V.2

On Arithmetization

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Hardly used today, the term "arithmetization" (*Arithmetisierung, arithmétisation*) was in use around 1900 as a generic description of various programmes which provided non-geometrical foundations of analysis, or other mathematical disciplines. These programmes included constructions of the continuum of real numbers from (infinite sets, or sequences, of) rational numbers, as well as clarifications of the notion of function, limit, etc.[1]

More or less detailed descriptions of arithmetization can be found in every history of XIX$^{\text{th}}$ century mathematics, and numerous special studies have been published.[2] The *raison d'être* of the present chapter in this book is the question whether (and in which way) Gauss's *Disquisitiones Arithmeticae*, and the image of arithmetic they created, influenced the arithmetization of analysis.[3] There is no simple-minded answer to this question because the arithmetization of analysis was a multi-faceted process which, at any given time, was represented by mathematicians with different, often conflicting agendas. For instance, the antagonism between Richard Dedekind's

1. The word "arithmetization" was taken up in other contexts in the 1930s: for the Gödelization of formalized theories, and by Oscar Zariski to describe his rewriting of Algebraic Geometry which was inspired by Wolfgang Krull's "arithmetic" theory of ideals and valuations. Such later developments will not be treated in this chapter.
2. From the early encyclopedia articles [Pringsheim 1898], [Molk 1909] to a special study like [Dugac 1976], and a more reflective general essay like [Jahnke, Otte 1981]. Among recent publications, we mention [Boniface 2002] and [Dugac 2003], and recommend particularly [Epple 1999/2003] as a concise introduction to the subject.
3. The present chapter is therefore a natural continuation of J. J. Ferreirós's chap. III.2 above – cf. [Bekemeier 1987]; the retrospective usage of the word arithmetization in reference to Cauchy, Ohm, and others was encouraged by Klein, see § 3.2 below – and is partly parallel to J. Boniface's chap. V.1.

approach and Leopold Kronecker's, which was discussed in § 3.4 of chap. I.2 above in the context of the further development of Kummer's theory of ideal numbers, reappears here via conflicting programmes of arithmetization.

In order to better understand the history of arithmetization, we distinguish major periods of it. The final answer to the initial question suggested by our investigation is that Gaussian elements become blurred to the point of being undetectable as soon as the Göttingen *nostrification* presented arithmetization as a unified movement in the last years of the XIX$^{\text{th}}$ century; see § 3.2 below.

1. The End of the Theory of Magnitudes[4] in 1872

The general post-XVII$^{\text{th}}$ century notion of number, commonly accepted until the middle of the XIX$^{\text{th}}$ century, was formulated for instance by Newton like this:

> By number we understand not a multitude of units, but rather the abstract ratio of any one quantity to another of the same kind taken as unit. Numbers are of three sorts; integers, fractions, and surds: an integer is what the unit measures, the fraction what a submultiple part of the unit measures, and a surd is that with which the unit is incommensurable.[5]

Numbers were thus defined in terms of magnitudes, or quantities; the foundation of the continuum was geometry, or at any rate not arithmetic.

The year 1872 saw the publication of four papers in Germany each of which presented a new arithmetic theory of the real numbers detaching numbers from magnitudes.[6] In § 1 and § 2, we recall salient features of these theories. We start with Charles Méray from Dijon who had already published his arithmetization in France slightly earlier.

1.1. Charles Méray

Charles Méray[7] seems to have been the first to publish an arithmetization of the irrational numbers. It appeared in 1870 as part of the report of the 1869 congress of the *Sociétés savantes* and seems to have gone unnoticed on what would soon be the other side of the war lines, in Germany.[8] Yet, there were analogies: Méray and

4. We borrow this very appropriate title from [Epple 1999/2003].

5. [Newton 1707], p. 2: *Per numerum non tam multitudinem unitatum quam abstractam quantitatis cujusvis ad aliam ejusdem generis quantitatem quæ pro unitate habetur rationem intelligimus. Estque triplex ; integer, fractus & surdus: Integer quem unitas metitur, fractus quem unitatis pars submultiplex metitur, & surdus cui unitas est incommensurabilis.*

6. [Kossak 1872] (containing an incomplete digest of Karl Weierstrass's introduction of real numbers), [Heine 1872] (based on what he had learned from Cantor), [Cantor 1872], and [Dedekind 1872].

7. See [Boniface 2002], pp. 48–56, for biographical notes on Méray (1835–1911).

8. [Méray 1869]. In his 1899 report on Méray to the Academy, Henri Poincaré described Méray and Weierstrass as working on different planets; see [Dugac 1973], p. 139: *les deux savants ont travaillé d'une façon aussi indépendante que s'ils avaient habité des planètes différentes.*

Dedekind (see § 1.3 below) shared the provincial situation within their countries and dissatisfaction with the lack of foundational rigour in the usual teaching of analysis. Both considered "such an elementary and arid subject" almost unfit for an ordinary mathematical publication.[9] But both insisted that formulas such as $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ had to be justified. Dedekind would actually claim in 1872 that such propositions "as far as I know have never been really proved,"[10] which corroborates his lack of awareness of [Méray 1869], p. 288. On the other hand, both Méray and Cantor (see § 1.2) used Cauchy sequences of rational numbers – called *variables progressives convergentes* in [Méray 1869] – with the same unpedagogical twist of calling them convergent even before they had served to define their own limit.[11] Méray also called them sequences "having a (fictitious) limit," as opposed to those "that have a (numerical [i.e., rational]) limit."[12]

Sequences that differ by a sequence tending to zero are called *équivalentes* by Méray, but he avoided treating the equivalence classes as objects. In fact, contrary to his successors Dedekind and Cantor, Méray did not construct the continuum from rational numbers, but wanted to eliminate "the rather obscure concept of irrational number."[13] A good deal of analysis thus turned merely symbolic in Méray's view:

> Finally, a sign adequate to recall both the nature of the calculations which define v_n and the rational values of the quantities with which they are to be performed, will conveniently designate in the language the fictitious limit. The same sign could represent in the formulas the undetermined rational number which represents its approximate value, computed to a higher and indefinitely growing degree of approximation, i.e., really, any progressive variable equivalent to v.[14] … [A]ny equation between rational or irrational quantities is really the abridged and picturesque enunciation of the fact that certain calculations performed on the rational value of those, on progressive variables which have the others as fictitious limits, and if necessary on integers tending to infinity, yield a progressive variable tending to zero independently of the relation established between these integers and independently of the way in

9. [Méray 1869], p. 281: *C'est ce que je me propose d'exposer aussi brièvement que le commande la nature élémentaire et aride d'un pareil sujet.*
10. [Dedekind 1872/1932], p. 330: *man gelangt auf diese Weise zu wirklichen Beweisen von Sätzen (wie z.B. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), welche meines Wissens bisher nie bewiesen sind.*
11. [Méray 1869], p. 284: *Il nous faut un terme spécial pour exprimer la propriété remarquable …* [$|v_{n+p} - v_p| \to 0$]: *je dirai que la variable progressive v est* convergente. See footnote 26 below.
12. [Méray 1869], p. 284: *pour exprimer la convergence de la variable, on dira simplement:* elle a une limite (fictive).
13. [Méray 1869], p. 281: … *on échappe à la nécessité d'introduire dans le raisonnement la conception assez obscure de nombre incommensurable.*
14. [Méray 1869], p. 285: *Enfin un signe quelconque propre à rappeler, à la fois, la nature des calculs qui définissent v_n et les valeurs numériques des quantités sur lesquelles on doit les exécuter, désignera commodément dans le langage, la limite fictive; le même signe pourra représenter dans les formules le nombre indéterminé qui en représente la valeur approchée, calculée à un degré d'approximation de plus en plus et indéfiniment élevé, c'est-à-dire, au fond, toute variable progressive équivalente à v.*

> which we change the nature of the progressive variables used for the calculation, provided they remain equivalent.[15]

The identity $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ is thus analyzed as saying that, "if α, β, γ are any rational sequences whose squares tend to a, b, ab, then the difference $\alpha\beta - \gamma$ tends to zero."[16] Méray's 1869 note ends with such explanations and never discusses the completeness of the continuum, even though one of the principles of the mathematical theory of limits isolated at the beginning of [Méray 1869] (p. 280) is the convergence of Cauchy sequences.[17] His goal to eliminate irrationals from the theory is a prominent point of contact with Kronecker's programme of arithmetization (see § 2 below), even though Méray's distinction betweeen *le langage*, i.e., the symbolic formalism of roots and other irrationals habitually used in analysis, and *le calcul*, performed exclusively in the domain of rational numbers, seems at odds with Kronecker's precise ideas about how to reduce analysis to general arithmetic.

The primal difference between Méray and his German successors, however, was that he was not seeking the *arithmetization* of analysis, but its *algebraization*. He shared no scientific ideal nurtured by number theory; he did not pretend, as Dedekind would, to have fathomed the essence of continuity. Méray's ideal of rigour was formal and algebraic; his hero was not the Gauss of the D.A., but Lagrange, the algebraic analyst. And when Méray pleaded for building function theory not on the turbid notion of continuity but on analyticity and the algebra of power series, this was again a reference to Lagrange, not to Weierstrass.[18]

1.2. Georg Cantor's Extension of a Result in the Theory of Trigonometric Series

According to his own *curriculum vitae*, Georg Cantor studied both Gauss's *Disquisitiones Arithmeticae* and Legendre's *Théorie des nombres* around 1866, and these readings inspired his 1867 doctoral dissertation[19] as well as his 1869 habilitation memoir.[20] One of the theses he proposed for his doctoral defense was: "In arithmetic, purely arithmetic methods are vastly superior to analytic ones."[21]

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15. [Méray 1869], p. 287: *une équation ... entre des quantités commensurables ou incommensurables: c'est l'énonciation abrégée et pittoresque du fait que certains calculs opérés sur la valeur numérique des unes, sur des variables progressives qui ont les autres pour limites fictives, et au besoin sur des nombres entiers croissant à l'infini, donnent une variable progessive qui tend vers zéro, quelque relation que l'on établisse entre ces nombres entiers et de quelque manière que l'on change la nature des variables progressives soumises au calcul, pourvu qu'elles restent équivalentes à elles-mêmes.*

16. [Méray 1869], p. 288: *signifie que α, β, γ étant des variables commensurables quelconques, dont les carrés tendent vers a, b, ab, la différence $\alpha\beta - \gamma$ tend vers zéro.*

17. The completeness of the continuum can be reformulated in terms of rational sequences with multiple indices; in this way it is at least implicitly treated in [Méray 1887], § 14.

