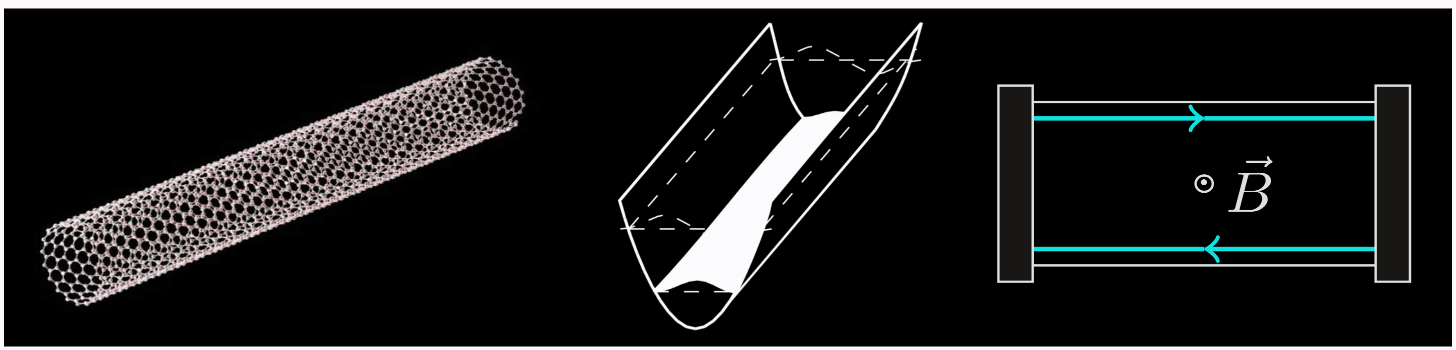
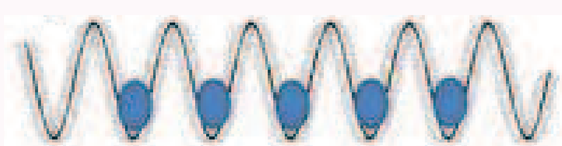
Heat waves in 1+1-dimensional Conformal Field Theory

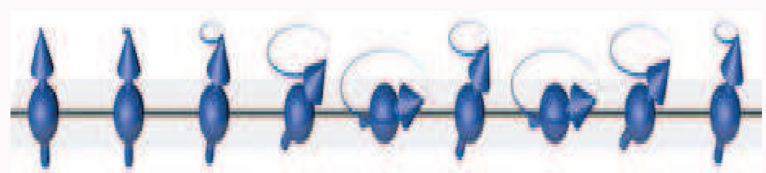
Krzysztof Gawędzki, Zurich, September 2018

- Many one-dimensional quantum systems have massless low-energy excitations described by **Conformal Field Theory**

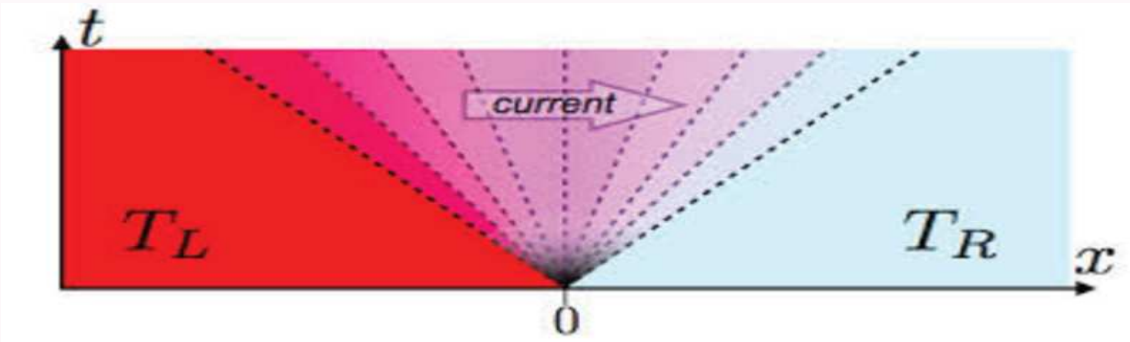
  Examples: carbon nanotubes, electrons or cold atoms trapped in $1d$ potential wells, quantum Hall edge currents, XXZ spin chains







- **1+1**-D **CFT** describes the low temperature equilibrium physics of such systems but also some of nonequilibrium situations as
  - "**quantum quenches**" to short-correlated states (reviewed by by **Calabrese-Cardy** in J. Stat. Mech. (2016), 064003)
  - "**partitioning protocol**" after two halves of a system prepared in different equilibrium states are joined together (reviewed by **Bernard-Doyon** in J. Stat. Mech. (2016), 064005, also **Hollands-Longo**, CMP 357 (2018))



