^3$ [BO16, Theorem 2].

Theorem 3.1. *Theorem 2 holds when the isotropic embeddings u_0, u_1 are replaced by symplectic embeddings $u_0, u_1 : D \hookrightarrow W$ such that $u_1^*\omega = u_0^*\omega = \omega_{\mathrm{st}}$.*

Proof of theorem 2: Let

$$\begin{array}{rcrcl} i_0 & : & D & \longrightarrow & \mathbb{C}\times\mathbb{C}\times\mathbb{C} = \mathbb{C}^3, \\ & & (x,y) & \longmapsto & (x,y,0) \end{array} \qquad \begin{array}{rcrcl} u_0 & : & D & \longrightarrow & \mathbb{C}\times\mathbb{C}\times\mathbb{C} \\ & & z & \longmapsto & (z,0,0) \end{array}$$

be the standard isotropic and symplectic embeddings of D into $\mathbb{C}^3$. Let also $f_k : D(2) \to D_{1/2^k}$ be an area-preserving immersion and

$$\begin{array}{rcrcl} u_k & : & D & \longrightarrow & \mathbb{C}\times\mathbb{C}\times\mathbb{C} \\ & & (x,y) & \longmapsto & (x,y,f_k(x+iy)). \end{array}$$

Then, u_k is a symplectic embedding of $\overline{D}$ into $\mathbb{C}^3$ with $d_{\mathcal{C}^0}(u_k, i_0) < \frac{1}{2^k}$. Let finally consider an isotropic embedding i_k^l of $\overline{D}$ into $\mathbb{C}^3$ with $d_{\mathcal{C}^0}(i_k^l, u_k) < \frac{1}{2^l}$. Although less explicit than the previous embedding in dimension 6, it certainly exists because one can approximate the standard symplectic embedding u_0 by isotropic ones of the form $z \mapsto (z, \overline{f_l(z)}, 0)$. We also define

$$W_k(\delta) := \{z \in \mathbb{C}^3 \mid d(z, \operatorname{Im} u_k) < \delta\}$$
$$\text{and } W^0(\varepsilon) := \{z \in \mathbb{C}^3 \mid d(z, \operatorname{Im} i_0) < \varepsilon\}.$$

It is enough to construct a sequence $\varphi_0, \varphi_1, \dots$ of compactly supported in $\mathbb{C}^3$ symplectic diffeomorphisms, such that for an increasing sequence of indices $k_0 = 0 < k_1 < k_2 < \dots$ we have $\varphi_i \circ u_{k_i} = u_{k_{i+1}}$, and such that moreover, the sequence $\Phi_i = \varphi_i \circ \varphi_{i-1} \circ \cdots \circ \varphi_0$ uniformly converges to a homeomorphism Φ of $\mathbb{C}^3$. We construct such a sequence φ_i by induction. Let $\mathbb{C}^3 = U_0 \supset U_1 \supset U_2 \supset \cdots \supset u_0(\overline{D})$ be a decreasing sequence of open sets such that $\cap U_i = u_0(\overline{D})$. In the step 0 of the induction, we let $k_1 = 1$, and choose φ_0 to be any symplectic diffeomorphism with compact support in $\mathbb{C}^3$ such that $\varphi_0 \circ u_0 = u_{k_1}$.

Now we describe a step $i \geqslant 1$. From the previous steps we get $k_1 < \cdots < k_i$, and symplectic diffeomorphisms $\varphi_0, \dots, \varphi_{i-1}$. Denote $\Phi_{i-1} = \varphi_{i-1} \circ \cdots \circ \varphi_0$. By the step $i-1$, we have $u_{k_i} = \Phi_{i-1} \circ u_0$ and $\Phi_{i-1}(U_{i-1}) \supset W^0(\varepsilon_i)$, where $\varepsilon_i = \frac{1}{2^{k_i}}$. The choice for ε_i implies that $W^0(\varepsilon_i) \supset u_{k_i}(\overline{D})$, and moreover by $u_{k_i} = \Phi_{i-1} \circ u_0$ we get $\Phi_{i-1}(U_i) \supset u_{k_i}(\overline{D})$, so we conclude $\Phi_{i-1}(U_i) \cap W^0(\varepsilon_i) \supset u_{k_i}(\overline{D})$. Hence we can choose a sufficiently large $l_i \geqslant k_i$ such that $\Phi_{i-1}(U_i) \cap W^0(\varepsilon_i) \supset W_{k_i}(\delta_i) \supset i^{l_i}_{k_i}(\overline{D})$, where $\delta_i = \frac{1}{2^{l_i}} \leqslant \varepsilon_i$. Note that

