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Structure preserving methods on staggered grids

Project report

M2 Calcul Scientifique et Mathématique de l'Information

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Contents

Introduction

Many physical models possess some conservation or symmetry properties that we want to conserve after discretization in order to build a physical consistent discrete model. As explained in the introduction of [11], when physical models are written in terms of differential calculus, their conservation properties often result of some differential operators properties, such as

$$\mathbf{curl}\ \mathbf{grad} = 0$$

or the fact that **curl** is self-adjoint for the $\mathcal{L}^2$ scalar-product in the case of periodic boundary conditions.

Then, a way to obtain a discrete model with the same properties is to build it along with discrete operators satisfying the same rules as their continuous equivalents. This is the starting point of mimetic discretization, which aims to produce a systematic way to build discrete spaces and discrete operators with good behaviours in view of the properties we want to preserve.

In this project, we mainly focus on the method explained in [6], which studies differential equations set on a compact manifold without boundary, say M. To the complex of differential forms it associates a complex of discrete spaces build using notions of algebraic topology, mainly chains and cochains. The resulting discrete spaces are linked to the continuous ones throught reduction and interpolation operators, leading to Galerkin projections on a space of finite dimension. We want to ensure that the discretization preserves certain topological properties of the studied manifold, in particular its de Rham cohomology. With this setting, to ensure that a discrete equivalent of an operator behaves on the discrete spaces as its countinuous counterpart in the continuous ones, we build it such that it commutes with the projection.

Another important aspect developed in [6] as well as in [2] and [11], is the preservation of orientation at the discrete level. In particular, methods developed in these papers care to take into account the two different types of orientation, inner and outer, that we can consider on a subset of a manifold. Differential forms are objects that we integrate on subspaces of M with inner orientation but some models involve what we call twisted forms, which has to be integrated on subspaces with outer-orientation.

Forms and twisted forms are linked with the Hodge-star operator. The twisted forms constitute another complex, dual of the previous one for the Hodge-star operator. These forms have almost similar behaviours but for our purpose, it is useful to distinguish them. As shown in [11], it allows to write exact relations in the discrete space which would otherwise be weakly formulated. In practice, we build two complexes of discrete spaces to discretize straight and twisted forms.

As it is explained in [11] and [6], these complexes are associated to two dictinct and staggered grids. To link them, a discrete Hodge-star operator is introduced. As for the other operators encountered so far, it is constructed in a way that the relations between the discrete spaces mimic as much as possible what occurs at the continuous level. In [6] and [11] are presented several ways to build these Hodge-star, reduction and interpolation operators. In [2], a different approach is taken, using mass matrices associated to the Riemannian metric and to the duality of Poincaré.

The aim of this project work was to understand the philosophy, the aims, the underlying theory and the methods of mimetic discretizations on staggered grids. As an example, we implemented Poisson's equations discretization using the formalism previously presented. This report starts with a brief presentation of topological and geometrical notions constituting the theorical background of mimetics methods, mainly de Rham cohomology, Poincaré duality and the Hodge star operator. Then, we introduce tools of algebraic topology and define an algebraic structure on the meshes while respecting the topology of the continuous spaces. On a third time, we show how these concepts gather to form a projection discretisation satisfying the desired properties. Finally, we apply all this framework to 1D Poisson's equation.

1 Continuous setting

1.1 De Rham cohomology

In the present work, we would like to present a way to discretize the sequence of differential forms

$$\Omega^0_n(M) \to \Omega^1_{n-1}(M) \to ...\Omega^n_0(M)$$

while preserving its structure. Before that, let us present some of its properties. We start by some general notions that we then apply to our case of interest.

Definition 1.1.1 (Module complex [9])**.** *Let R be a ring. A* complex of R-modules *is a sequence of $(C^i, d_i)_{i\in\mathbb{Z}}$ where C_i are modules on R linked by linear operators $d_i : C^i \to C^{i+1}$ verifying $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$.*

Since $d \circ d = 0$, the sequence given by $C^k = \Omega^k(M)$ and $d_k = d$ for $0 \le k \le n$ and $C^k = 0$ for other k is a $\mathbb{R}$-module complex (more precisely, a $\mathbb{R}$-vector space complex). Note that the condition $d_i \circ d_{i+1} = 0$ in the definition is equivalent to $\text{im} d_i \subset \ker d_{i+1}$. We can then define :

Definition 1.1.2 (Cohomology of a module complex [9])**.** *The* cohomology *of a complex of R-modules is the sequence of R-modules given by*

$$H^i(C, d) = {}^{\ker d_i}\big/_{\text{im} d_{i-1}}.$$

In our case, $\ker d_i$ and $\text{im} d_{i-1}$ respectively contains the closed and exact forms of $\Omega^k(M)$. The associated cohomology is called *de Rham cohomology* [9].

We give now some examples of de Rham cohomology, taken from [7].

Example 1.1.1 (de Rham cohomology of maximum degree)**.** Assume that M is a n-dimensional compact and orientable manifold without boundary. Then, $\dim H^n_{dR}(M) = 1$. The fact that $\dim H^n_{dR}(M) \ge 1$ can be show using Stockes theorem. Because M is assumed to be orientable, there exists $\lambda \in \Omega^n(M)$ which never vanishes on M. As $d\lambda$ is an element of $\Omega^{n+1}(M) = \{0\}$, λ is closed. Suppose that λ is exact and write it as $d\alpha$. On the one hand, since $\partial M = \varnothing$ we should have,

$$\int_M \lambda = \int_{\partial M} \alpha = 0.$$

On the other hand, it must be

$$\int_M \lambda > 0.$$

These two statements are in contradiction so we deduce that λ is closed but not exact and so that $H^n_{dR}(M)$ is not trivial. The proof that its dimension is exactly 1 can be found in [7].

On the contrary, let us see a case where we are sure to find $H^k(M) = \{0\}$.

Theorem 1.1.1 (Poincaré lemma [7])**.** *If U is a starred domain in $\mathbb{R}^k$, then every closed form on U is exact*

$$H^k_{dR}(U) = \{0\}.$$

Every exact form is closed but the reverse is false in general. However, Poincaré lemma states that every closed form is locally exact. The de Rham cohomology then says something about the global properties of the manifold M.

1.2 L^2 space of differential forms

For our purpose, the condition that ω_x varies differentiably with x is too strong. The appropriate work space here is in fact $L^2\Omega^k(M)$, that we define on this section from [10].

Consider a Riemannian metric g on a n-dimensional manifold M. On each tangent space T_xM, choose a base $(e_1(x), ..., e_n(x))$ orthonormal by respect to g. Then, the family $\{e_{i_1}(x) \wedge ... \wedge e_{i_k}(x)\}_{1 \leq i_1 \leq ... \leq i_k}$ forms an orthonormal basis of $\Omega^k(M)$ for the scalar product induced by g_x on $\Lambda^k T_xM$ [4], that is

$$(v_1 \wedge ... \wedge v_k, w_1 \wedge ... \wedge w_k)_x = det\Big([g_x(v_i, w_j)]_{ij} \Big).$$

Indeed, if $I = \{i_1, ..., i_k\}$ and $J = \{j_1, ..., j_k\}$ are different, there exists $1 \leq l \leq k$ such that i_l is not in $\{j_1, ..., j_k\}$. Since the $e_i(x)$ are orthonormal, $g_x(e_{i_l}(x), e_{j_m}(x)) = 0$ for all $1 \leq m \leq k$ and $(e_I(x), e_J(x))_x = 0$. On the contrary, if $I = J$, then $[g_x(v_i, w_j)]_{ij}$ is the identity and $(e_I(x), e_I(x))_x = 1$.