18. [Méray 1872], pp. XI–XXIII. Méray called an analytic function *fonction olotrope*.

19. [Cantor 1932], p. 31. The dissertation is about the integral zeros of ternary quadratic forms, and picks up from D.A., art. 294.

20. [Cantor 1932], pp. 51–62, on the transformation of ternary quadratic forms.

21. [Cantor 1932], p. 31: *In arithmetica methodi mere arithmeticae analyticis longe praestant.*

However, contrary to Dedekind's foundational motivation – see subsection 1.3 below – Cantor's theory of "numerical magnitudes in a large sense" (*Zahlengrößen im weiteren Sinne*) was a necessary technical ingredient to formulate the main theorem of [Cantor 1872/1932], and its presentation is accordingly "sketchy" (p. 92).[22] Cantor's main theorem (§ 3, p. 99) says that a Fourier series which is zero everywhere except possibly in a point set "of the ν^{th} kind" (*Punktmenge der* ν^{ten} *Art*) is in fact identically zero. Cantor called a point set of the ν^{th} kind, if ν successive "derivations" of the set, i.e., passing to the set of its accumulation points ν times, leaves only a finite set of points.

Cantor, who had attended Weierstrass's and Kronecker's lectures in Berlin, did point out that his "definitions and operations may serve to good purpose in infinitesimal analysis" (p. 96). One gathers from §§ 1–2 that they amount to a "general, purely arithmetical theory of magnitudes, i.e., one which is totally independent of all geometric principles of intuition." But Cantor first stated this fact explicitly only in 1882.[23] In his 1872 paper, Cantor's *Zahlengrößen* serve a *dual purpose*: they allow him to define the real numbers via (equivalence classes of) Cauchy sequences, but they also give rise to particular sets of rational numbers in the continuum whose limit points are "of the ν^{th} kind."[24]

An "infinite series given by a law" $a_1, a_2, \ldots, a_n, \ldots$ such that "the difference $a_{n+m} - a_n$ becomes infinitely small as n grows" is said to "have a definite limit," or that it is a numerical magnitude in the large sense (§ 1). Given several such series, Cantor associated symbols to them, $b, b', b,'' \ldots$, and defined relations:

$$(1) \quad b = b', \qquad (2) \quad b > b', \qquad (3) \quad b < b',$$

as $a_n - a'_n$ for growing n becomes infinitely small (case 1), stays bigger than a certain positive number (case 2), or stays smaller than a certain negative number (case 3). However, the equality relation $b = b'$ thus defined does *not* mean that Cantor used the symbols $b, b', \ldots$, or the words *Zahlengröße, Grenze*, etc., for equivalence classes of Cauchy sequences: "the identification of two numerical magnitudes $b, b' \ldots$ does not include their identity, but only expresses a certain relation which takes place between the series to which they refer."[25] Cantor wrote $b * b' = b''$, for $*$ denoting any one of the operations $+, -, \times, /$, if the elements of the corresponding series satisfy $\lim(a_n * a'_n - a''_n) = 0$. From these definitions it follows in particular that $b - a_n$ becomes infinitely small for growing n. This justifies *a posteriori* the initial parlance of the "definite limit."[26]

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22. Page- or §-numbers in this subsection refer to [Cantor 1872/1932]: *nur andeutungsweise*.
23. [Cantor 1879–1884/1932], p. 156, note: *eine allgemeine, rein arithmetische, d.h. von allen geometrischen Anschauungsgrundsätzen vollkommen unabhängige Größenlehre.*
24. [Cantor 1872], pp. 98–99. For a presentation which focusses exclusively on the construction of real numbers, see [Cantor 1879–1884], part IV, § 9.
25. [Cantor 1872/1932], p. 95: … *indem ja schon die die Gleichsetzung zweier Zahlengrößen* b, b' *aus B ihre Identität nicht einschließt, sondern nur eine bestimmte Relation ausdrückt, welche zwischen den Reihen stattfindet, auf welche sie sich beziehen.*
26. This unpedagogical twist was later avoided in [Cantor 1879–1884/1932], p. 186f. There

If B is the domain of all numerical magnitudes thus obtained, Cauchy sequences of elements of B can be formed because the condition that $b_{m+n} - b_n$ becomes infinitely small as n grows "is conceptually completely determined by the previous definitions" (p. 95).[27] After the obligatory definitions of relations and operations in the domain C thus obtained, one may again form Cauchy sequences from elements of C, and so forth. Cantor called those numerical magnitudes "of the λ^{th} kind" which are obtained as the result of exactly λ subsequent limit processes. He pointed out that, for $\lambda \geq 1$, each numerical magnitude of the λ^{th} kind can be "set equal" to a numerical magnitude of the μ^{th} kind, for all $1 \leq \mu \leq \lambda$ (the continuum is complete).[28] But he insisted on the conceptual difference between the ways in which magnitudes of different kinds are given; a magnitude of the λ^{th} kind will in general be a λ-fold infinite array of rational numbers.[29]

Given a unit, a point on an oriented line with origin o is "conceptually determined" (*begrifflich bestimmt*) by its abscissa. This is unproblematic, if the abscissa is rational. Conversely, if the point is effectively given, "for instance by a construction," then there will be a sequence of points with rational abscissas a_n which will "get infinitely close, as n grows, to the point which is to be determined." In this case, Cantor says that "the distance from o of the point to be determined equals b," where b is the numerical magnitude given by the sequence (a_n) (§ 2, p. 96). One verifies that the topological ordering of the distances to o coincides with the ordering of the corresponding numerical magnitudes. The statement that *every* numerical magnitude (of any order) also determines a point on the line with the corresponding abscissa, is postulated by Cantor as an *axiom*, "since it is in the nature of this statement that it cannot be proven."[30] It endows

> the numerical magnitudes *a posteriori* with a certain objectivity, from which they are, however, totally independent.[31]

Cantor called "fundamental series" (*Fundamentalreihen*) what we call Cauchy sequences today.

27. Cantor's set phrase *begrifflich ganz bestimmt* sounds like a preemptive defense against constructivist criticism. According to a letter from Cantor to Hermann Amandus Schwarz (see [Cantor 1991], p. 24: March 30, 1870), Leopold Kronecker had doubts about the "Weierstrass-Bolzano Theorem" to the effect that a continuous function on a closed interval attains the boundaries of its range. For Cantor, this theorem was fundamental, and Schwarz needed it to complete a proof of Cantor's first identity theorem for Fourier series; cf. [Cantor 1870], p. 141. It was also the main goal of [Heine 1872]. The fact that the article [Cantor 1872] did not appear in *Journal für die reine und angewandte Mathematik* like most of his preceding articles on the subject, but in *Mathematische Annalen*, may be related to Kronecker's criticism.
28. Here Cantor never read "=" as "equal," but rather as "set equal" or the like. Dedekind failed to appreciate the interest of this distinction: [Dedekind 1872/1932], p. 317.
29. [Cantor 1872/1932], p. 95f: *im allgemeinen λfach unendlichen Reihen rationaler Zahlen.*
30. [Cantor 1872/1932], p. 97: *weil es in seiner Natur liegt, nicht allgemein beweisbar zu sein.*
31. [Cantor 1872/1932], p. 97: *Durch ihn wird denn auch nachträglich den Zahlgrößen eine*

Admitting this axiom, the points on the line correspond precisely to the equivalence classes of Cauchy sequences.

Already in 1872, Cantor had in mind the transfinite extension of his hierarchy of numerical magnitudes "of the λ^{th} kind":

> The results of analysis (except for a few known exceptions) can all be reduced to such identifications [of numerical magnitudes of different kinds], even though (we just touch upon this here, with a view to those exceptions) the concept of number, as far as it has been developed here, carries in itself the germ for an inherently necessary and absolutely infinite extension.[32]

The transfinite ordinals[33] thus appear as the true completion of Cantor's programme of arithmetization of analysis. They are in contradiction with Gauss's rejection of completed infinites,[34] and thus also a long way from Cantor's arithmetic beginnings in the spirit of the D.A. The theory expounded in [Cantor 1872] also violated Kronecker's constructivity requirement; Cantor gave names ("$b, b', b'', \ldots$") to objects of which it may not be decidable in a finite number of steps whether two of them can be "set equal" to one another.

Cantor's 1872 paper not only defended the freedom to form new concepts, even non-constructively, but also tried to demonstrate the usefulness of distinguishing between sets of various "kinds." The first, methodological aspect makes it similar to [Dedekind 1872]. Their respective axiomatic treatments of the relationship between the arithmetized continuum and points on a line are also quite analogous.[35] The main difference from Dedekind is Cantor's concern for hierarchies according to the way the real numbers are given.[36]

1.3. Richard Dedekind on Continuity and Irrational Numbers

It was under the influence of Dirichlet and Riemann that Richard Dedekind developed his markedly conceptual approach to mathematics. He also traced this "decision for

gewisse Gegenständlichkeit gewonnen, von welcher sie jedoch ganz unabhängig sind.

32. [Cantor 1872], p. 95: *Auf die Form solcher Gleichsetzungen lassen sich die Resultate der Analysis (abgesehen von wenigen bekannten Fällen) zurückführen, obgleich (was hier nur mit Rücksicht auf jene Ausnahmen berührt sein mag) der Zahlenbegriff, soweit er hier entwickelt ist, den Keim zu einer in sich notwendigen und absolut unendlichen Erweiterung in sich trägt.*

33. See [Cantor 1879–1884/1932], part IV, p. 167: Cantor had originally regarded them as "infinite whole numbers."

34. [Cantor 1991], p. 148f: Cantor to Lipschitz, November 19, 1883. [Cantor 1879–-1884/1932], p. 189.