$$d_{\mathcal{C}^0}(i^{l_i}_{k_i}, i_0) \leqslant d_{\mathcal{C}^0}(i^{l_i}_{k_i}, u_{k_i}) + d_{\mathcal{C}^0}(u_{k_i}, i_0) < \frac{1}{2^{l_i}} + \frac{1}{2^{k_i}} \leqslant 2\varepsilon_i,$$

and moreover $i_0(\overline{D}), i^{l_i}_{k_i}(\overline{D}) \subset W^0(\varepsilon_i)$. Hence by the convexity of $W^0(\varepsilon_i)$ and by theorem 1, there exists a Hamiltonian diffeomorphism φ'_i supported in $W^0(\varepsilon_i)$ such that $i_0 = \varphi'_i \circ i^{l_i}_{k_i}$ and $d_{\mathcal{C}^0}(\varphi'_i, \mathrm{Id}\,) < 4\varepsilon_i$. Note that in particular, $\varphi'_i(W_{k_i}(\delta_i)) \supset i_0(\overline{D})$.

We claim that there exists a homotopy of *a small size* between the (symplectic) disc $\varphi'_i \circ u_{k_i}$ and the (isotropic) disc i_0, inside $\varphi'_i(W_{k_i}(\delta_i))$. Indeed, the open set $W_{k_i}(\delta_i)$ contains the discs $u_{k_i}(\overline{D}), i^{l_i}_{k_i}(\overline{D})$. Also we have $d_{\mathcal{C}^0}(u_{k_i}, i^{l_i}_{k_i}) < \delta_i$. Hence the linear homotopy $\rho_i(z,t) := (1-t)u_{k_i}(z) + t i^{l_i}_{k_i}(z)$, $(z \in \overline{D},\ t \in [0,1])$, satisfies $d_{\mathcal{C}^0}(u_{k_i}(z), \rho_i(z,t)) < \delta_i$ for all $z \in \overline{D}$, $t \in [0,1]$, and so by definition of the neighbouhood $W_{k_i}(\delta_i)$, this homotopy ρ_i lies inside $W_{k_i}(\delta_i)$. We moreover conclude that the size of ρ_i is less than δ_i, and therefore the homotopy $\varphi'_i \circ \rho_i$ between $\varphi'_i \circ u_{k_i}$ and $\varphi'_i \circ i^{l_i}_{k_i} = i_0$, lies inside $\varphi'_i(W_{k_i}(\delta_i))$, and has size less than $\delta_i + 8\varepsilon_i \leqslant 9\varepsilon_i$ (recall that $d_{\mathcal{C}^0}(\varphi'_i, \mathrm{Id}\,) < 4\varepsilon_i$).

We therefore have $\varphi'_i(W_{k_i}(\delta_i)) \supset i_0(\overline{D})$, and moreover the homotopy $\varphi'_i \circ \rho_i$ between $\varphi'_i \circ u_{k_i}$ and i_0, lies inside $\varphi'_i(W_{k_i}(\delta_i))$, and is of size less than $9\varepsilon_i$. Hence by choosing a sufficiently large $k_{i+1} > k_i$ and denoting $\varepsilon_{i+1} = \frac{1}{2^{k_{i+1}}}$, we get

$$\varphi'_i(W_{k_i}(\delta_i)) \supset W^0(\varepsilon_{i+1}) \supset u_{k_{i+1}}(\overline{D}),$$