Integrating $(\cdot, \cdot)_x$ on M, we obtain a scalar product on $\Omega^k(M)$:

$$(\cdot, \cdot)_{L^2\Omega^k(M)} : (\omega, \eta) \mapsto \int_M (\omega_x, \eta_x)_x d\mu,$$

with μ the natural Riemannian mesure induced by g. Developping $(\omega, \eta)_{L^2}$ for ω, η in $\Omega_0^k(M)$, we have

$$(\omega, \eta)_{L^2} = \int_M (\sum_I \omega_I(x) e_I(x), \sum_I \eta_I(x) e_I(x))_x = \int_M \sum_I \omega_I(x)\eta_I(x) = \sum_I (\omega_I, \eta_I)_{L^2(M)}.$$

It happens [10] that $(\cdot, \cdot)_{L^2\Omega^k(M)}$ does not depend on the bases $(e_1(x), ..., e_n(x))$ we have chosen on the T_xM.

Let us define $L^2\Omega^k(M)$ as the completion of $\Omega^k(M)$ by respect to the norm $\| \cdot \|_{L^2}$ induced by $(\cdot, \cdot)_{L^2\Omega^k(M)}$. We also denote this space by $\Omega_0^k(M)$. By definition, it is a Hilbert space and it contains the set of differential forms with coefficients L^2.

In the same way, we want to define $\Omega_p^k(M)$ the space of k-forms with coefficients in H^p for $p \geq 1$. More precisely, we say that $u \in \Omega_0^k(M)$ is in $\Omega_p^k(M)$ if for all chart $(U, \phi : U \to \mathcal{B})$ and all function ρ in $\mathcal{C}_c^\infty(U, \mathbb{R})$, the coefficients v_I such that $\phi_*(\rho u) = \sum_{|I|=k} v_I dx_I$ are in $H^p(\mathcal{B})$. It is equivalent to ask that the previous condition is verified one for elements (U_i, ϕ_i) of a given atlas and for functions ρ_i of a given subordinate partition of the unity. Suppose that this is the case, let (U, ϕ) be another chart and ρ be an element of $\mathcal{C}_c^\infty(U, \mathbb{R})$. We have

$$\phi_*(\rho u) = \phi_*((\sum_i \rho_i)\rho u) = \sum_i \phi_*(\rho_i \rho u) = \sum_i (\phi \circ \phi_i^{-1})_*(\phi_i)_*(\rho_i \rho u) = \sum_i (\phi_i \circ \phi^{-1})^*(\phi_i)_*(\rho_i \rho u)$$

Let us note $\psi = \phi_i \circ \phi^{-1}$. The assumption we made on the atlas and on the partition of the unity makes that for all i such that $U_i \cap U \neq \varnothing$,

$$\psi^*(\psi_i)_*(\rho_i \rho u) = \psi^*(\sum_{|I|=k} v_I^j dx_I).$$

By linearity and definition of the pullback,

$$[\psi^*(\psi_i)_*(\rho_i \rho u)]_x (v_1, ..., v_k) = \sum_{|I|=k} (v_I^i \circ \psi)(x) dx_I(x)(d_x\psi(v_1), ..., d_x\psi(v_k)).$$

Since ψ is a diffeomorphism, the forms $dx_I(d\psi(\cdot), ..., d\psi(\cdot))$ are linear combinations of the dY_I with bounded coefficients in $U_i \cap U \neq \varnothing$. For the same reason, $v_I^i \circ \psi$ is in $H^k(\mathcal{B})$. Then, the coefficients of $psi^*(\psi_i)_*(\rho_i \rho u)$ and so those of $\phi_*(\rho u)$ in the basis dY_I are in $H^k(\mathbb{R}^n)$ and are bounded. We deduce that $\phi_*(\rho u)$ has the desired property. As this is true for any ρ and any ϕ, and since the reverse is obviously true, we have the desired equivalence.

We immediately see that $\Omega_p^k(M)$ is a vector space. It is a Hilbert space for the scalar product

$$(u,v)_{\Omega_p^k(M)} = \sum_{i,I} (u_I^i, v_I^i)_{H^p(M)}.$$

The application $(\cdot,\cdot)_{\Omega_p^k(M)}$ is indeed a scalar product and does not depend on the charts. It is bilinear since the maps $u \mapsto u_I^i$ are linear and the scalar product in $H^p(M)$ is (obviously) bilinear. Since $(\cdot,\cdot)_{H^p(M)}$ is definite positive, $(u,u)_{\Omega_p^k(M)} \geq 0$ for all u. If it is equal to zero, then it means that the u_I^i are all zeros and so that u is 0 in $\Omega_p^k(M)$. It remains to see that $\Omega_p^k(M)$ is complete for the induced norm. Consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$. We immediateley see that the coefficients $(u_I^i)_n$ also form Cauchy sequences in $H^p(M)$. Since $H^p(M)$ is complete, they all admit a limit u_I^i in $H^p(M)$. By definition, the form u build with those coefficients is in $\Omega_p^k(M)$ and the fact that u_n converges to u for $\|\cdot\|_{H^p(M)}$ is straightforward. As this argument is true for any Cauchy sequence $(u_n)_n$, we deduce that $\left(\Omega_p^k(M), (\cdot,\cdot)_{H^p(M)}\right)$ is indeed a Hilbert space.

Now we have defined the $\Omega_p^k(M)$, it makes sense to consider the differential operator $d : \Omega_p^k(M) \to \Omega_{p-1}^{k+1}(M)$. We set it as the extension by continuity of the derivation operator defined on $\Omega^k(M)$, which is linear in finite dimension so uniformely continuous.

1.3 Poincaré duality and Hodge star operator

We know that the dimension of $\Omega^k(M)$ is $\binom{n}{k}$. Then, it happens that $\Omega^k(M)$ and $\Omega^{n-k}(M)$ are isomorphic. In a 3-dimensional space, 1 and 2-forms are isomorphic. Given a orthonormal basis $\{dx_1, dx_2, dx_3\}$ of $\mathbb{R}^3$, we can build such an isomorphism by taking the linear application which send the 1-form dx_i to the 2-form $dx_j \wedge dx_k$, were $(i,j,k) = (\sigma(1), \sigma(2), \sigma(3))$ with σ is a permutation with positive signature. These two kinds of objects can be represented by a vector in $\mathbb{R}^3$. In physics, the first are called *polar vectors* and the second *axial vector*. They seem to behave the same way except in some situations such as a change of orientation, which occurs for instance if we switch two vectors in the basis we have chosen, where axial vectors changes sign while polar vector do not. We can verify it on the example below by replacing all mentions of dx_2 by dx_3 and vice-versa. Forms dx_1, dx_2 and dx_3 are now associated with the two opposite forms of those they were previously associated with. Althought the symmetry does not change coordinates of polar vectors, it changes axial vectors one. A better explanation of this phenomenon is described in [3]. The map we have build in the previous example is in fact the Hodge star operator, which we define now in any dimension.

Let μ^n be a volume form on an compact and orientable manifold M without boundary. The bilinear form

$$\begin{aligned} \Omega_0^k(M) \times \Omega_0^{n-k}(M) \to &\quad \mathbb{R} \\ (\omega, \eta) \mapsto &\quad \langle \omega, \eta \rangle := \int_M \omega \wedge \eta \end{aligned}$$

is non-degenerate so for all $\eta \in \Omega_0^k(M)$, there exists an unique element, denoted $\star\eta \in \Omega_0^{n-k}(M)$ such that

$$\langle \omega, \star\eta \rangle = (\omega, \eta)_{\Omega_0^k(M)}$$

and the operation $\eta \mapsto \star\eta$ is linear.