35. Cf. [Cantor 1889]. In 1882, Cantor claimed that the "hypothesis of the continuity of space" could only mean that the space underlying the phenomena of our experience was in perfect one-to-one correspondence with the "purely arithmetical continuum (x, y, z)," and he referred to [Dedekind 1872] and [Cantor 1872]; see [Cantor 1879–1884/1932], part III, p. 156. To be sure, this hypothesis itself was for Cantor an arbitrary one: *die an sich willkürliche Voraussetzung.*

36. It is tempting but anachronistic to interpret this in the light of later criticism of impredicative definitions, like for instance in [Weyl 1918].

the intrinsic against the extrinsic" back to his reading of Gauss's *Disquisitiones Arithmeticae*.[37] His little brochure [Dedekind 1872] – a present to his father on the occasion of his 50 years in office, rather than an article in a mathematical journal – is a showcase example of his method; Dedekind exhibited a conceptual analysis of the continuity of the line, and the way in which "the discontinuous domain of the rational numbers has to be completed into a continuous one" follows from it by necessity (p. 323).[38] In particular, Dedekind considered his analysis not as a purely *ad hoc* construction but was convinced that he had discovered a fundamental principle:[39]

> If all the points of the line fall into two classes in such a way that each point of the first class lies left of each point of the second class, then there is one and only one point which produces this partition into two classes, this cutting up of the line.[40]

For Dedekind this was an unprovable axiom "by which we invest the line with the idea of continuity."[41] Its validity relied on the fact that "everybody" will find it compatible with his "idea of the line." This implies neither the reality of space nor its actual continuity, if space has indeed an independent existence (p. 323).

Following this lead, Dedekind constructed the irrational real numbers by "creating" one for each cut of the rationals not produced by a rational number, and extended the order relation from rational numbers (where he had carefully analyzed it before) to these new numbers, and found that "this domain $\mathcal{R}$ now also enjoys continuity:"

> If the system $\mathcal{R}$ of all real numbers splits up in two classes $\mathcal{A}_1, \mathcal{A}_2$ in such a way that each number α_1 of the class $\mathcal{A}_1$ is smaller than each α_2 of the class $\mathcal{A}_2$, then there exists one and only one number α which gives rise to this partition.[42]

In conclusion, he proved that this "principle of continuity" is equivalent to the convergence of all bounded monotone sequences, and to the convergence of all Cauchy

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37. See C. Goldstein's and N. Schappacher's chap. I.2, § 1, footnote 52, Dedekind's quote on D.A., art. 76 cited there, and the references given.
38. Simple page numbers in this subsection refer to [Dedekind 1872]. Another example of such conceptual work, analyzed in O. Neumann's chap. II.1, § 3, and alluded to in chap. I.2, § 3.2, is Dedekind's emphasis on the notion of irreducibility for sec. 7 of the D.A.
39. This principle has been interpreted as Dedekind's attempt to contribute to Riemann's notion of continuous manifold; see [Ferreirós 1999], p. 73. Be this as it may, Cantor did try to find such a higher-dimensional generalization; see [Cantor 1991], p. 83: Cantor to Dedekind, September 15, 1882.
40. [Dedekind 1872/1932], p. 322: *Zerfallen alle Punkte der Geraden in zwei Klassen von der Art, daß jeder Punkte der ersten Klasse links von jedem Punkte der zweiten Klasse liegt, so existiert ein und nur ein Punkt, welcher diese Einteilung aller Punkte in zwei Klassen, diese Zerschneidung der Geraden in zwei Stücke hervorbringt.*
41. [Dedekind 1872/1932], p. 323: *durch welches wir die Stetigkeit in die Linie hineindenken.* The fact that, between two distinct points, there are infinitely many others appeared unproblematic for Dedekind; see Kronecker's criticism in his 1891 lectures (§ 2.2 below).
42. [Dedekind 1872/1932], p. 329: *IV. Zerfällt das System $\mathcal{R}$ aller reellen Zahlen in zwei Klassen $\mathcal{A}_1, \mathcal{A}_2$ von der Art, daß jede Zahl α_1 der Klasse $\mathcal{A}_1$ kleiner ist als jede Zahl α_2 der Klasse $\mathcal{A}_2$, so existiert eine und nur eine Zahl α, durch welche diese Zerlegung hervorgebracht wird.*

sequences, thus completing his sketch of a "purely arithmetical and completely rigorous foundation of the principles of infinitesimal analysis."[43] At the same time, he had successfully dissociated the definition of number from the nowhere rigorously defined notion of extensive magnitude (p. 321), and based infinitesimal analysis on (infinite sets of) rational numbers, i.e., ultimately on sets of integers.

The analogy between Dedekind's introduction of irrational numbers via cuts and his replacing Kummer's ideal numbers by ideals, i.e., by infinite sets of algebraic numbers – in other words, the analogy between the ordering of rational numbers according to size, and of algebraic numbers according to divisibility – and the subsequent formal investigation of the sets obtained, strongly suggests that we view these two Dedekindian theories as variations of the same foundational theme. In this sense, Dedekind's "arithmetization"[44] is closely associated with number theory.[45]

2. Arithmetization in the Berlin Way

The publications discussed in § 1 all originated in the province. At the same time and even before 1870, Karl Weierstrass and Leopold Kronecker in Berlin had their own ideas about arithmetization, and had conveyed some of them to their students (like Georg Cantor). But Heine's, Cantor's, and Dedekind's 1872 publications, and possibly other factors, would provoke a greater explicitness in Berlin in the 1870s and 1880s.

2.1. Karl Weierstrass

Weierstrass would later be considered the central figure of arithmetization, in view of the many ambiguities he had cleared up in real and complex function theory, by counterexamples and rigorous exposition. This is why we briefly discuss him here, even though we see at least no *specific* relationship between his approach and the notion of arithmetic initiated by Gauss's *Disquisitiones Arithmeticae*.

Weierstrass's introduction of positive real numbers[46] starts from finite or infinite "aggregates" (*Aggregate*) a of positive fractions $\frac{1}{n}$, i.e., collections of (possibly multiple) copies of such fractions. Finitely many positive multiples of various $\frac{1}{n}$ can be transformed into a multiple of $\frac{1}{d}$, for a common denominator d. This effectively defined equality of finite aggregates and their linear ordering: $a_1 \leq a_2$ if a_1 is transformable by fractional arithmetic into a subaggregate of a_2. Infinite aggregates will in general not admit a common denominator. For two of them, Weierstrass defined $A_1 \leq A_2$ to mean that every finite aggregate which is equal, in the above

43. [Dedekind 1872/1932], p. 316: *rein arithmetische und völlig strenge Begründung der Prinzipien der Infinitesimalanalysis.*

44. In the preface of [Dedekind 1872], he spoke about "discovering the proper origin in the elements of arithmetic" (*seinen eigentlichen Ursprung in den Elementen der Arithmetilk zu entdecken*).

45. To discover the coherence of Dedekind's contributions to various domains is one of the main goals of [Dugac 1976]. See also the thesis [Haubrich 1992], which starts with a chapter on arithmetization, and [Ferreirós 1999].

46. Weierstrass may have had *some* such theory as early as 1841: [Kopfermann 1966], p. 80.

sense, to a finite subaggregate of A_1, is also equal to a finite subaggregate of A_2. If $A_1 \leq A_2$ and $A_2 \leq A_1$, the two infinite aggregates are said to be equal. If all finite aggregates are less than or equal to A, then A equals infinity. All other aggregates (the finite ones and those infinite aggregates which are not infinity) make up Weierstrass's domain of positive real numbers. Negatives are obtained by working with two units opposite to each other; the complex numbers etc. require even more units.

By tracing over the years the growing weight given to foundations in Weierstrass's introductory course on the theory of analytic functions, one gets a first understanding of the way the movement of arithmetization was catching on. We know four versions of this course: Wilhelm Killing's notes from Spring 1868, Georg Hettner's notes of Spring 1874, Adolf Hurwitz's of Spring 1878, and Kurt Hensel's notes, probably from the Winter 1882–1883.[47]

The whole chapter "Introduction to the concept of number" (*Einführung in den Zahlbegriff*) in Killing's notes [Weierstrass 1868] gives the impression of recalling known facts, based on the notion of magnitude or quantity (*Größe* in German). For instance: "If the numerical quantity is given by an infinite series, then it will equal another quantity, if …"[48] We conclude that arithmetization was at least not for the students in 1868.

A keener interest in arithmetization is evident in a letter, written in December 1873 to Paul du Bois-Reymond, where Weierstrass distinguished between various approaches to analysis: either "with the notion of extensive magnitude, or coming from algebra, i.e., from the notion of number and the basic arithmetic operations necessarily implied by it. I myself hold this last path to be the only one by which analysis can be founded with scientific rigour and all difficulties can be solved."[49] In the 1874 lecture notes we read about the theory of complex numbers:

> However, for analysis we need a purely arithmetical foundation which has already been given by Gauss. Even though the geometric presentation of the complex quantities is an essential tool for their investigation, we must not use it here because analysis has to be kept clean of geometry.[50]

47. See [Ullrich 1988], pp. xi–xiv, for the structure of Weierstrass's regular lecture cycle. Hettner's notes may have been worked out only after 1889; at any rate, the copy uses post-1880 orthography. Only Hurwitz's notes date the individual lectures. Hensel's notes in the IRMA library at Strasbourg are undated; we associate them to 1882–1883 on the basis of Hensel's curriculum and a comparison with other notes he took.