- Purpose of this talk: to show how **CFT** describes the dynamics of states with a preimposed smooth temperature profile

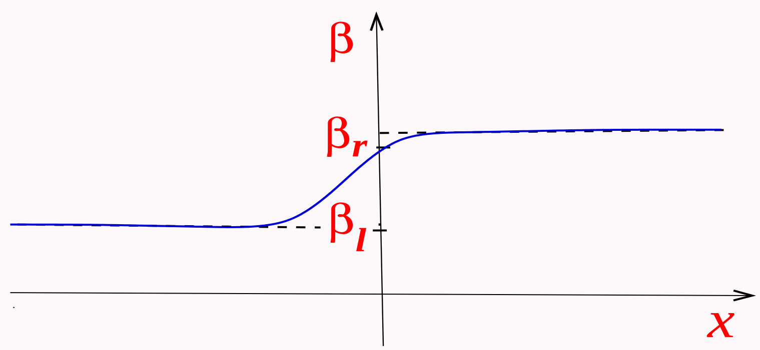
- Based on joint work with **E. Langmann** and **P. Moosavi**, J. Stat. Phys. 172 (2018), 353-378, and my article in preparation

- Inspired by **Lebowitz-Langmann-Mastropietro-Moosavi**, Phys. Rev. B 95 (2017)

LLMM studied in the **Luttinger** model of interacting $1d$ electrons the time evolution of the nonequilibrium state

$$\omega^{\rm neq}(A) = \frac{\mathrm{Tr}\big(\mathrm{e}^{-G}A\big)}{\mathrm{Tr}\big(\mathrm{e}^{-G}\big)} \qquad \text{for} \qquad G = \int \beta(x)\,\mathcal{E}(0,x)\,dx$$

where $\mathcal{E}(t,x)$ is the energy density and $\beta(x)$ is a smooth inverse-temperature profile with the values β_ℓ and (β_r) far on the left (right)



- By resumming the perturbation series in powers of $(\beta_r - \beta_\ell)$, **LLMM** showed that for the model with local interactions (which is a **CFT**)

$$\omega^{\text{neq}}(\mathcal{E}(t,x)) \;=\; \tfrac{1}{2}\big(F(x-vt)+F(x+vt)\big)$$
$$\omega^{\text{neq}}(\mathcal{J}(t,x)) \;=\; \tfrac{v}{2}\big(F(x-vt)-F(x+vt)\big)$$

where $\mathcal{J}(t,x)$ is the heat current, v is the effective **Fermi** velocity, and

$$F(x) = \frac{\pi}{6v\beta(x)^2} - \frac{v}{12\pi}\mathcal{S}_\beta(x)$$

for

$$\mathcal{S}_\beta(x) = -\frac{\beta''(x)}{\beta(x)} + \frac{1}{2}\Big(\frac{\beta'(x)}{\beta(x)}\Big)^2$$