The operator $\star$ is called the *Hodge star operator* and the bilinear form $\langle\cdot,\cdot\rangle$ *Poincaré duality.*

Proposition 1.3.1 (Properties of the Hodge operator [10])**.** *The Hodge star operator verifies:*

1. $\star dx_I = \epsilon(I, {}^cI) dx_{{}^cI}$, *where* $I \sqcup {}^cI = \{1, ..., n\}$, *for all* I *such that* $|I| = k$,
2. $\star\star = (-1)^{k(n-k)} id$ *for all* $1 \leq k \leq n$.

Proof. To prove the first point, we have to see that

$$\omega \wedge \epsilon(I, {}^cI) dx_{{}^cI} = (\omega, dx_{{}^cI})\mu^n$$

for all $\omega \in \Omega_0^k(M)$. It suffices to show that this is true for all the others dx_J with $|J| = k$. This is true since

$$dx_J \wedge \epsilon(I, {}^cI) dx_{{}^cI} = \epsilon(I, {}^cI)\delta_{IJ} dx_I \wedge dx_{{}^cI} = \delta_{IJ}\mu^k.$$

By linearity, we only have to establish the equality on the dx_I to prove the second point. For that, we successively use what we have just proved on dx_I and $dx_{{}^cI}$:

$$\star \star dx_I = \star\epsilon(I, {}^cI) dx_{{}^cI} = \epsilon(I, {}^cI) \star dx_{{}^cI} = \epsilon({}^cI, I)\epsilon(I, {}^cI) dx_{{}^{cc}I} = \epsilon({}^cI, I)\epsilon(I, {}^cI) dx_I.$$

Then, we see that $\epsilon({}^cI, I) = \epsilon(\sigma)\epsilon(I, {}^cI)$, where $\epsilon(\sigma)$ is the signature of the permutation which switches I and ${}^cI$. This permutation is obtain by transposing all the k elements of I with all the $n-k$ elements of ${}^cI$, one by one. The signature of a permutation is equal to the number of transposition we can use to perform it. Here, we have $\epsilon(\sigma) = (-1)^{k(n-k)}$, hence the desired result. □

1.4 Hodge decomposition

Let us now introduce the formal adjoint of d for $(\cdot,\cdot)_{\Omega_0^k(M)}$, the *coderivation operator* $d^* : \Omega_0^k(M) \to \Omega_0^{k-1}(M)$. Obviously, we have $d^*d = 0$.

Proposition 1.4.1 ([10])**.** $d^* = (-1)^{n(k-1)+1} \star d \star$

Proof. Let $\omega \in \Omega_0^k(M)$ and $\eta \in \Omega_0^{k-1}(M)$. First recall that

$$d(\eta \wedge \star\omega) = d\eta \wedge \star\omega + (-1)^{k-1}\eta \wedge d \star \omega.$$

Integrating this equality on M gives

$$0 = (d\eta \wedge \omega)_{\Omega_0^k} + (-1)^{k-1}\langle \eta, d \star \omega\rangle$$

by Stockes theorem since M has no boundary and Poincaré duality definition. Then, from a previous proposition

$$\begin{aligned}
(\star d \star \omega, \eta)_{\Omega_0^{k-1}} &= \langle \eta, \star \star d \star \omega\rangle \\
&= (-1)^{(k-1)(n-k+1)}\langle \eta, d \star \omega\rangle \\
&= (-1)^{(k-1)(n-k+1)+k}(d\eta, \omega)_{\Omega_0^k} \\
&= (-1)^{n(k-1)+1}(\eta, d^*\omega)_{\Omega_0^k}.
\end{aligned}$$

As it stands for any η and ω, we conclude that $d^* = (-1)^{n(k-1)+1} \star d\star$. □

From the derivative and the cderivative, we now define a Laplacian on the differential forms.

Definition 1.4.1 (Laplacian [10])**.** *The* Laplacian *is the operator given by* $\Delta := dd^* + d^*d$.

The study of this operator leads to an import result, the Hodge decomposition theorem.

Theorem 1.4.1 (Hodge decomposition theorem)**.** *Let M be a compact and orientable manifold without boundary. The space of k-differential forms decomposes as*

$$\Omega^k(M) = d\Omega^{k-1}(M) \bigoplus d^*\Omega^{k+1}(M) \bigoplus \mathcal{H}^k,$$

with $\mathcal{H} := \ker \Delta$ is the space of harmonic functions.

A proof of this theorem can be found in [10], we present here its main ideas. It relies on the study of the Laplacian Δ. That this operator is autoadjoint and positive comes from the properties and definitions of d and d^*. However, it is not definite. To circumvent this difficulty, we would rather study $id + \Delta$ but this operator is not compact, which finally leads us to study $T := (id + \Delta)^{-1}$.

We apply Lax-Milgram theorem on the problem

$$(id + \Delta)\omega = \eta$$

for $\eta \in \Omega^k(M)$. The coercivite of the bilinear form involved is prove thanks to Garding inequality. We then have that T is compact and so that there exists a Hilbert basis on $\Omega_0^k(M)$ made of eigenvectors of T. Morevoer, T eigenvalues are positive, appear a finite number of times and decrease to 0.

Coming back to the Laplacian, we find that $\Omega_0^k(M)$ can be decomposed as the direct sum of the kernel of Δ, which is of finite dimension, and of Δ eigenspaces, all of finite dimension. The eigenvalues of the Laplacian increases to ∞.

The theorem then follows from the elliptic regularity lemma, which states that for all $\phi \in \Omega^k(M)$, the weak solution $u \in \Omega_0^k(M)$ of

$$\Delta u = \phi$$

is in fact in $\Omega^k(M)$.

2 Discrete setting

2.1 Chains

Let M a n-dimensional manifold. Here, we are interested in numerically computing differential forms defined on M. To do that, we first need to discretize it and we want to do it in a way that will allows us to establish a discrete equivalent for the de Rham diagramm. This is achieved using some tools of algebraic topology. We define below the notions used to set up this discretization.

Definition 2.1.1 (singular k-cube [8])**.** *A singular* k-cube *is a continuous application* $\tau_k : [-1,1]^k \to M$ *for* $0 \leq k \leq n$*. If there exists an indice* $0 \leq i \leq k$ *such that* $\tau_k(x_1, ..., x_k)$ *does not depend on* x_i*, the cube is said to be degenerate.*

From now on, we consider non-degenerate k-cubes. By abuse of notation, we will also call a k-cube and write τ_k the image of τ_k.

Definition 2.1.2 (k-cube faces [8])**.** *Given a* k*-cube* τ_k *with* $k \geq 1$ *and an integer* $1_leqi \leq k$*, we define respectively the* front i-face *and the* back i-face *of* τ_k *as the maps*

$$A_i(\tau_k) : (x_1, ..., x_{k-1}) \in [-1,1]^{k-1} \mapsto \tau_k(x_1, ..., x_{i-1}, -1, x_i, ..., x_{k-1}) \in M$$

and

$$B_i(\tau_k) : (x_1, ..., x_{k-1}) \in [-1,1]^{k-1} \mapsto \tau_k(x_1, ..., x_{i-1}, 1, x_i, ..., x_{k-1}) \in M.$$

Notice that from the definition of cubes, the faces of a k-cube are themselves $(k-1)$-cubes.

Definition 2.1.3 (Cubical complex [6])**.** *A complex of cubes in the manifold* M *is a finite collection* D *of cubes such that :*

1. M *is included in the union of the elements that are in* D*,*
2. *all faces of all cubes in* D *are also in* D*,*
3. *if two* k*-cubes* τ_k *and* μ_k *are distinct and if their intersection is not empty, then it exists* $l < k$ *and a* l*-cube*ν_l *in* D *such that* ν_l *is include both in* τ_k *and* μ_k*.*

In the context of numerical analysis, a cubical complex is nothing more than a mesh. In this section, we consider them from the algebraic toppology point of view and endow it with a group structure.