48. [Weierstrass 1868], p. 3: *Ist die Zahlengrösse durch eine unendliche Reihe gegeben, so wird sie einer andern Grösse gleich sein, wenn …*

49. [Weierstrass 1923], p. 203f: *je nachdem man von geometrischen und physikalischen Vorstellungen ausgehend, also mit dem Begriff der extensiven Größe, das Gebiet der Analysis betritt oder von der Algebra aus, d.h. dem Zahlbegriff und den mit demselben notwendig gegebenen arithmetischen Grundoperationen. Ich halte den letzteren Weg für den, auf welchem allein sich die Analysis mit wissenschaftlicher Strenge begründen läßt und alle Schwierigkeiten sich beseitigen lassen.*

50. [Weierstrass 1874], p. 6: *Wir bedürfen jedoch für die Analysis einer rein arithmetischen Begründung, die schon Gauss gegeben hat. Obgleich die geometrische Präsentation der complexen Grössen ein wesentliches Hülfsmittel zur Untersuchung derselben ist, können*

And the strongest arithmetization programme is formulated in the 1882–1883 lectures:

> For the foundation of pure analysis all that is required is the concept of number, while geometry has to borrow many notions from experience. We will try here to construct all of analysis from the concept of number.[51]

As of 1874, the lecture courses all develop the theory which we have briefly sketched in the 2nd paragraph of this subsection. A crucial point explicitly made in all three courses is that infinite sums have no meaning other than the one they receive from definitions involving only operations with finite subaggregates.[52] For instance in the 1882–1883 lectures, Weierstrass insisted that the idea of a number determined by infinitely many elements[53] is itself not any more difficult than that of the infinite sequence of natural numbers. And after the general definition of equality, special mention was made of the case where a certain law assures us of the existence of all of its elements, even if we are not able to effectively specify them.[54]

The relationship between the arithmetized numbers and points on a line was still treated as unproblematic in [Weierstrass 1874], p. 41. That each line segment corresponds to a numerical quantity was mentioned there in passing (p. 76). The problem whether to each numerical quantity also corresponds a point, was spirited away by the convention that "a single value of a complex quantity be called a point."[55] The 1878 lectures were more elaborate in this respect. For the existence of a point (on a line with marked origin P and unit) corresponding to a given numerical quantity, say a, Weierstrass considered (for a particular example) all the points X for which the segment PX is smaller than the segment corresponding to some finite quantity contained in a, and all points Y for which PY is greater than all the segments corresponding to a finite quantity contained in a. He then argued directly (without explicitly alluding to Dedekind for this intuitive cut-argument):

> The points X and the points Y now form one continuous series of points. There

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wir sie hier nicht anwenden, da die Analysis von der Geometrie rein erhalten werden muss.

51. [Weierstrass 1883], p. 1: *Die reine Analysis bedarf zu ihrer Begründung nur des Begriffes der Zahl, während z.B. die Geometrie viele Begriffe der Erfahrung entlehnen muß. Wir wollen versuchen, hier die gesammte Analysis aus dem Begriffe der Zahl zu construiren.*

52. Cf. Cantor's compliment to Weierstrass on this point in [Cantor 1879–1884/1932], part IV, p. 185.

53. Weierstrass's expression "element" and other features of his presentation may well go back to the tradition of algebraic analysis. See for instance [Stern 1860], p. 9. Cf. [Dirksen 1845], chap. 3, *Abschnitt* 2.

54. [Weierstrass 1883], p. 26f. The example of $\sqrt{2}$ given thereafter carries Hensel's note in the margin: "for rational numbers one can specify <u>all</u> the elements, for numbers with infinitely many elements <u>every</u> required element can be specified." (*bei rat. Zahlen kann man <u>alle</u> bei den Zahlen mit unendl. vielen Elementen <u>jedes</u> verlangte Element angeben.)*

55. [Weierstrass 1874], p. 116: *Wir werden häufig einen einzelnen Wert einer complexen Grösse einen Punkt nennen.*

> must therefore be a point which affords the transition from the point series of X to the point series of Y.[56]

In [Weierstrass 1883], however, only the numerical quantity corresponding to a given ratio of segments is explained in detail, whereas the inverse problem is dismissed with the remark that one may "imagine" the necessary construction "done."[57]

In conclusion, by 1874 Weierstrass's theory was built exclusively on an arithmetized notion of quantity. The relationship to extensive quantities is discussed, albeit less profoundly than by Dedekind or Cantor. One may read the evolution of Weierstrass's presentations over the years as a movement towards a more constructivist point of view, where "ideas" or "imaginations" (*Vorstellungen*), i.e., processes of the mind, are appealed to in order to smoothen the acceptance of infinites. This may have been the result of his ongoing dialogue with Kronecker.[58]

2.2. Leopold Kronecker

In Part I of this book, Leopold Kronecker's role in the history of Gauss's *Disquisitiones Arithmeticae* has been discussed under two headings: in chap. I.1, § 4.3, he appeared as a representative of the field of arithmetic algebraic analysis, whereas his theory of algebraic numbers and functions was described as one of the alternatives to Dedekind's theory of ideals in chap. I.2, § 3.4. Kronecker's programme of arithmetization was based on the same method as his theory of algebraic numbers and functions – i.e., the adjunction of indeterminates and the reduction of the polynomials obained with respect to module systems – also to incorporate all of analysis into a unified "General Arithmetic."[59]

Unlike his Berlin colleague Weierstrass, Kronecker was not concerned with designing a coherent, up-to-date presentation of function theory, including pathological counterexamples etc., for he was perfectly happy with the parts of classical analysis, especially elliptic and modular functions, that he had himself enriched. Nor was he interested in Cantor's set theoretical innovations and the completed infinites involved in them. And unlike Dedekind, Kronecker was not looking for conceptual analyses (even less, if they employed completed infinites) of such notions as continuity which for him were germane to geometry or mechanics.

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56. [Weierstrass 1878], p. 22: *Die Punkte X und die Punkte Y bilden nun eine stetige Reihe von Punkten, es muß also einen Punkt geben, der den Übergang von der Punktreihe X zur Punktreihe Y vermittelt.*

57. [Weierstrass 1883], p. 197: *Dann läßt sich jede Z-Gr [Zahlengrösse] geometrisch dadurch darstellen, daß man eine Strecke gebildet denkt, welche aus der Längeneinheit a und deren genauen Theilen gerade so zusammengesetzt ist, wie die zu repräsentierende Z-Gr aus der Haupteinheit und deren genauen Theilen.*

58. Their joint criticism of Riemann's use of Dirichlet's principle in the 1860s is not only confirmed by Casorati's papers (see [Neuenschwander 1978], p. 27), but is also alluded to by [Heine 1870], p. 360.

59. *Allgemeine Arithmetik*. Kronecker also chose this as the title of his standard lecture course at the end of his life; see the beginning of Hensel's preface to [Kronecker 1901], p. V.

Kronecker published his views on arithmetization only in the 1880s.[60] In 1886, the 63 year old Kronecker published his programmatic article "On the concept of number" in a *Festschrift* dedicated to the philosopher Eduard Zeller:

> [A]rithmetic bears a similar relationship to the other two mathematical disciplines, geometry and mechanics, as mathematics as a whole bear to astronomy and the other natural sciences; arithmetic likewise renders manifold services to geometry and mechanics and receives from its sister disciplines a wealth of inspirations in exchange. Here, however, the word "arithmetic" has to be taken not in the usual restrictive sense, but one has to subsume under it all mathematical disciplines except geometry and mechanics, in particular algebra and analysis. And I actually believe that one day one will succeed in "arithmetizing"[61] the complete content of all these mathematical disciplines, i.e., to found them solely and exclusively on the notion of number taken in the strictest sense, thereby peeling away the modifications and extensions of this notion,[62] most of which have been prompted by applications to geometry or mechanics. The fundamental difference between geometry and mechanics on the one hand, and the mathematical disciplines on the other which are here being collected under the label of "arithmetic," is, according to *Gauss*, that the object of the latter, number, is *only* a product of our mind, whereas space as well as time also have a *reality outside* of our mind whose laws we cannot completely impose *a priori.*[63]

60. As in chap. I.2, one has to consult (aside from the more philosophical [Kronecker 1887b] and his last lecture course [Kronecker 1891]) his *Grundzüge* [Kronecker 1881], the paper [Kronecker 1886] where module systems (*Modulsysteme*, introduced in 1881) are applied to algebra, and [Kronecker 1888]. Cf. J. Boniface's chap. V.1.

61. According to his student Jules Molk, Kronecker was the first to use this verb transitively; see [Molk 1909], p. 158, note 78.

62. Kronecker's note: "I here mean in particular the inclusion of irrational numbers and of the continuous quantities."

63. We quote from the extended printing [Kronecker 1887b/1895–1931], vol. 3(1), p. 253: *In der Tat steht die Arithmetik in ähnlicher Beziehung zu den anderen beiden mathematischen Disciplinen, der Geometrie und Mechanik, wie die gesammte Mathematik zur Astronomie und den anderen Naturwissenschaften; auch die Arithmetik erweist der Geometrie und Mechanik mannigfache Dienste und empfängt dagegen von ihren Schwester-Disziplinen eine Fülle von Anregungen. Dabei ist aber das Wort "Arithmetik" nicht in dem üblichen beschränkten Sinne zu verstehen, sondern es sind alle mathematischen Disciplinen mit Ausnahme der Geometrie und Mechanik, also namentlich die Algebra und Analysis, mit darunter zu begreifen. Und ich glaube auch, dass es dereinst gelingen wird, den gesammten Inhalt aller dieser mathematischen Disciplinen zu "arithmetisieren," d.h. einzig und allein auf den im engsten Sinne genommenen Zahlbegriff zu gründen, also die Modificationen und Eweiterungen dieses Begriffs* (Kronecker's footnote: *Ich meine hier namentlich die Hinzunahme der irrationalen sowie der continuirlichen Grössen*) *wieder abzustreifen, welche zumeist durch die Anwendungen auf die Geometrie und Mechanik veranlasst worden sind. Der prinzipielle Unterschied zwischen der Geometrie und Mechanik einerseits und zwischen den übrigen hier unter der Bezeichnung "Arithmetik" zusammengefassten mathematischen Disciplinen andererseits besteht nach* Gauss *darin, dass der Gegenstand der letzteren, die Zahl,* bloss *unseres Geistes Product ist, während der Raum ebenso wie die Zeit auch* ausser *unserem Geiste eine* Realität *hat, der*

Kronecker went on to quote Gauss's letter to Bessel of April 9, 1830.[64] Also the method with which Kronecker would eliminate at least the *algebraic* irrationalities is described as being directly inspired by Gauss:

> [W]ith the *systematic* introduction of indeterminates (*indeterminatae*), which stems from *Gauss*, the special theory of integers was expanded into the general arithmetic theory of entire functions of indeterminates with integral coefficients. This general theory allows to eliminate all notions which are alien to arithmetic proper: that of negative, fractional, real, and imaginary algebraic numbers.[65]

Negative numbers are then multiples of an indeterminate s which is taken modulo $s+1$; a fraction $\frac{1}{b}$ is represented by q_b modulo $b \cdot q_b - 1$, and an algebraic number is handled by working with polynomials modulo its minimal equation.[66] What remains somewhat unclear is how this Kroneckerian programme was to affect the practice of analysis. At least the formal developments in Kronecker's long series of papers on elliptic functions of the late 1880s seem unaffected by the radical arithmetization one might expect from the preceding quotes. A first clue is provided by what Kronecker told the young David Hilbert when the latter paid him a visit in 1888:

> Equal is only $2 = 2$. Irrational and transcendental numbers are given 1.) by implicit representation $\sin x = 0$, $x^2 = 5$, or 2.) by approximation. In general, it is not at all difficult to build everything rigorously on this basis, avoiding Weierstrass's notion of equality and continuity. But at certain critical junctures it is difficult, and there one can never be precise and rigorous enough. Only the discrete and the singular have significance. But the rest one can also obtain, by interpolation. Therefore his goal for the elliptic functions is to admit only the singular moduli,[67] and then build everything arithmetically from there.[68]

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wir a priori ihre Gesetze nicht vollständig vorschreiben können.

64. See J. Ferreirós, chap. III.2 above, § 1.

65. [Kronecker 1887b/1895–1931], vol. 3(1), p. 260: *[M]it der* principiellen *Einführung der "Unbestimmten"* (indeterminatae)*, welche von* Gauss *herrührt, hat sich die specielle Theroie der ganzen Zahlen zu der allgemeinen arithmetischen Theorie der ganzen ganzzahligen Functionen von Unbestimmten erweitert. Diese allgemeine Theorie gestattet alle der eigentlichen Arithmetik fremden Begriffe, den der negativen, der gebrochenen, der reellen und der imaginären algebraischen Zahlen, auszuscheiden.*

66. Cf. [Kronecker 1887a]. [Kronecker 1887b], § 5, III, shows how to separate real conjugates.

67. The singular moduli are analogous to algebraic numbers in that they yield algebraic values for modular and elliptic functions; the arithmetic properties of these values were one of Kronecker's central domains of research. See chap. I.1, § 4.3. Cf. [Schappacher 1998], [Vlăduţ 1991]. At the end of his life, contrary to what he told Hilbert, Kronecker studied a more general theory, deriving invariants of binary quadratic forms which also contained continuous parameters from Fourier developments of elliptic functions for *nonsingular* moduli; see [Kronecker 1932]; cf. [Kronecker 1895–1931], vol. 5, pp. 65–83.

68. See BNUS, Cod. Ms. D. Hilbert 741, "Kronecker," pp. 1/2–1/3: *Gleich sei nur* $2 = 2$. *Irrationale und transzendente Zahlen seien 1.) durch die implizite Darstellung* $\sin x = 0, x^2 = 5$ *gegeben oder 2.) durch Annäherung. Im allgemeinen sei es gar nicht schwer*

But which transcendental functions would be accepted as implicitly defining transcendental numbers, like in the equation "$\sin x = 0$" quoted by Hilbert? It seems that what really gave meaning to transcendental functions for Kronecker, was their role as invariants of some general equivalence relation. Molk gave the concrete example of the function $\pi \cot(\pi u) = \sum_{k=-\infty}^{k=+\infty} \frac{1}{u+k}$ as an invariant of the relation which identifies u with $u+k$ for any integer k.[69]

At any rate, there would hardly be room in Kronecker's arithmetized analysis for Cantor's theory of the continuum or the budding functional analysis, and Kronecker also wanted to keep geometry relegated to its own domain: "In opposition to the notion of continuity, which is present in a certain way in geometry and mechanics, stands the discontinuity of the sequence of numbers."[70] This turns the real line (which was still referred to by way of comparison in [Kronecker 1881/1895–1932], vol. 2, p. 354) into an attempt "to somehow conjure up continuity within arithmetic," and in 1891 Kronecker explicitly stated that it was impossible to order all fractions "in a straight line,"[71] presumably because this would require infinitely many unknowns and relations.[72]

One of Kronecker's most basic *methodological tenets* was *concreteness*: defini-

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auf dieser Grundlage bei Vermeidung des Weierstrass'schen Begriffs der Gleichheit und der Continuirlichkeit alles strenge aufzubauen. Aber bei gewissen kritischen Stellen sei es schwer und da könne man nicht genau und strenge genug sein. Nur das Diskrete und Singuläre habe Bedeutung. Das übrige könne man aber auch erhalten, nämlich durch Interpolation. So sei es bei den elliptischen Funktionen sein Ziel, nur die singulären Moduln zuzulassen und dementsprechend alles arithmetisch aufzubauen.

69. See [Kronecker 1891], p. 238, and [Molk 1909], p. 162. See also the discussion of the invariance of dimension in [Kronecker 1891], pp. 246–247. Cf. [Kronecker 1901], pp. 132–142.

70. [Kronecker 1891], p. 227; see J. Boniface's chap. V.1, footnote 64 for the original quote. In [Kronecker 1888/1895–1931], vol. 3(2), pp. 89–97, and [Kronecker 1891], p. 265, Kronecker discussed approximating a ball by cubes. He concluded more radically in 1891: "There is a volume – the ball – … But there is no number to which the convergent series of fractions tend." (*[W]ohl ist ein Volumen da – nämlich die Kugel – … Nicht aber ist eine Zahl da, welcher die konvergierenden Bruchreihen zustreben.)*

71. [Kronecker 1891], p. 227 (see J. Boniface's chap. V.1, footnote 64) and p. 257: *Unmöglich aber ist es, nach dieser Ausdehnung der Begriffe größer und kleiner auf die Brüche diese in einer geraden Linie ihrer Größe nach anzuordnen. Für eine endliche Zahl gegebener Brüche ist diese Ordnung wohl möglich. Für eine endliche Zahl ist sie aber nicht nötig, für alle Brüche nicht möglich.*

72. In [Kronecker 1886/1895–1931], vol. 3(1), p. 155, he did mention the "purely logical" possibility of "module systems with infinitely many elements," asking however that this "arithmetically not sufficiently precise notion" be reduced to finite module systems in "special arithmetical applications." He added a note where he explained that this caution was not followed in Dedekind's general theory of modules and ideals, and criticized in the same way the definitions of irrational numbers, thinking apparently of Heine, Dedekind, and Cantor.

tions are to be given along with an effective criterion to decide whether or not they apply to a given object;[73] infinite series have to be given so that they are amenable to computation; indirect existence proofs are either avoidable or anathema.[74] Likewise, Kronecker repeated in his 1891 lectures that there was really no need for a "theory of infinite series which define irrational numbers."[75]

Another recurrent methodological topos of Kronecker's was to use adequate equivalence relations, the celebrated model being the equivalence and composition of quadratic forms in Gauss's *Disquisitiones Arithmeticae*.[76] In [Kronecker 1888/1895–1931], vol. 3(1), p. 90, and [Kronecker 1891], p. 266, the author even stretched this idea to the mode "2.) by approximation" in which irrationals can be given (as he had told Hilbert); approximations would be identified if they were in the same "equivalence interval."

This last idea matches the "mathematics of approximation" that Felix Klein propagated as of the early 1870s. To be sure, Klein's theory of "function strips" (*Funktionsstreifen*) was not guided by Gauss's D.A.,[77] but by an analysis of the inherent lack of precision of our spacial intuition, and thus fitted well with the contemporary interest in the psychology of perception. But Klein did acknowledge an indebtedness to a conversation with Kronecker about the impossibility of effectively giving infinitely many terms of a series, which had provided "the first occasion to develop his ideas."[78]

3. Discourses on Arithmetization

By excluding extensive magnitudes from the foundation of analysis, arithmetization modified the relation between mathematics and its applications in the empirical sciences. Concretely, the move looked at first like the retreat into the ivory tower; in an academic speech in 1891, the rector of Innsbruck university, Otto Stolz, refrained from going into details about arithmetization, not just for lack of time, but because he felt "that pure mathematics has not gained in popularity by the immersion into itself in which it currently indulges."[79] (One may recall Méray's and Dedekind's initial

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73. See for instance the discussion of irreducibility in [Kronecker 1881/1895–1931], vol. 2, p. 256–257. In [Kronecker 1891], p. 240, he suggested that definitions be gathered from experience, and mathematics recognize itself as a natural science.

74. See [Kronecker 1891], p. 240; cf. footnote 80 below.

75. [Kronecker 1891], p. 269–271, where he went on to declare that the series $\sum \frac{c_n}{n!}$ with integers c_n such that $0 \leq c_n \leq n-1$ "really exist."

76. [Kronecker 1881/1895–1931], vol. 2, p. 324, and [Kronecker 1891], p. 261f (esp. note 62). Cf. the way in which Kronecker paraded this arithmetic idea in his quarrel with Camille Jordan, e.g. in [Kronecker 1895–1931], vol. 1, p. 418: *En appliquant les notions de l'Arithmétique à l'Algèbre, on peut appeler équivalentes ....*

77. He did, however, refer to Gauss the astronomer in this context; see [Klein 1921–1922], vol. 2, p. 245.

78. [Klein 1873/1921–1922], vol. 2, p. 216, note 6: *Ich bin hierauf gelegentlich von Herrn Kronecker gesprächsweise aufmerksam gemacht worden; in seiner Bemerkung lag für mich wohl der erste Anlaß, mir die in* § *1, 2 des Textes niedergelegte Auffassung zu bilden.*

79. [Stolz 1891], p. 4: *Auch kann ich mir denken, dass die reine Mathematik durch die*

hesitations about their publications.) Stolz, a former Berlin student, mentioned as protagonists of arithmetization Weierstrass and Kronecker, who "have not arrived at agreeing opinions."[80]

3.1. Philosophical Points of View

Contrary to Stolz's hesitations, a number of philosophically inclined mathematicians, and philosophers with their own image of mathematics, were publishing their views as the movement of arithmetization gathered momentum. Within the limits of the present article, we only mention a few names here, as scattered evidence of an ongoing, if apparently unstructured debate. The history of this whole debate is worth looking into and remains to be written.