and moreover the homotopy between $\varphi'_i \circ u_{k_i}$ and $u_{k_{i+1}}$, given by the concatenation of $\varphi'_i \circ \rho_i$ and of the linear homotopy between i_0 and $u_{k_{i+1}}$, lies in $\varphi'_i(W_{k_i}(\delta_i))$ and still has size less than $9\varepsilon_i$. Applying the quantitative h-principle for symplectic discs [BO16], we get a Hamiltonian diffeomorphism φ''_i supported in $\varphi'_i(W_{k_i}(\delta_i))$, such that $\varphi''_i \circ \varphi'_i \circ u_{k_i} = u_{k_{i+1}}$ and $d_{\mathcal{C}^0}(\varphi''_i, \mathrm{Id}\,) < 18\varepsilon_i$.

As a result, the composition $\varphi_i := \varphi''_i \circ \varphi'_i$ is supported in $W^0(\varepsilon_i) \subset \Phi_{i-1}(U_{i-1})$, we have $\varphi_i \circ u_{k_i} = u_{k_{i+1}}$,

$$\varphi_i \circ \Phi_{i-1}(U_i) = \varphi''_i \circ \varphi'_i \circ \Phi_{i-1}(U_i) \supset \varphi''_i \circ \varphi'_i(W_{k_i}(\delta_i)) = \varphi'_i(W_{k_i}(\delta_i)) \supset W^0(\varepsilon_{i+1}),$$

and

$$d_{\mathcal{C}^0}(\mathrm{Id}\,,\varphi_i) \leqslant d_{\mathcal{C}^0}(\mathrm{Id}\,,\varphi_i') + d_{\mathcal{C}^0}(\mathrm{Id}\,,\varphi_i'') < 22\varepsilon_i.$$

This finishes the step i of the inductive construction.

To summarize, we have inductively constructed a sequence of Hamiltonian diffeomorphisms $\varphi_0, \varphi_1, \dots$ with uniformly bounded compact supports in $\mathbb{C}^3$, such that:

(i) φ_i has support in $W^0(\varepsilon_i) \subset \Phi_{i-1}(U_{i-1})$ where $\Phi_i = \varphi_{i-1} \circ \cdots \circ \varphi_0$,

(ii) $d_{\mathcal{C}^0}(\mathrm{Id}\,,\varphi_i) < 22\varepsilon_i = \frac{22}{2^{k_i}}$,

(iii) $u_{k_{i+1}} = \varphi_i \circ u_{k_i}$.

It follows by (ii) that Φ_i is a Cauchy sequence in the $\mathcal{C}^0$ topology, hence uniformly converges to some continuous map $\Phi : \mathbb{C}^3 \to \mathbb{C}^3$. Next, since $u_{k_{i+1}} = \varphi_i \circ u_{k_i}$ for every $i \geqslant 0$, we have $i_0 = \Phi \circ u_0$. Finally, we claim that Φ is an injective map, hence a homeomorphism. To see this, consider two points $x \neq y \in U_0 = \mathbb{C}^3$. If $x, y \in u_0(\overline{D})$, then by (iii), $\Phi(x) = i_0 \circ u_0^{-1}(x) \neq i_0 \circ u_0^{-1}(y) = \Phi(y)$. If $x, y \notin u_0(\overline{D})$, then $x, y \in {}^cU_i$ for i large enough, so by (i), $\Phi_i(x) = \Phi_{i+1}(x) = \Phi_{i+2}(x) = \dots = \Phi(x)$, and similarly $\Phi_i(y) = \Phi(y)$ (because for each $j > i$, the support of φ_j lies in $\Phi_{j-1}(U_{j-1}) \subset \Phi_{j-1}(U_i)$), so $\Phi(x) = \Phi_i(x) \neq \Phi_i(y) = \Phi(y)$. Finally, if $x \in u_0(\overline{D})$ and $y \notin u_0(\overline{D})$, then $y \in {}^cU_i$ for i large enough, and so $\Phi(y) = \Phi_i(y) \in \Phi_i({}^cU_i) \subset {}^cW^0(\varepsilon_{i+1})$ by (i). Since $\Phi(x) \in \mathrm{Im}\, i_0 \subset W^0(\varepsilon_{i+1})$, we conclude that also in this case we have $\Phi(x) \neq \Phi(y)$. □