Definition 2.1.4 (k-chain [6])**.** *Given a cubical complex* D*, the space of* k-chains *of* D *is define as the formal Abelian group generated by the* k*-cubes in* D*. We will write this group additively and note it* $C_k(D)$*.*

Let N the number of k-cubes in a complex D and denote these elements by $\tau_{k,i}$ for $1 \leq i \leq N$. A k-chain $\mathbf{c}_k$ of $C_k(D)$ will be written

$$\mathbf{c}_k = \sum_{i=1}^{N} c^i \tau_{k,i},$$

with $c^i \in \mathbb{Z}$.

In this work, we will only use chains with coefficients in $\mathbb{Z}/3$. We can then think of a chain as collection of oriented cubes, with positive orientation if the corresponding coefficient is 1 and negative if it is -1. An important example of a chain is given by the boundary of a cube, that we define now.

Definition 2.1.5 (Boundary operator [6]). *We call* boundary *of a k-cube τ_k the k-chain*

$$\partial\tau_k = (-1)^i \big[A_i(\tau_k) - B_i(\tau_k)\big].$$

The k-boundary operator *is the homomorphism ∂_k which associates to each k-cube its boundary.*

In order to simplify the notations, we omit the integer k and note ∂ the k-boundary operator for all k.

If we think of cubes as subsets of M, the boundary of a k-cube according to this definition and its boundary understood as the set difference between its closure an its interior in M coincide. The coefficients chosen in the definition of $\partial\tau_k$ then give to it a consistent orientation as a subset of M. For instance, if we take for τ_2 a full square, $\partial\tau_2$ will be the union of the square edges positively oriented.

This is also true for chains composed of more than one cube, assuming that the coefficients in front of all k-cubes in a k-chain $\mathbf{c}_k$ are equal. Since each $(k-1)$-cube part of two adjacent cubes will appear one time with positive coefficient and antoher time with a negative one, it only remains the $(k-1)$-cube located on the "outside" of the chain in the expression of $\partial\mathbf{c}_k$, which corresponds to the boundary of $\mathbf{c}_k$ as a subset of M.

For the same reason, the boundary of a boundary of a k-chain is the nul element of $C_{k-2}(D)$ and we have the relation $\partial\partial = 0$, which is analogous to the one that holds for the topological boundary.

2.2 Cochains

In the previous section, we introduced chains, whose link with the discretisation of a continuous domain is easily seen. However, the usefulness of such a formalism is less obvious at first sight. The following section allows us to understand its logic for our purpose. It introduces the cochains, which will then be used to construct discrete equivalents to the differential forms we want to compute numerically.

Definition 2.2.1 (Cochains [6]). *Given a cubical complex D, we call space of k*-cochains *and note $C^k(D)$ the space of homomorphisms between $(C_k(D), +)$ and $(\mathbb{R}, +)$.*

The cochains spaces are themselves Abelian groups and we also write them additively. Let $(\tau_{k,i})_{1\le i\le N}$ be the set of k-cubes that generates $C_k(D)$ and note $\tau^{k,i}$ the element of $C^k(D)$ which associates 1 to the cube $\tau_{k,i}$ and 0 to the others. The set $(\tau^{k,i})_{1\le i\le N}$ generates $C^k(D)$. Then, the result of the action of a cochain $\mathbf{c}^k$ on a chain $\mathbf{c}_k$ is equal to $\sum_{i=1}^N c^i c_i$, where c_i and c^i are the coefficients of $\mathbf{c}^k$ and $\mathbf{c}_k$ in the bases $(\tau^{k,i})_i$ and $(\tau_{k,i})_i$.

We now present an important exemple for our futur discretizations. Let ω be a k-differential form defined on M. Following [1], we define the integration of ω on a k-cube τ_k as

$$\int_{\tau_k} \omega := \int_{[-1,1]^k} \tau_k^* \omega.$$

We then define the integration of ω on all chains of $C_k(D)$ by defining the "integration of ω operator" $\mathbf{c}_k \in C_k(D) \mapsto \int_{\mathbf{c}_k} \omega \in \mathbb{R}$ as the homomorphism equal to $\int_{\tau_k} \omega$ on the generators $\tau_{k,i}$. Then, each k-form on M induces a k-cochain via integration. The coefficients of this cochain in the basis $\tau^{k,i}$ are exactly the mean value taken by ω on each cube of the complex. With this in mind, we can think of cochains as discrete versions of differential forms. This will turn out to be crucial in the subsequent discretization process.

This example also motivates the introduction of the coboundary operator. In the differential context, Stockes theorem links the integration of the derivative of a differential form on a domain and the integration of this form on the domain boundary. If we take a chain and a cochain for discrete equivalents of a domain and a differential form, we would like to establish a discrete counterpart of Stockes formula.

We still have a boundary operator for chains. We now consider its dual homomorphism. Recall first that given three groups A, B and G and a homomorphism $\alpha : A \to B$, the dual homomorphism $\alpha^* : Hom(B, G) \to Hom(A, G)$ is defined by $\alpha^*(f) = f \circ \alpha$ [5]. In our case,

Definition 2.2.2 (Coboundary operator [5])**.** *The* k-coboundary operator *is the operator* $\partial^* = \delta^k : C^k(D) \to C^{k+1}(D)$ *verifying*

$$\delta^k \mathbf{c}^k(\mathbf{c}_k) = \mathbf{c}^k(\partial \mathbf{c}_k)$$

for all k*-cochain* $\mathbf{c}^k \in C^k(D)$ *and all* k*-chain* $\mathbf{c}_k \in C_k(D)$*.*

As for the boundary operator, we will skip the integer k and simply note δ the coboundary operator for all k when the context makes no confusion possible on which operator is used.

From the above general definition of the dual, we immediately find that $(\partial\partial)^* = \partial^*\partial^*$ and so that $\delta\delta = 0$.

2.3 Chains and cochains complexes

The nilpotency of the boundary and coboundary operators allows us to define the *complex of chains* and the *complex of cochains*, respectively

$$C_0(D) \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} C_{k-1}(D) \overset{\partial}{\leftarrow} C_k(D) \overset{\partial}{\leftarrow} C_{k+1}(D) \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} C_n(D)$$

and

$$C^0(D) \overset{\partial}{\rightarrow} \dots \overset{\partial}{\rightarrow} C^{k-1}(D) \overset{\partial}{\rightarrow} C^k(D) \overset{\partial}{\rightarrow} C^{k+1}(D) \overset{\partial}{\rightarrow} \dots \overset{\partial}{\rightarrow} C^n(D).$$

If we have in mind that a cochain represents a discrete equivalent for a differential form, the last diagramma can be seen as discrete analoguous for the de Rham diagramm.

A k-chain (respectively a k-cochain) is called a k-cycle (resp. a k-cocycle) when it is in $\ker \partial$ (resp. $\ker \delta$), a *k-boundary* (resp. *k-coboundary*) when it is part of $\mathrm{im}\partial_{k+1}$ (resp. $\mathrm{im}\delta^{k-1}$) and a *harmonic k-chain* (resp. *harmonic k-cochain*) when it is a k-cycle (resp. k-cocycle) without being a k-boundary (resp. k-coboundary). We note by $Z_k(D)$ (resp. $Z^k(D)$) the subgroup of k-cycles (resp. k-cocycles) and by $B_k(D)$ (resp. $B^k(D)$) the subgroup of k-boundaries (resp. k-coboundaries). We finally note

$$H_k(D) = {Z_k(D)} \big/ {B_k(D)} \quad \text{and} \quad H^k(D) = {Z^k(D)} \big/ {B^k(D)}$$

the homology groups.