- They noticed that $\mathcal{S}_\beta(x)$ is the **Schwarzian** derivative

$$\{f(x), x\} = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

  of the map

$$x \mapsto \int_0^x \frac{dx'}{\beta(x')} \equiv f_\beta(x)$$

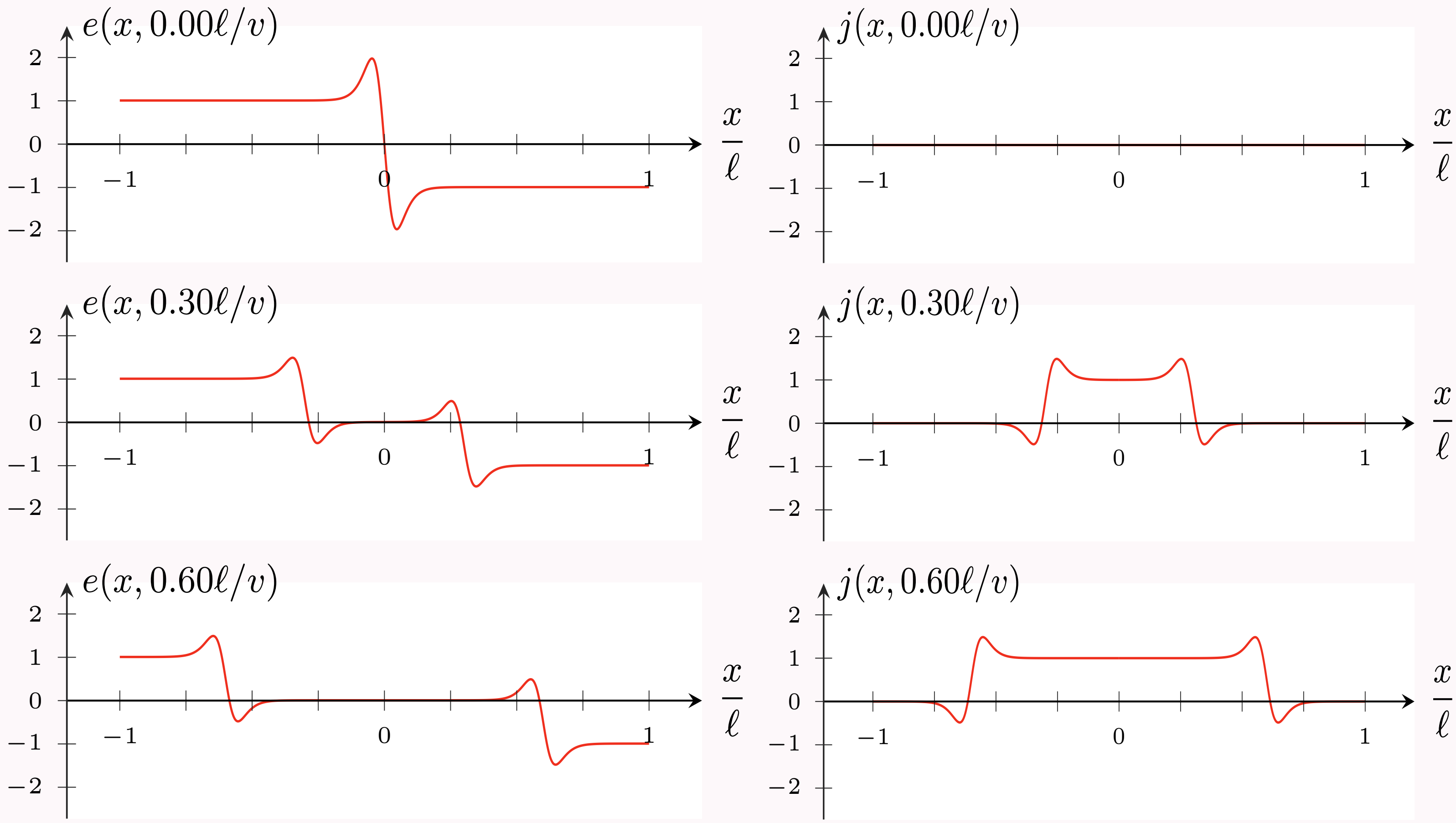
  and $\{f(x), x\}$ appears in the **CFT** formula for the transformation of the energy-momentum tensor suggesting a **CFT** origin of their result

- The formulae of **LLMM** imply that

$$\omega^{\mathrm{neq}}(\mathcal{E}(t,y)) \underset{t\to\infty}{\longrightarrow} \frac{\pi}{12v}\left(\beta_\ell^{-2} + \beta_r^{-2}\right) \equiv \mathcal{E}_0$$

$$\omega^{\mathrm{neq}}(\mathcal{J}(t,y)) \underset{t\to\infty}{\longrightarrow} \frac{\pi}{12}\left(\beta_\ell^{-2} - \beta_r^{-2}\right) \equiv \mathcal{J}_0 \neq 0$$

  but also shows a nontrivial evolution of the nonequilibrium expectations of $\mathcal{E}(t,x)$ and $\mathcal{J}(t,x)$ with traveling heat waves



Evolution of the mean energy density minus $\mathcal{E}_0$ (left)
and of the mean heat current (right)

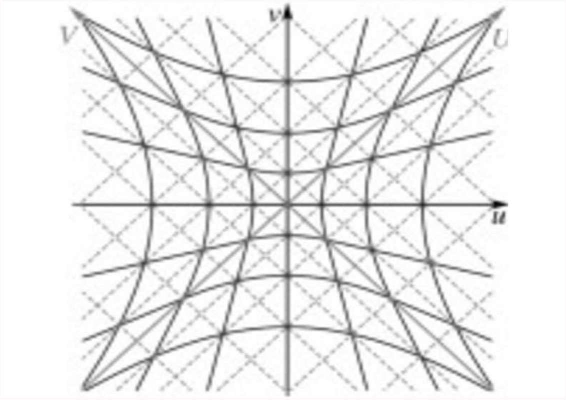
- **General theory**

  - Set $x^{\pm} \equiv x \pm vt$. The conformal transformations in $1{+}1$-D spacetime are:

$$(x^-, x^+) \mapsto (f_+(x^-), f_-(x^+))$$

since

$$v^2 dt^2 - dx^2 = dx^- dx^+ \mapsto d\,f_+(x^-) df_-(x^+) = f'_+(x^-) f'_-(x^+) dx^- dx^+$$



- In a **CFT** the infinitesimal conformal symmetries in the **Hilbert** space $\mathbb{H}$ of states are generated by the components $T_{--}(x^-)$ and $T_{++}(x^+)$ of the energy-momentum tensor s.t.