Hermann Hankel, a student of Riemann's, already published before 1872 (and died in 1873 at age 34). In [Hankel 1867], p. 46, he explicitly doubted the scientific relevance, and in fact the possibility, of defining irrational numbers without appeal to magnitudes. At the same time, however, importing ideas from the British logical school, he began to build a "general arithmetic," i.e., an axiomatic theory of algebraic composition laws which gave him a general, formal notion of number; see [Hankel 1867], pp. VIII, 47. In spite of this modern, formal theory, the gap between arithmetic and analysis appears even wider in his encyclopedia article [Hankel 1871] where the notion of limit separates analysis from arithmetic and algebra, and appears to render any actual arithmetization impossible.

Paul du Bois-Reymond presented in his book [Bois-Reymond 1882] a dialogue between the "idealist" and the "empiricist" with the intention to help mathematicians to greater philosophical clarity, specifically about the existence of the limit of an infinite decimal fraction. The presentation is strongly influenced by the time-honoured interest in processes of thinking and perception. Magnitudes are maintained as a source of inspiration and application of analysis;[81] arithmetization is invited to formally ascertain rigorous proofs (p. 290). Hankel and Bois-Reymond were analogous in that they saw the potential of formal, structural mathematics,[82] but reacted to it conservatively. In Bois-Reymond's case, this reaction is also motivated by the conviction that mathematical analysis "is in truth a natural science." In spite of the later date of [Bois-Reymond 1882], its author (like Hankel) really reacted essentially to pre-1872 forms of arithmetization.[83]

Versenkung in sich selbst, der sie sich gegenwärtig hingiebt, an Popularität nicht gewonnen hat. [Daum 2002] suggests that mathematics remained largely untouched by the big wave of German literature popularizing the natural sciences since about 1850.

80. [Stolz 1891], p. 4: *Dabei sind sie jedoch nicht zu übereinstimmenden Ansichten gelangt.* (Dedekind and Cantor *are* mentioned in a footnote on p. 16.) Incidentally, this speech contains the missing quote in [Kronecker 1891], p. 240, footnote 41: [Stolz 1891], p. 9. See also [Stolz, Gmeiner 1904], p. 148.
81. [Bois-Reymond 1882], p. 54, appeals to Gauss for having called mathematics, "so correctly and profoundly, the science of magnitudes" (*die von Gauss so wahr und so tief Grössenlehre genannte Wissenschaft*).
82. [Bois-Reymond 1882], p. 54, speaks of a symbolic game (*Zeichenspiel*).
83. See [Bois-Reymond 1882], pp. 53–55. His comment on [Heine 1872] on p. 55 is a way

When the physicist Gustav Robert Kirchhoff opposed metaphysical reflections in physics and defined as the goal of mechanics (somewhat vaguely), to describe natural movements as simply and completely as possible, this gave a tremendous boost to the German empiricist, positivist philosophy of science.[84] Even though all variants of arithmetization could probably be made to comply with this philosophy, it was Kronecker who explicitly followed his Berlin colleague Kirchhoff and presented his own programme of arithmetization within Kirchhoff's mould, stressing the analogy of mathematics with other sciences and dividing mathematics into general arithmetic on the one hand and geometry and mechanics on the other.[85]

Against Gauss's, Kirchhoff's, and Kronecker's separation of arithmetic from geometry and mechanics, Leo Königsberger, in his speech [Königsberger 1895], pleaded for a return to Kant's foundation of all mathematics on pure intuitions.

Sightless Eugen Karl Dühring was as of 1877 the most universally hated philosopher on the Berlin academic scene.[86] Dühring considered mathematical notions to be touchstone cases for epistemology. Having determined early on the impossibility of thinking an infinite number,[87] his interest in mathematics increasingly turned into wild criticism of allegedly untenable mathematical notions and tendencies.[88] In [Dühring 1878], pp. 249–265, however, he developed a sort of philosophical programme of arithmetization turning analysis into a perfect form of arithmetic. This, however, did not diminish his polemics: against mathematics in general that he found overrated, and against higher arithmetic in particular.

Another very prolific philosopher, Wilhelm Wundt in Leipzig, tried to justify modern mathematical trends, in particular Dedekind's and Cantor's, against Berlin restrictions to potential infinites; see [Wundt 1883]. His position was neokantian: pure intuition is taken as an abstract notion, not a *Vorstellung*. For him, the fundamental theme of mathematics for the last 2000 years was the mediation between discrete numbers and the continuum. At the same time, he held a similarly skeptical

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of not taking the arithmetization of irrational numbers seriously.

84. See [Kirchhoff 1876] for a concise formulation; cf. [Cornelius 1903].

85. [Kronecker 1881/1895–1931], vol. 2, p. 354; [Kronecker 1891], pp. 226, 252. In [Kirchhoff 1865], p. 5, Kirchhoff had called geometry and mechanics two closely related and equally certain applications of pure mathematics.

86. [Köhnke 1986], pp. 373, 519. He would flirt with socialism (albeit not very successfully; recall Friedrich Engels's *Anti-Dühring* of 1878), and, at least after his removal from Berlin University, would be openly antisemitic, and finally founded a sect.

87. [Dühring 1865], p. 115. Felix Klein was duly impressed by this; see [Klein 1873/1921–1922], vol. 2, p. 215, note 5. We do not know if Kronecker reacted to this early Dühring.

88. His would-be historical treatise [Dühring 1877] has Lagrange as its absolute hero, and is in many respects written from the point of view of French mathematics of the first third of the XIXth century; see pp. 545–549. (This may remind one of Méray, but the contexts, professional identities and styles of both authors do not suggest a fruitful comparison.) Developments originating from the D.A. are described as "pleasures of speculation" (*Speculationsvergnügungen*) without real relevance, and the contemporary analysis and algebra is ridiculed for its hollowness. The same continues in the joint book with his son [Dühring, Dühring 1884].

position on proofs by contradiction as Kronecker, and – possibly under the influence of Klein? – he also insisted on the importance of intuition for mathematics.

Benno Kerry – a young *Privatdozent* of philosophy at Strassburg University, who died in 1899 at age 31 – is usually known today for Gottlob Frege's 1892 replique to him.[89] But in his long series of papers on intuition and its psychic processing (in Wundt's influential journal), he also dealt with Kronecker's arithmetization, criticizing the narrowness of Kronecker's concept of number in general, and the introduction of negative and fractional numbers via indeterminates and congruences in particular, quoting Cantor, Dedekind, and Elwin B. Christoffel for their criticism of Kronecker.[90]

Fig. V.2. Collegiengebäude, Kaiser-Wilhelm-Universität Strassburg (1879–1884) Four German scholars. Kant, Gauss, and J. Müller incarnate the domains entering into the discussions sketched in § 3.1: philosophy, mathematics, and physiology.

Finally, Adolf Elsas published a fundamental criticism of Fechner's psychophysics [Elsas 1886] where (starting p. 53) he criticized the mathematicians for giving up magnitudes. It is voices like his that provide some evidence *ex contrario* for the thesis in [Jahnke, Otte 1981], p. 45, that arithmetization was in fact "a response to the changed relationship between mathematics and the empirical sciences," since new sciences, treating new kinds of magnitudes, asked to be mathematized.

89. In Frege's *Über Begriff und Gegenstand.*

90. [Kerry 1889], pp. 89–92; [Kerry 1890], pp. 319–324. His second point seems to be misguided insofar as it tries to argue with values of the newly adjoined unknown; see [Kerry 1889], pp. 90–91. Kerry calls (p. 92) Kronecker's method "exceedingly cumbersome and complicated" (*überaus schwerfällige und umständliche Weise*).

3.2. The Göttingen Nostrification *of Arithmetization*

As the turn of the century approached, arithmetization, in one form or the other, seemed well established, and the dominant question was no longer, whether analysis should be founded independently of the notion of magnitude, but what arithmetization meant for the unity of mathematics, for the relation among the mathematical disciplines – in particular arithmetic against geometry – and for the applications of mathematics to the sciences. Indeed, the movement of arithmetization could potentially threaten the unity of mathematics, separate arithmetic from geometry, and mathematics from the sciences. It was with this potential threat in mind that Felix Klein gave his address on arithmetization to the Göttingen Academy [Klein 1895]; its timeliness, and the growing importance of the author, is underlined by the prompt translations of it that followed.[91] The speech also marked the beginning of the *nostrification* of arithmetization by the newly emerging mathematical centre at Göttingen.[92]

The starting point of the talk was Weierstrass's 80$^{\text{th}}$ birthday.[93] Klein presented Weierstrass as "the principal representative" of arithmetization.[94] Recalling that the XVIII$^{\text{th}}$ century had been a "century of discoveries" in mathematics, Klein first described the XIX$^{\text{th}}$ century as an aftermath:

> Gradually, however, a more critical spirit asserted itself and demanded a logical justification for the innovations with such assurance, the establishment, as it were, of law and order after the long and victorious campaign. This was the time of Gauss and Abel, of Cauchy and Dirichlet.[95]

Although it may already seem unusual to liken these extremely creative mathematicians to administrators,[96] Klein carried on in the same vein:

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91. Already in his Leipzig inaugurational lecture, which he published 15 years after the event, [Klein 1880/1895], Klein had warned against losing the unity of mathematics.
92. For the Göttingen concept of "nostrification," cf. [Corry 2004], sec. 9.2.
93. October 31, 1895, three days before the address. Weierstrass would die in 1897.
94. In preparatory notes for his 1880–1881 Leipzig classes, he had called Weierstrass's introduction of irrational numbers "arithmetical," and Dedekind's cuts "geometrical" [Klein 1880–1881], p. 264
95. [Klein 1895], p. 966 (English); p. 232 (German).
96. Minkowski would play on Klein's metaphor in his famous Dirichlet centennial speech [Minkowski 1905], p. 451: "One hears about the progressive arithmetization of *all* mathematical disciplines, and some therefore take arithmetic to be nothing but a convenient constitution for the extensive empire of mathematics. Well, in the end some will see it only as the high police which is authorized to check on all unlawful incidents in the widely ramified commonwealth of magnitudes and functions. – *... man hört von der fortschreitenden Arithmetisierung* aller *mathematischen Wissenszweige sprechen, und manche halten deshalb die Arithmetik nur noch für eine zweckmäßige Staatsverfassung, die sich das ausgedehnte Reich der Mathematik gibt. Ja, zuletzt werden einige in ihr nur noch die hohe Polizei sehen, welche befugt ist, auf alle verbotenen Vorgänge im weitverzweigten Gemeinwesen der Größen und Funktionen zu achten.*

> But this was not the end of the matter. Gauss, taking for granted the continuity of space, unhesitatingly used the intuition of space as a basis for his proofs; but closer investigation showed not only that many special points still needed proof, but also that the intuition of space had led to the too hasty assumption of the generality of certain theorems which are by no means general. Hence arose the demand for *exclusively arithmetical methods of proof*... This is the Weierstrassian method in mathematics, the *Weierstrass'sche Strenge*, as it is called.[97]

Klein then simply called "arithmetization" *all* developments of this kind, from Gauss to Weierstrasss, from Kronecker to Peano, and he went even further:

> For since I consider that the essential point is not the mere putting of the argument into the arithmetical form, but the more rigid logic obtained by means of this form, it seems to me desirable – and this is the positive side of my thesis – to subject the remaining divisions of mathematics to a fresh investigation based on the arithmetical foundation of analysis. On the other hand I have to point out most emphatically – and this is the negative part of my task – that it is not possible to treat mathematics exhaustively by the method of logical deduction alone, but that, even at the present time, *intuition* has its special province.[98]

In this way, Klein dismissed any special role of number theory for arithmetization, and reduced this movement to what he saw as its "essence," i.e., to a matter of logical tidying up to ensure the necessary rigour. This point of view stresses continuous progress, and does not invite the search for historical fault lines. In fact, Klein's agenda was not history at all. A passing reference to contemporary textbooks (p. 233) suggests that he considered the arithmetization of basic analysis as accomplished, and went out to promote research in geometry and mathematical physics which would take this most modern, arithmetized analysis into account.[99] Furthermore, he pleaded the case of well-trained mathematical intuition, which "*is always ahead of logical reasoning*."[100] He hailed (p. 238) the new appeal to intuition that Minkowski's geometry of numbers brought to arithmetic,[101] and he insisted that intuition has to be trained in university courses for beginners, scientists and engineers. Klein closed his address with a holistic metaphor of mathematics as a tree for which deep roots are just as vital as high branches.

When Klein gave this speech, David Hilbert had just started his second semester of teaching in Göttingen. Back in Königsberg, in his 1891 lectures on geometry,

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97. [Klein 1895], p. 966 (English); p. 233 (German).

98. [Klein 1895], p. 967 (English); p. 234 (German).

99. Let us mention in passing the measure-theoretic turn that Felix Bernstein would give to this kind of approach with his "axiom of the restricted arithmetizability of observations" in [Bernstein 1911].

100. [Klein 1895], p. 237: ... daß die so verstandene mathematische Anschauung auf ihrem Gebiete überall dem logischen Denken voraneilt und also in jedem Momente einen weiteren Bereich besitzt als dieses. (Emphasis in the original.) See also the last few sentences of [Klein 1890/1921–1922], vol. 1, p. 382, where Klein insisted on the necessity of arithmetizing irrational numbers first, in order to then sharpen our intuition by transferring the abstract notions thus found into geometry.

101. See J. Schwermer's chap. VIII.1 below.

Hilbert had still faithfully echoed the separation of arithmetic from geometry which can be traced back to Gauss.[102] Over the following decade, Hilbert's position changed significantly. By 1897, he was in tune with Felix Klein's very general, nostrified concept of arithmetization when he emphasized in the preface to his *Zahlbericht* the similar level of abstractness of all mathematical disciplines once they are treated "with that rigour and completeness … which is actually necessary."[103] The same is repeated along Klein's lines, and with a criticism of Kronecker's position, in the introduction to his 1900 Mathematical Problems:

> While insisting on rigour in the proof as a requirement for a perfect solution of a problem, I should like, on the other hand, to oppose the opinion that only the concepts of analysis, or even those of arithmetic alone, are susceptible of a fully rigorous treatment. This opinion, occasionally advocated by eminent men, I consider entirely erroneous. Such a one-sided interpretation of the requirement of rigour would soon lead to the ignoring of all concepts arising from geometry, mechanics and physics, to a stoppage of the flow of new material from the outside world, and finally, indeed as a last consequence, to the rejection of the ideas of the continuum and of the irrational number.[104]

This was written the year after the publication of his *Foundations of Geometry*, which open with the following declaration of Hilbert's arithmetization-via-axiomatization:

> Geometry, just like arithmetic, needs only a few simple basic facts to be built up from systematically. These basic facts are called *axioms*.[105]

The 1890s thus took David Hilbert from a position marked by arithmetic as the model discipline of pure mathematics to an egalitarian programme of axiomatization (which he would in fact try to extend all the way to physics). His sweeping declarations

102. [Hilbert 2004], p. 22–23 (where Kronecker ought to have been mentioned in note 6). Note the simultaneity with [Kronecker 1891]. Cf. [Toepell 1986], p. 21.
103. Our transl. of [Hilbert 1897/1932], p. 64: *Ich bin der Meinung, daß alle die anderen Wissensgebiete der Mathematik wenigstens einen gleich hohen Grad von Abtraktionsfähigkeit … verlangen – vorausgesetzt, daß man auch in diesen Gebieten die Grundlagen überall mit derjenigen Strenge und Vollständigkeit zur Untersuchung zieht, welche tatsächlich notwendig ist.*
104. [Hilbert 1900a], p. 294–295: *Wenn ich die Strenge in den Beweisen als Erfordernis für eine vollkommene Lösung eines Problems hinstelle, so möchte ich andererseits zugleich die Meinung widerlegen, als seinen etwa nur die Begriffe der Analysis oder gar nur diejenigen der Arithmetik der völlig strengen Behandlung fähig. Eine solche bisweilen von hervorragenden Seiten vetretene Meinung halte ich für durchaus irrig; eine so einseitige Auslegung der Forderung der Strenge führt bald zu einer Isolierung aller aus der Geometrie, Mechanik und Physik stammenden Begriffe, zu einer Unterbindung des Zuflusses von neuem Material aus der Außenwelt und schließlich sogar in letzter Konsequenz zu einer Verwerfung der Begriffe des Kontinuums und der Irrationalzahl.*
105. Our transl. of [Hilbert 2004], chap. 5, p. 436: *Die Geometrie bedarf – ebenso wie die Aritmnetik – zu ihrem folgerichtigen Aufbau nur weniger und einfacher Grundthatsachen. Diese Grundthatsachen heissen die* Axiome *der Geometrie.*

on arithmetization in the preface to the 1897 *Zahlbericht*[106] are best read with this evolution in mind. This preface builds up to the notion of arithmetization through a list of interactions of number theory with other mathematical disciplines. First he points to

> the close connection between number-theoretic questions and algebraic problems … The central reason for this connection is nowadays completely clear. Namely, the theory of algebraic numbers and the Galois theory of equations both have their roots in the theory of algebraic fields, and the theory of number fields has come to be the most essential part of modern number theory.[107]

Then Hilbert mentions five fruitful interactions between number theory and function theory: the analogies between number fields and function fields, the relation between the distribution of primes and the zeros of the Riemann zeta function, the transcendence of e and π, Dirichlet's analytic class number formula, and the theory of complex multiplication. All these examples motivate the inthronisation: "Thus we see how far arithmetic, the 'Queen' of mathematics, conquers broad areas of algebra and function theory and takes the lead in them."[108] Arithmetization is then added on top:

> Finally, there is the additional fact that, if I am not mistaken, the modern development of pure mathematics takes place chiefly *under the sign of number*: Dedekind's and Weierstrass's definitions of fundamental concepts of arithmetic and Cantor's general construction of the concept of number lead to an *arithmetization of function theory* and serve to realize the principle that even in function theory a fact can count as proved only when in the last resort it is reduced to relations between rational integers. The *arithmetization of geometry* is accomplished by the modern investigations in non-euclidean geometry in which it is a question of a strictly logical construction of the subject and the most direct possible and completely satisfactory introduction of number into geometry.[109]

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106. The essential building blocks of this preface to [Hilbert 1897] date back to 1895 and 1896; see [Hilbert 2004], pp. 153–156.

107. I.T. Adamson's transl. of [Hilbert 1897], p. 64.

108. Slight modification of I.T. Adamson's transl. of [Hilbert 1897], p. 65: *So sehen wir, wie die Arithmetik, die "Königin" der mathematischen Wissenschaft, weite algebraische und funktionentheoretische Gebiete erobert und in ihnen die Führerrolle übernimmt.*

109. Our emendation (we correct in particular the erroneous replacement of "Weierstrass" by "Kronecker") of I.T. Adamson's transl. of [Hilbert 1897], p. 66: *Es kommt endlich hinzu, daß, wenn ich nicht irre, überhaupt die moderne Entwickelung der reinen Mathematik vornehmlich* unter dem Zeichen der Zahl *geschieht:* Dedekinds *und* Weierstrass' *Definitionen der arithmetischen Grundbegriffe und* Cantors *allgemeine Zahlgebilde führen zu einer* Arithmetisierung der Funktionentheorie *und dienen zur Durchführung des Prinzips, daß auch in der Funktionentheorie eine Tatsache erst dann als bewiesen gilt, wenn sie in letzter Instanz auf Beziehungen für ganze rationale Zahlen zurückgeführt worden ist. Die* Arithmetisierung der Geometrie *vollzieht sich durch die modernen Untrersuchungen über Nicht-Euklidische Geometrie, in denen es sich um einen streng logischen Aufbau derselben und um die möglichst direkte und völlig einwandfreie Einführung der Zahl in die Geometrie handelt.*

The ubiquity of number thus meets Hilbert's syncretist notion of arithmetization in 1897. This arithmetization touches algebra, analysis, and geometry alike. Hilbert does insist on the tremendous success of higher arithmetic, transformed into the mature theory of algebraic number fields. Arithmetization, however, is only added at the end of the argument as a general logical approach to the foundations of mathematical theories. Hardly three years later, Hilbert will speak of axiomatisation instead. His specific agenda was thus obviously different from Klein's, but the two Göttingen accounts of arithmetization resemble each other in that they retain essentially a very general idea about rigorous foundations, and brush over differences between specific arithmetization programmes.[110] We have seen in chap. I.2, § 3.6, above that inside the *Zahlbericht*, Hilbert also freely navigated between Dedekind's and Kronecker's approaches.