An equivalent of Hodge decomposition for chains and cochains is given by [6]

$$C_k(D) = B_k(D) \bigoplus H_k(D) \bigoplus Z_k(D)^c \tag{2.1}$$

and

$$C^k(D) = B^k(D) \bigoplus H^k(D) \bigoplus Z^k(D)^c. \tag{2.2}$$

The main challenge of mimetic discretization is to make the cochain complex and the decomposition given above consistent with those taking in the space of differential forms.

2.4 Chains and cochains maps

We now consider how the objects we have presented in the previous sections behaves under continuous transformations of the manifold M. In this section, N denotes a n-dimensional manifold, E a cubical complex on N and ϕ an homeomorphism between M and N such that the cubes in D are send to the cubes in E. Let us first see how it acts on chains.

Definition 2.4.1 (Chain map [6])**.** *We call* cochain map *and note* $\phi_\#$ *the homomorphism going from* $C_k(D)$ *to* $C_k(E)$ *which verifies*

$$\phi_\#(\tau_{k,i}) = \phi \circ \tau_{k,i}$$

on the cubes composing D*.*

It is easy to see that the chain map commute with the boundary operator. First, notice that

$$A_i(\phi_\#\tau_k)(x_1,...,x_k) = (\phi_\#\tau_k)(x_1,...,x_{i-1},1,x_{i+1},...,x_k) = \phi(\tau_k(x_1,...,x_{i-1},1,x_{i+1},...,x_k)) = \phi_\#(A_i(\tau_k))(x_1,...,x_k)$$

for all $(x_1,...,x_k) \in [-1,1]^k$ and for each cube τ_k. Because it is an homomorphism, the chain map commutes with the boundary operator on the cubes. Then, as they are both homomorphisms, $\phi_\# \circ \partial$ and $\partial\phi_\#$ are also homomorphisms and as they coincide on the cubes, they coincide on the whole group. This property can be linked to the fact that an homeomorphism send a boundary on a boundary, that is $\phi(\partial M) = \partial\phi(M)$.

An immediate consequence of this result is the fact that $\phi_\#$ maps the cycles of D on the cycles of E, the boundaries of D on the boundaries of E and the harmonic chains of D on the harmonic chains of E. The decomposition 2.1 is preserved. Moreover, $\phi_\#$ induces a complex morphism between the chain complex on D and the chain complex on E.

Now, as we have done for the boundary operator, we define the *cochain map* $\phi^\#$ as the dual homomorphism of the chain map. Because of

$$\phi^\#\delta = (\partial\phi_\#)^* = (\phi_\#\partial)^* = \partial^*\phi_\#^* = \delta\phi^\#,$$

the cochain map also commutes with the coboundary operator. For the differential form, this property is equivalent to $\phi^* \circ d = d \circ \phi^*$, where ϕ^* denotes the pullback of ϕ.

Similarly to the chain map, the cochain map preserves cocycles, coboundaries and coharmonic cochains and so decomposition 2.2. It also induces a complex of morphisms between the cochain complexes of D and E.

2.5 Dual complexes

In the four previous section, we have considered one cubic complex. In practise, we will consider two grids to discretize our manifold M. Fr om the point of view of algebraic topology, these two grids can be linked using the notion of *dual complex*. Before giving its definition, we first need to introduce *cofaces*.

Definition 2.5.1 (Coface [6])**.** *For* $0 \le k \le n-1$ *and a* k*-cube* τ_k*, a* cofaces *of* τ_k *is a* $(k+1)$*-cube which has* τ_k *as a face.*

Definition 2.5.2 (Dual cube [6])**.** *For* $0 \le k \le n-1$ *and a* k*-cube* τ_k*. A* $(n-k)$*-cube is dual to* τ_k *if its cofaces are the dual cubes of* $\partial\tau_k$*.*

This definition is quite convoluted. Let us clear it up with a simple example in two dimensions. Consider a subset of $\mathbb{R}^2$ and define on it two cubical complexes whose 2-cells are full squares. We assume that all the squares in both complexes have the same size. The cofaces of a 0-cube, that is a point, are the four segments starting at this point. The cofaces of a 1-cube, that is a segments, are the two squares separated by this segment. For a 0-cell, its dual cube is a 2-cube, which has no cofaces. We impose then that the dual cube of a point is the square on which lies this point. Conversely, the dual cube of a square in one of the two complex is the vertex of the other complex which is in this square. Now, as the boundary of a segment is composed of its two end points, their cofaces are the two squares that the segment crosses. Then, its dual segment is the segment of the dual complex which intersects it.

Definition 2.5.3 (Dual complex [6])**.** *Given a cubical complex* D*, we define its* dual complex *as the smallest cubical complex which contains the dual cubes of the cubes of* D*. We write it* $\tilde{D}$*.*

Let us note the *coface operator* by ∂^*. Let $* \in Hom(C_k(D), C_k(\tilde{D}))$ the homomorphism which associate to each cube its dual one. The definition of dual cubes exactly states that $*\partial = \partial^* *$.

The operator $*$ can be view as an equivalent of the Hodge star operator for chains. The notion of duality we used in the previous definition can then be considered as a kind of Poincaré duality for chains. In the differential case, we have presented some results for the case of a manifold without boundary. We know consider their counterparts for chains.

Definition 2.5.4 (Cubical complex without boundary)**.** *A cubical complex is said to be with no boundary if the set of all its dual cubes is equal to its dual complex.*

If a complex has no boundary, it is also the case of its dual complex, for which it is also dual modulo orientation. The operator $*$ is then invertible, of inverse $\pm *$. This allows us to express the coface operator in terms of $*$ and ∂ :

$$\partial^* = \pm * \partial * .$$

2.6 Notations

We finish this section with some useful notations. As we choose to note additively the group $C_k(D)$, we can represent a k-cube using an element of $\mathbb{Z}^{N_k}$ and a element of $Hom(C_k(D), C_l(D))$ by a element of $\mathcal{M}_{N_l,N_k}(\mathbb{Z})$, with N_k and N_l the number of k- and l-cubes in D. In particular, the matrix of the boundary operator has $k-1$ lines and k columns. On the j-th column, the coefficients corresponding to front i-faces of the j-th k-cube of D are equal to 1. Those corresponding to i-back faces of the same cube are equal to -1 and all the others are equal to 0. This matrix is called [6] *incidence matrix* and will be noted by $E_{k-1,k}$.

In the same way, k-cochains and cochains homomorphisms are also represented by elements in $\mathbb{Z}^{N_k}$ and $\mathcal{M}_{N_l,N_k}(\mathbb{Z})$. The action of a cochain $\mathbf{c}^k$ on a chain $\mathbf{c}_k$ is then equal to the matricial multiplication of their vectorial representations. The incidence matrix for the cochains is written $E^{k-1,k}$ and is equal to $Ek, k-1^T$.

3 Discretisation

The aim of this work is to present a way of discretizing the space of differential forms in a way that discrete operators between discrete spaces behaves in the same way than their continuous equivalents [6]. More precisely, given a domain $\Gamma \subset M$, we are looking for finite dimensional spaces $\Omega_h^k(\Gamma) \subset \Omega^k(\Gamma)$ and projections $\pi^k : \Omega^k(\Gamma) \to \Omega_h^k(\Gamma)$ such that continuous operators $T : \Omega^k(\Gamma) \to \Omega^l(\Gamma)$ and associated discrete operators $T_h : \Omega_l^k(\Gamma) \to \Omega_h^l(\Gamma)$ satisfies

$$T_h \circ \pi^k = \pi^l \circ T_h. \tag{3.1}$$

In the following, we explain how we can build the projections π^k as the composition of a reduction and an interpolation operators with the spaces of cochains as intermediate spaces.