3.2 Relative Eliashberg-Gromov $\mathcal{C}^0$-rigidity

Here we address the following question which appeared in our earlier work [BO16]:

Question 1. *Assume that a symplectic homeomorphism h sends a smooth submanifold N to a submanifold N', and that $h_{|N}$ is smooth. Under which conditions $h^*\omega_{|N'} = \omega_{|N}$?*

Of course, that question is non-trivial only when $\dim N$ is at least 2, which we assume henceforth. The question is particularly interesting in the setting of pre-symplectic submanifolds. Recall that a submanifold $N \subset (M, \omega)$ is called pre-symplectic if ω has constant rank on M. The *symplectic dimension* $\dim^\omega N$ of a pre-symplectic submanifold N is the minimal dimension of a symplectic submanifold that contains N. One checks immediately that $\dim^\omega N = \dim N + \mathrm{Corank}\,\omega_{|N}$.

In [BO16], we answered question 1 in various cases of the pre-symplectic setting. Theorem 2 allows to address almost all the remaining cases. Our next result incorporates these remaining cases, together with those verified in [BO16]:

Theorem 3. *Let $N \subset (M^{2n}, \omega)$ be a pre-symplectic disc. Then the answer to question 1 is*

- *Negative if* $\dim^\omega N \leqslant 2n - 4$*, or if* $\dim^\omega N = 2n - 2$ *and* $\mathrm{Corank}\,\omega_{|N} \geqslant 2$.

- *Positive if* $\dim^\omega N = 2n$*, or if* $\dim^\omega N = 2n-2$ *and* $\operatorname{Corank}\omega_{|N} = 0$.

The only case that remains open is when $\dim^\omega N = 2n-2$ and $\operatorname{Corank}\omega_{|N} = 1$ (i.e. $\dim N = 2n-3$, $\operatorname{Corank}\omega_{|N} = 1$).

Proof of theorem 3: When $\dim^\omega N \leqslant 2n-4$ and N is not isotropic, the answer is negative because we can find a symplectic homeomorphism that fixes N and contracts the symplectic form (by [BO16]). When $\dim^\omega N \leqslant 2n-2$ and $r := \operatorname{corank}\omega_{|N} \geqslant 2$, there is a local symplectomorphism that takes N to $[0,1]^r \times D^k \times \{0\} \subset \mathbb{C}^r_{(z)} \times \mathbb{C}^k_{(z')} \times \mathbb{C}^m_{(w)}$, where $m \geq 1$ and $r \geq 2$. By theorem 2, we can find a symplectic homeomorphism $f(z_1, z_2, w_1)$ of $\mathbb{C}^2 \times \mathbb{C}$ which takes $[0,1]^2 \times \{0\}$ to a symplectic disc. The induced map

$$\begin{array}{rccl} \tilde{f} & : & \mathbb{C}^2_{(z_1,z_2)} \times \mathbb{C}_{(w_1)} \times \mathbb{C}^{r-2} \times \mathbb{C}^k \times \mathbb{C}^{m-1} & \longrightarrow & \mathbb{C}^n \\ & & (z_1, z_2, w_1, z_3, \dots, z_r, z'_1, \dots, z'_k, w_2, \dots, w_m) & \longmapsto & f(z_1, z_2, w_1) \times \mathrm{Id} \end{array}$$

is obviously a symplectic homeomorphism which takes N to a submanifold on which the co-rank of the symplectic form is reduced by 2. Note that this argument also works when $\dim^\omega N \leqslant 2n-4$ and N is isotropic. The second item of the theorem was proved in [BO16].$\square$

References

[BHS18] Lev Buhovsky, Vincent Humilière, and Sobhan Seyfaddini. A C^0 counterexample to the Arnold conjecture. *Invent. Math.*, 213(2):759–809, 2018.

[BO16] Lev Buhovsky and Emmanuel Opshtein. Some quantitative results in $\mathcal{C}^0$ symplectic geometry. *Invent. Math.*, 205(1):1–56, 2016.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[Gro86] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

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