Reading Gauss's *Disquisitiones Arithmeticae* had provided Leopold Kronecker with a precise methodology of arithmetization, and Dedekind had interpreted the same source as a call for a particular type of mathematical conceptual analysis. Hilbert's *Zahlbericht* and his other foundational projects from the turn of the century do conjure up a very general principle of arithmetic and rigour; but the application of this principle in various parts of mathematics has emancipated itself from any Gaussian model and from the various types of arithmetization proposed since 1872; in fact, Hilbert described all of these as "genetic" in 1900, and preferred the "axiomatic" method instead.[111]

3.3. Looking Back on Arithmetization

The parallel German and French editions of the *Enzyklopädie der mathematischen Wissenschaften* provide an interesting snapshot capturing the differences between the German and the French outlook on mathematics before WW I.[112] In the case of arithmetization, however, the German text written by Alfred Pringsheim and its French arrangement by Jules Molk bear a more complicated relation to each other because Molk was not only French but also Kronecker's former student. Pringsheim's original German text on the arithmetization of irrationals focuses first on the axiomatisation of the relationship between numbers and points on the line initiated by Cantor and Dedekind. Then follow brief discussions of Paul "du Bois-Reymond's fight against the arithmetical theories," and of the "total arithmetization according to Kronecker." These two positions are described as deviating from the majority consensus and Kronecker's programe is flatly dismissed as impracticable; see [Pringsheim 1898], pp. 53–58. The French version, not surprisingly, discusses Charles Méray's approach in greater detail, insisting on its priority. Furthermore, Molk added more than four

110. See also [Corry 1996/2004], chap. 3, and [Rowe 1989].

111. [Hilbert 1900b], pp. 180–181.

112. See C. Goldstein's chapter VI.1 below for a discussion of French reactions to arithmetization at the end of the century, esp. in connection with Charles Hermite's reading of the D.A. As for other countries, some initial references on the interesting Italian case can be found in [Pringsheim 1898], p. 53, note 18; p. 55, note 27; p. 57, note 37, as well as [Bohlmann 1897], p. 110.

pages describing Kronecker's programme quite carefully, explicitly stressing the constructivist principles behind it.[113]

A precious textbook reflecting the movement of arithmetization is [Stolz, Gmeiner 1902], i.e., the 2nd revised edition of Stolz's *Vorlesungen über allgemeine Arithmetik* of 1885. It is precious precisely because it looks less modern than one might expect in 1902, but covers a largely pre-set-theoretic panorama beginning with (abstract) magnitudes according to Grassmann.[114] The "Theoretical Arithmetic" treated here is characterized as the part of the foundations of analysis which does not require the notion of continuous function.[115] Both the point of view of magnitudes (sec. 5, pp. 99–119), and the arithmetization of the continuum "according to G. Cantor and Ch. Méray" (which the authors consider easiest to explain; sec. 7, pp. 138–184) are treated. Weierstrass's method is treated in exercises (e.g., pp. 177–179, 270). Kronecker's arithmetization is dismissed on the strength of the majority opinion among mathematicians. Probably the most original part for a textbook – which reminds us of the encyclopedia – is the historical sec. 6 (pp. 120–137) which takes the reader from Euclid's Book 5 – i.e., Eudoxus's theory of proportions – to Descartes, Newton, etc., and to the contemporary period.

By clearly exhibiting this traditional approach through magnitudes as a substantially different alternative to the arithmetization of the continuum, Stolz and Gmeiner displayed a keener historical sense than several of their colleagues, including even professionals of the history of mathematics. In fact, the arithmetization of real numbers was often seen as a modern replay of Eudoxos's theory of ratios. This strikes us as symptomatic of the rapidity with which arithmetization was not only nostrified in Göttingen, but lost the appearance of an innovative rearrangement of the hierarchy of mathematical disciplines, at the very time when the paradoxes of set theory began to potentially undermine the Dedekind-Cantor definitions of the continuum.

Rudolf Lipschitz would write to Dedekind already on June 8, 1876 with reference to book V of Euclid's *Elements*: "But I think that your definition of irrational numbers differs only in form, not in content from what the ancients have found."[116] Dedekind in his prompt reply naturally disliked the appeal to magnitudes, and also stressed – admitting for the sake of the argument[117] that Euclid's ratios of magnitudes

113. [Molk 1909], pp. 147–163. In passing, Molk defends Kronecker's procedures against criticism by Couturat; see [Molk 1909], p. 160, notes.

114. We do not attempt to survey the textbook reception of arithmetization in general. The work has not yet been done, as far as we know. The unbalanced [Bohlmann 1897], which was surely written upon Klein's request, takes arithmetization to start somehow with Euler, and fails to work out the last period for lack of time on the part of the author.

115. [Stolz, Gmeiner 1902], p. IV: *Das von den soeben erwähnten Gegenständen gebildete Gebiet lässt sich dadurch kennzeichnen, das zur Behandlung desselben der Begriff der stetigen Function nicht erforderlich ist.*

116. [Lipschitz 1986], p. 58 (cf. [Dedekind 1932], p. 469): *... ich aber der Meinung bin, dieselbe [Dedekinds Definition der irrationalen Zahlen] unterscheide sich nur in der Form des Ausdruckes aber nicht in der Sache von dem was die Alten festgestellt haben.* For Dedekind's reply, see pp. 64–68.

117. Which one should not admit, because even ratios of integers were treated by Euclid as

were meant to define numbers – the absence of any discussion of completeness and continuity.

Other authors made the same observation as Lipschitz, apparently deriving a special satisfaction from this alleged coming together of great minds over many centuries. For Max Simon[118] for instance, prop. 24 of Euclid's Book V, clearly showed that Book V was really about "the foundation of the rules of computation for irrational numbers, and that Eudoxus's method differs only inessentially from that of our Weierstrass."[119] Sir Thomas Heath retorted by explaining that Dedekind's definition of his cuts was formally much closer to Euclid, Book V, def. 5, than Weierstrass's.[120]

4. Conclusion

There are a few respects which Cantor's, Dedekind's, and Kronecker's arithmetization programmes share, in spite of all their manifest incompatibility with respect to finitist or constructivist requirements. First, all three authors considered mathematics as a science with a clearly defined domain of objects: as mentioned before, Kronecker viewed mathematics as a natural science;[121] Dedekind considered his analysis of continuity via cuts as expressing the essence of this concept; Cantor seems to have considered even his transfinite numbers as something that he discovered, rather than invented.[122] For all three authors arithmetization reduced the irrational numbers to the rational – or all the way to natural – numbers whose existence was taken to be evident. Second, they all executed this reduction to elementary given objects in a way that they considered naturally adequate for the problem at hand: for Kronecker, this meant indeterminates and congruences *à la Gauss*, for Dedekind grouping together sets of primary objects was just as naturally adequate a procedure as the consideration of series of rational numbers was to Cantor. The overall image that this suggests of the movement of arithmetization in the 1870s and 1880s is therefore that of a novel theory of objects that had formerly been understood in terms of extrinsic notions (magnitudes), this novel theory being founded on an independently accepted basis (the natural numbers), and proceeding with ingredients or methods deemed to be

relations rather than objects; see [Vitrac 1992], p. 150.

118. A teacher in Strasbourg who had obtained his docorate with Weierstrass and was from 1903 also *ordentlicher Honorarprofessor* for the history of mathematics at *Kaiser Wilhelm Universität Strassburg.*

119. [Simon 1901], p. 122: *S[atz] 24 zeigt mit größter Schärfe, ... daß es sich im fünften Buch um nichts anderes handelt, als um die strenge Begründung der Rechnungsregeln für Irrationalzahlen, und daß der Gang des Eudoxus von dem unseres Weierstraß nur unwesentlich abweicht.* See also [Simon 1901], p. 108–110, where he acknowledged Hieronymus Zeuthen's similar observation; see [Zeuthen 1893/1902], § 16 of the part on Greek mathematics.

120. [Heath 1926], p. 124–126. Cf. [Vitrac 1994], p. 548–551. According to [Simon 1906], p. 49, Cantor and Dedekind deluded themselves when they thought to have defined continuity arithmetically, without recourse to geometry. Cf. footnote 94 above.

121. [Kronecker 1891], last paragraph on p. 232.

122. [Cantor 1991], letter to Veronese, November 17, 1890, p. 330.

acceptable. From this point of view, arithmetization, in spite of all its novelty, appears not as an expression of modernity – indeed, as far as new objects were created, they were not purely formal, nor were they objectivized tools, but regularly formed from existing integers – but as a new type of solid building, erected on a traditional base in a controlled and supposedly innocuous and stable construction.

Our periodization has allowed us to isolate a transitional phase of arithmetization where Gaussian influence is detectible at least in two of the major authors. This influence operated via diverging fundamental positions (Kronecker's constructivism vs. Dedekind's completed infinites), but always in the direction of a novel but object-oriented rewriting of analysis. Gauss's after-effect ended with the onset of purely set-theoretic, axiomatic or logicist approaches, i.e., at the same time as the Göttingen nostrified image of arithmetization took shape.

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