In the previous section, we have presented some tools of algebraic topology which have led us to the construction of a complex sharing some fundamental properties with the de Rham complex. In particular, there exists relations similar to the Stockes theorem and the Hodge decomposition. Elements of this complex, cochains, act on their spaces of definition as homomorphisms and this action can be represented by matricial multiplications. These objects are then adapted to numerical computations while retaining a structure comparable to the one that interests us in the continuous framework. Thus, chains and cochains are the objects we will manipulate at the discrete level.

Going from a continuous domain $\Gamma \subset M$ to the space of chains is made by discretising Γ with a cubical complex D. It now remains to find an explicit way to link differential forms and cochains. For that, it is useful to consider the space of cochains as a vector space on $\mathbb{R}$ and the family $(\tau^{k,i})_i$ as one of its bases. To make sure that the Hodge decomposition and its discrete counterpart are consistent, we then look for a linear *reduction operator* $R : \Omega^k(\Gamma) \to C^k(D)$ such that the image of close and exact forms are respectively cycles and boundaries.

We have already mentionned the link between integration of k-forms and cochains. Let us now show that the reduction by integration satisfies the desired properties.

3.1 Projections

3.1.1 Reduction operator

Definition 3.1.1 (Reduction operator [6])**.** *We define the* reduction operator *as* $R : \Omega^k(\Gamma) \to C^k(D)$ *such that* $R(\omega^k)(\tau_k) = \int_{\tau_k} \omega^k$ *for all k-form* ω^k *and all k-cube* τ_k *in D. This operator is also called the* de Rham operator*.*

Since the k-cubes are compact and since $C_k(D)$ is of finite dimension, $R(\omega^k)(\tau_k)$ is well defined for a differential form ω^k in $L^2(\Omega^k(\Gamma))$.

Let us now study the properties of R.

Proposition 3.1.1 (Reduction properties [6])**.** *The reduction operator verifies that:*

1. *R is a surjective but non-injective operator,*

2. *it commutes with the differentiation :* $Rd = \delta R$,
3. *it commutes with the pullback :* $R\phi^* = \phi^{\#}R$ *for all continuous map* $\phi : M \to N$.

Proof. The non-injectivity is obvious. The surjectivity of R is proven by the fact that each cochain $\mathbf{c}^k$ of given coefficients c_i in the basis $\tau^{k,i}$ can be obtained as the image of the form constant at c_i on each cube $\tau_{k,i}$.

The commutation of the reduction operator with the differentiation comes from Stockes theorem and its equivalent on cochains : for a k-form ω^k we have for all k-chain $\mathbf{c}_k$

$$(Rd\omega^k)(\mathbf{c}_k) = \int_{\mathbf{c}_k} d\omega^k = \int_{\partial \mathbf{c}_k} \omega^k = (R\omega^k)(\partial \mathbf{c}_k) = (\delta R\omega^k)(\mathbf{c}_k)$$

so $Rd\omega^k = \delta R\omega^k$. As it is true for any ω^k, we have $Rd = \delta R$.

The commutation with the pullback is obtained the same, using first properties of the pullback operator and then properties of the cochain map :

$$(R\phi^*\omega^k)(\mathbf{c}_k) = \int_{\mathbf{c}_k} \phi^*\omega^k = \int_{\phi_{\#}\mathbf{c}_k} \omega^k = (R\omega^k)(\phi_{\#}\mathbf{c}_k) = \phi^{\#}(R\omega^k)(\mathbf{c}_k)$$

for all k-chain $\mathbf{c}_k$, all k-form ω^k and all homeomorphism ϕ from M to another manifold. We deduce that $R\phi^*\omega^k = \phi^{\#}R\omega^k$ for all ω^k and so that $R\phi^* = \phi^{\#}R$ for all ϕ. □

The last two points ensure that any operator T which can be written as sums or compositions of pullbacks and differentiations commutes with R, in the sense of the following formula: $RT = \mathbb{T}R$, where $\mathbb{T}$ is build by replacing in the expression of T the occurences of d by δ and those of ϕ^* by $\phi^{\#}$.

The second point also guarantees that the respective images by R of the close and exact k-forms are the k-cocycles and the k-coboundaries of $C^k(D)$, as desired. Equivalence classes of $H(\Omega^k)$ are the send to equivalence classes of $H^k(D)$, which means that the reduction by integration preserves the de Rham cohomology.

3.1.2 Interpolation operator

To build the projection, it remains to choose an injective linear operator I called *interpolation operator* with the same conservation properties. Let us define the properties we expect I to satisfy. In order to ensure 3.1 for operators T given by sums or compositions of pullbacks and differentiations, we impose on I the same commutation conditions as for R, that is $dI = I\delta$ and $\phi^*I = I\phi^{\#}$ for all homeomorphism ϕ. The same condition ensures that k-cocycles and k-coboudaries of $C^k(D)$ are respectively send to close and exact k-forms by I. Therefore, the resulting projection preserves the de Rham cohomology.

We also want the resulting projection operator to be projection in that $\pi\pi = \pi$. To ensure it, we impose $RI = id$. As the numerical error depends on the precision of the projection, we want to take it close to the identity. If we note h the fineness of the considered mesh and take p as a positive integer, we finally want

$$\|\pi - id\| = \mathcal{O}(h^p). \tag{3.2}$$

The choice made on [2] is to take for I the Lagrange interpolation of order 0. We will use the same I in our numerical experiments in dimension 1, presented in the following section.

3.1.3 Projection operator

When we mesh Γ with a primal D and a dual $\tilde{D}$ complex, we obtain two projections, one on the primal mesh and another on the dual one, respectively $\pi := IR$ and $\tilde{\pi} := \tilde{I}\tilde{R}$, with $R, \tilde{R}, I$ and $\tilde{I}$ the reductors and interpolators on D and $\tilde{D}$. We note $\Omega_h^k(\Gamma)$ and $\tilde{\Omega}_h^k(\Gamma)$ the images of $\Omega^k(\Gamma)$ by π and $\tilde{\pi}$. To be rigourous, we should precise in our notations for the reduction, the interpolation and the projection operator on which space they are defined and use for instance π^k or $\tilde{\pi}^k$ rather than π and $\tilde{\pi}$. To simply the notations, we will forget the exponant when confusions are unlikely.

An important property of the projections we have defined is the following. Let f be an exact $(k+1)$-form and consider the problem

$$da = f.$$

From the first section, we know that it exists a k-form a solution of it and that this solution is unique up to a close k-form. Thanks to the properties of the projections we have just defined, the discrete equation

$$da_h = f_h$$

with f_h an element of $\Omega^{k+1}(\Gamma)$ is also well-posed in $\Omega^k(\Gamma)$.

Theorem 3.1.1 ([6]). *For $f_h \in \Omega^{k+1}(\Gamma)$, the problem*

$$da_h = f_h$$

admits a solution in $\Omega_h^k(\Gamma)$, unique up to a close form of $\Omega_h^k(\Gamma)$.

Proof. Let us consider f_h as an element of $\Omega^k(\Gamma)$. From the continuous setting, there exists a k-form a in $\Omega^k(\Gamma)$ such that

$$da - f_h = 0.$$

Applying the projection operator and using its properties, we get

$$\pi da - \pi f_h = I(Rda - Rf_h) = I(\delta Ra - Rf_h) = 0.$$

Since I is a bijection, it is equivalent to find a study the equation

$$\delta \mathbf{a} - \mathbf{f} = 0$$

with $\mathbf{f}$ in $B^{k+1}(D) = R(B^{k+1}(\Gamma))$ for $\mathbf{a}$ in $C^k(D)$. Such a $\mathbf{a}$ exists from the definition of $B^{k+1}(D)$. The linearity of δ ensures that $\mathbf{a}$ is unique up a k-cocycle : for $\mathbf{a}_1$ and $\mathbf{a}_2$ two solutions of the problem, we have

$$\delta(\mathbf{a}_1 - \mathbf{a}_2) = \mathbf{f} - \mathbf{f} = 0$$

so $\mathbf{a}_1 - \mathbf{a}_2$ is a k-cocycle. □

This shows that it makes sense to approximate the solution of the differential equation $da = f$ by the solution of the finite dimensional problem $da_h = f_h$. In practice, it is more convenient to use the formalism of the cochains and the notations introduces on section 2.6. We will then solve the problem $\delta \mathbf{a} = \mathbf{f}$. This is also true on the dual complex $\tilde{D}$.

3.2 Discrete Hodge star operator

On section 2.5, we have presented the operator $*$, which links chains on the primal and on the dual complexes. We now want an operator that associates the cochains of the two discrete spaces in a way similar to what the Hodge operator does between continuous spaces. We present below two choices we can make for this discret Hodge star operator. Both are presented in [2]. The first, denoted $\mathbf{H}$ and called the *natural Hodge operator*, is the one classically used in the mimetic discretization literature. The second one, noted $\star$ and called the *Galerkin Hodge operator*, is proposed by the authors.

The natural Hodge operator is build such that the following diagramm commutes:

$$\begin{array}{ccc} \Omega^k(\Gamma) & \xrightarrow{\star} & \Omega^{n-k}(\Gamma) \\ R^k \downarrow & & \downarrow \tilde{R}^{n-k} \\ C^k(D) & \xrightarrow{\mathbf{H}^k} & C^{n-k}(\tilde{D}) \end{array}$$

Definition 3.2.1 (Natural Hodge star operator [2])**.** *The* natural Hodge operator *is defined as* $\mathbf{H}^k = \tilde{R}^{n-k} \star I^k$.

Applying $\mathbf{H}^k$ on the cochain $\mathbf{c^k}$ consists in taking the k-form associated to $\mathbf{c^k}$, then considering it as an element of $\Omega^k(\Gamma)$, applying the Hodge star operator and coming back to the cochain space using the reduction in the dual complex. Thanks to the identity $IR = id$, we indeed have

$$\mathbf{H}R^k = \tilde{R}^{n-k} \star I^k R^k = \tilde{R}^{n-k} \star .$$

The Galerkin hodge operator is build in a different way and uses mass matrix, as in the finite elements setting. Consider $\omega_h \in \Omega_h^k(\Gamma)$, $\tilde{\nu}_h \in \tilde{\Omega}_h^{n-k}(\Gamma)$ and their expressions in the bases $(\omega^{k,i})_i$ and $(\tilde{\omega}^{n-k,i})_i$, images of the $\tau^{k,i}$ $\tilde{\tau}^{k,i}$ by I and $\tilde{I}$. The bilinearity of the wedge product and the linearity of the integration make that

$$\langle \omega_h, \tilde{\nu}_h \rangle = \int_\Gamma \omega_h \wedge \tilde{\nu}_h = \int_\Gamma \left(\sum_{i=1}^{N_k} \omega_h^i \omega^{k,i} \right) \wedge \left(\sum_{i=1}^{N_{n-k}} \tilde{\nu}_h^i \tilde{\omega}^{n-k,i} \right) = \int_\Gamma \sum_{i=1}^{N_k} \sum_{j=1}^{N_{n-k}} \omega_h^i \tilde{\nu}_h^j \left(\omega^{k,i} \wedge \tilde{\omega}^{n-k,j} \right)$$

If we take $\tilde{M}_{k,n-k}$ the matrix of $\tilde{\mathcal{M}}_{k,n-k}(\mathbb{R})$ whose coefficients are $(\tilde{M}_{k,n-k})_{i,j} = \omega^{k,i} \wedge \tilde{\omega}^{n-k,j}$, we then have

$$\langle \omega_h, \tilde{\nu}_h \rangle = \omega^T \tilde{M}_{k,n-k} \tilde{\nu}$$

where ω and $\tilde{\nu}$ are the algebraic representations of ω_h and $\tilde{\nu}_h$ in bases $(\omega^{k,i})_i$ and $(\tilde{\omega}^{n-k,i})_i$. In the same way,

$$\langle \tilde{\omega}_h, \nu_h \rangle = \tilde{\omega}^T M_{k,n-k} \nu$$

with $\tilde{\omega}_h \in \tilde{\Omega}_h^k(\Gamma)$, $\nu_h \in \Omega_h^{n-k}(\Gamma)$, $\tilde{\omega}$ and ν their algebraic representations in bases $(\omega^{k,i})_i$ and $(\tilde{\omega}^{n-k,i})_i$ and $M_{k,n-k} = \left[\tilde{\omega}^{k,i} \wedge \omega^{n-k,j} \right]_{1 \le i \le \tilde{N}_k, 1 \le j \le N_{n-k}}$.

The same argument proves

$$(\omega_h, \eta_h) = \omega^T M_{k,k} \eta$$

and

$$(\tilde{\omega}_h, \tilde{\eta}_h) = \tilde{\omega}^T \tilde{M}_{k,k} \tilde{\eta}$$

for $\eta_h \in \Omega_h^k(\Gamma)$, $\tilde{\eta}_h \in \tilde{\Omega}_h^k(\Gamma)$ and $M_{k,k}, \tilde{M}_{n-k,n-k} \in \mathcal{M}_{N_k,N_k}(\mathbb{R})$ such that $(M_{k,k})_{i,j} = \left(\omega^{k,i}, \omega^{k,j} \right)$. Since $(\cdot,\cdot)$ is a scalar product, $M_{k,k}$ is a member of $S_k^{++}(\mathbb{R})$. In particular, it is invertible.

From section 1, we know that,

$$(\omega_h, \star \tilde{nu}_h) = \langle \omega_h, \star \star \tilde{nu}_h \rangle = (-1)^{k(n-k)} \langle \omega_h, \tilde{nu}_h \rangle.$$

Using the previous matricial expressions, this gives

$$\omega^T M_{k,k} \bar{\nu} = (-1)^{k(n-k)} \omega^T M_{k,n-k} \tilde{\nu}$$

where $\bar{\nu}$ is the algebraic representation of $\star\tilde{\nu}$ in the same basis than previously. As this has to be true for all ω_h in $\Omega_h^k(\Gamma)$ and in particular for all basis form $\omega^{k,i}$, we deduce

$$\bar{\nu} = (-1)^{k(n-k)} M_{k,k}^{-1} M_{k,n-k} \tilde{\nu}.$$

Then, a possible choice $\star^k$ for a discrete Hodge operator between $\Omega_h^k(\Gamma)$ and $\tilde{\Omega}_h^{n-k}(\Gamma)$ is to take the operator whose matricial representation in the bases $(\omega^{k,i})_i$ and $(\tilde{\omega}^{n-k,i})_i$ is

$$(-1)^{k(n-k)} \tilde{M}_{k,k}^{-1} \tilde{M}_{n-k,k}.$$

The two discrete Hodge operators that we have just defined are in fact close in the following sense (see [2] for the proof).

Proposition 3.2.1 ([2])**.** *The Galerkin Hodge operator and the natural Hodge operators are equal up to discretization errors :*

$$\|\mathbb{H}^k - \star^k\| = \mathcal{O}(h^p) \quad \text{and} \quad \|\tilde{\mathbb{H}}^k - \tilde{\star}^k\| = \mathcal{O}(h^p)$$

where p is the same as in 3.2.

4 Application to 1D Poisson's equation

4.1 Periodic boundary conditions

We now apply the method described previously on Poisson's equation set on $[0,1]$, given by

$$-u'' = f$$

for a given $f \in L^2(0,1)$. We will successively consider periodic

$$u(0) = u(1)$$

and Dirichlet conditions

$$u(0) = \alpha, \quad u(1) = \beta.$$

The segment $[0,1]$ is divided in N parts c_i of equal lengths, that we will consider as our primal 1-cells (or 1-cubes). Points x_i, for $1 \le i \le N+1$, delimiting those cells are the 0-cells (or 0-cubes) of the primal mesh. In the middle of each cell c_i, we put a point $x_{i+1/2}$. The $x_{i+1/2}$ are the 0-cells of the dual mesh and the segments they delimit are the dual 1-cells. In the case of periodic boundary conditions, we associate the points x_0 and x_N. We then have the same number of 0-cells than of 1-cells, in the primal mesh as well as in the dual one.

Now, let us reformulate the equation as it is done in [11]. Introducing $q = u'$, we have

$$u' = q \quad \text{and} \quad -q' = f.$$

We will reduce one of the equation on the primal and the other on the dual mesh. Here, both pairings can be chosen. We start by reducing the primal grid for the first equation and the dual for the second one. Applying R^1 on the first equation, we get

$$R^1 du = R^1 q.$$

Thanks to the properties of the reduction, this becomes

$$\delta R^0 u = R^1 q.$$

Now, we have seen on the section devoluted to cochains that the matrix of the discrete derivative operator $\mathbb{G}$ is nothing else than the incidence matrix E_{10}. In our case, we also have $\tilde{E}_{01} = E_{10}^T$.

On the other hand, if we reduce the second equation on the dual mesh, we get

$$-\tilde{R}^1 dq = \tilde{R}^1 f$$

and then

$$-\delta \tilde{R}^0 q = \tilde{R}^1 f.$$

We combine these two equations using the Hodge star operator. In what follows, we use the natural Hodge operator which is here the identity divided by the mesh size, here $1/N$ [11]. In fact, this amounts to consider the relations

$$q_i^1 = \int_{x_i}^{x_{i+1}} q(x) \approx \frac{1}{N} q(x_{i+1/2}) = \frac{1}{N} \tilde{q^0}.$$

Notice that all relations are exact except the one that involves the Hodge operator.

We finally get

$$\tilde{\mathbf{f}}^1 = -\tilde{\mathbb{G}} \tilde{\mathbf{q}^0} = -\tilde{\mathbb{G}} \mathbf{q}^1 = \tilde{\mathbb{G}} \mathbb{G} \mathbf{u}^0,$$

where $\tilde{\mathbf{q}^0}, \mathbf{q}^1, \mathbf{u}^0$ $\tilde{\mathbf{f}}^1$ are the coefficients of q, u, f in the cochains space. To find u, it then remains to solve this system and apply the interpolation operator on the result. The operator we have to invert is nothing else than the Laplacian we use in finite differences with boundary conditions :

$$\begin{pmatrix} 2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ -1 & & -1 & 2 \end{pmatrix}$$

Using exactly the same argument, but inverting the mesh on which we reduce the two equations, we also find

$$\mathbf{f}^1 = \tilde{\mathbb{G}} \mathbb{G} \tilde{\mathbf{u}}^0.$$

We also get the discrete Laplacian for finite differences.

Note that the solution we obtain is not unique since the problem is not well-posed : if u is a solution for the Poisson's equation with periodic boundary conditions, then $u + cste$ is solution of the same problem for any constant $cste$. This is not the case for Dirichlet boundary conditions, a case we examinate now.

4.2 Dirichlet boundary conditions

When we have Dirichlet boundary conditions, we do not need to compute the value at the points x_0 and x_N since the solution is given at theses points. In out implementation, we remove these points from the primal mesh, but we keep the first and the last 1-cells of the primal mesh. Then, we have :

- $(N-2)$ 0-cells on the primal mesh,
- $(N-1)$ 0-cells on the dual mesh,
- $(N-1)$ 1-cells on the primal mesh,
- $(N-2)$ 1-cells on the dual mesh.

Thus, we still have $\mathbb{G} = \tilde{\mathbb{G}}^T$. To ensure that the resulting operator is invertible, we do not have the choice on which equation is reduced on which mesh : we must make this choice in order to obtain $\tilde{\mathbb{G}}\mathbb{G}$ because $\mathbb{G}\tilde{\mathbb{G}}$ is not invertible. From what we said for periodic boundary conditions, we must reduce the first equation on the primal mesh. Again, we obtain the finite differences discrete Laplacian for problem with Dirichlet boundary conditions, that is

$$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{pmatrix}$$

We add boundary conditions to the system by adding them to its second member, as we do for finite differences, so we solve

$$\tilde{\mathbb{G}} \mathbb{G} \mathbf{u}^0 = \tilde{\mathbf{f}}^1_{\alpha,\beta},$$

where $(\tilde{\mathbf{f}}^1_{\alpha,\beta})_1 = (\tilde{\mathbf{f}}^1)_1 - \alpha$, $(\tilde{\mathbf{f}}^1_{\alpha,\beta})_{N-2} = (\tilde{\mathbf{f}}^1)_{N-2} - \beta$ and $(\tilde{\mathbf{f}}^1_{\alpha,\beta})_i = (\tilde{\mathbf{f}}^1)_i$ for the other $1 \leq i \leq N-2$.

Conclusion

Mimetic discretization is mainly aimed at three things. The first is to construct discrete spaces that faithfully represent the original continuous space from a topological point of view, more precisely from the point of view of de Rham cohomology. The tools to achieve this are the chains and cochains of algebraic topology. The algebraic point of view allows to provide the discrete space with a structure from which a coherent discrete space for the forms defined on the continuous space follows naturally. This formalism allows to consider a discrete equivalent to the de Rham cohomology, from which the differential point of view is absent, making explicit the transformations that the reduction makes.

This reduction, by integration, is naturally imposed, which is not the case for interpolation. The conditions imposed on the latter are mainly aimed at ensuring the commutation of the projection with the differential operator and the pullbacks. These properties ensure that the actions of operators constructed from derivations or pullbacks on discrete space are similar to, or "mimic", the effect of these same operations on continuous space.

Finally, another important aspect of the method studied in this project is the fact of reducing the equations on two dual meshes. The simple fact of considering two grids instead of one already makes it possible to formulate certain equations strongly, where discretization on a single grid would have imposed a weak formulation. Another advantage is that it allows to differentiate at the discrete level physical objects of slightly different nature whose behaviour with respect to the type of orientation chosen on the definition space is not quite the same. The choice of working on two grids allows us to take advantage of another tool from the continuous framework, the Poincaré duality. By defining a discrete Hodge operator, we establish a link in the space of cochains similar to what exists between differential forms. Mimetism thus refers here to several aspects and several stages of discretisation.

Although the study of the continuous framework proved necessary to understand the construction of the proposed methods and their interest, the practical application of these methods quickly reaches some limits of this theory. Indeed, the results found in the various references cited often focus on compact varieties without edges. Probably for this reason, the construction of the mimetic operators presented in [6] or [2] are also restricted to the same types of varieties. However, it may happen that one has to work on a variety with an edge, as was the case in our second example. In these cases, the management of the conditions at the edges becomes a bit tedious, in particular because of the double grid.

To improve the present work , we could conduct more experiements and in particular on higher dimension. In fact, discrete operators that we have build for Poisson's equation in one dimension are the same as the one that we obtain with a classical finite difference setting. Without exploring higher dimensions, the theorical apparatus we used seems too convoluted.

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