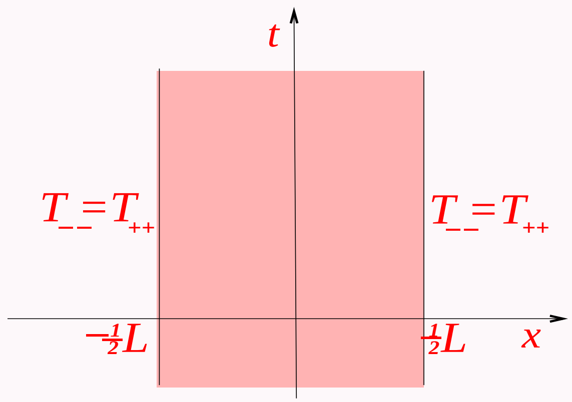
$$[T_{\substack{--\\++}}(x), T_{\substack{--\\++}}(x')] = \mp 2\mathrm{i}\,\delta'(x-x')T_{\substack{--\\++}}(x') \pm \mathrm{i}\,\delta(x-x')T'_{\substack{--\\++}}(x') \pm \frac{c\,\mathrm{i}}{24\pi}\,\delta'''(x-x')$$

  where c is the **central charge** of the theory

- The energy density and heat current in a **CFT** are

$$\mathcal{E}(t,x) = v\big(T_{--}(x^-) + T_{++}(x^+)\big)$$
$$\mathcal{J}(t,x) = v^2\big(T_{--}(x^-) - T_{++}(x^+)\big)$$

- It is convenient to work in a finite box $[-\frac{1}{2}L, \frac{1}{2}L]$ with the boundary conditions that guarantee that $T_{--}(x^-) = T_{++}(x^+)$ for $x = \pm\frac{1}{2}L$



- There is then only one independent component of the energy-moment. tensor $T_{--}(x) = T_{--}(x+2L)$ with $T_{++}(x) = T_{--}(x \pm L)$

$$T_{--}(x) = \frac{\pi}{2L^2} \sum_{n=\infty}^{\infty} e^{\frac{\pi i}{L} n(x+\frac{1}{2}L)} \left(L_n - \frac{c}{24}\delta_{n,0}\right) \equiv T(x)$$

where L_n satisfy the **Virasoro** algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}$$

- T generates a unitary projective representation $f \mapsto U_f$ of $Diff_+^{\sim} S^1$ for $f(x+2L) = f(x) + 2L$ with $f'(x) > 0$ such that

$$U_f\, T(x)\, U_f^{-1} \;=\; f'(x)^2\, T(f(x)) \,-\, \tfrac{c}{24\pi}\{f(x), x\}$$

- If f_s is the flow of a vector field $-\zeta(x)\partial_x$ with $\zeta(x+2L) = \zeta(x)$, i.e.

$$\partial_s f_s(x) = -\zeta(f_s(x))\,, \qquad f_0(x) = x$$

then

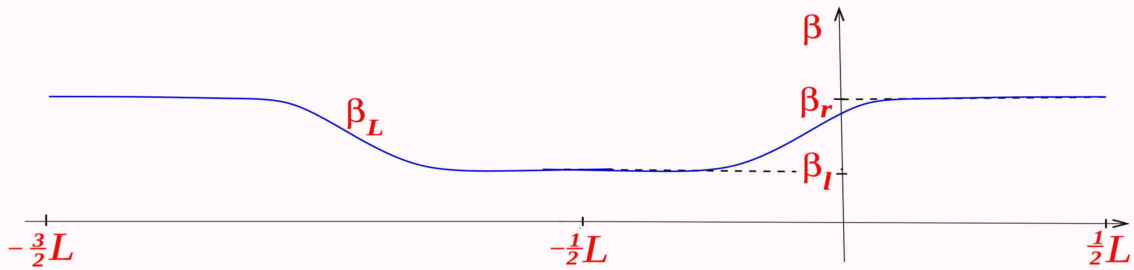
$$U_{f_s} \;=\; c_{s,\zeta}\, \exp\left[\mathrm{i}s\int_{-L}^{L} \zeta(x)\, T(x)\, dx\right]$$

E.g. for translations $f_s(x) = x - s$

$$U_{f_s} \;=\; c_{s,1}\, \mathrm{e}^{\frac{\pi \mathrm{i}}{L} s(L_0 - \frac{c}{24})}$$

- For L big enough let $\beta_L(x) = \beta_L(x+2L)$ be defined by

$$\beta_L(x) \;=\; \begin{cases} \beta(x) & \text{for} \quad x \in [-\tfrac{1}{2}L, \tfrac{1}{2}L] \\ \beta(-x-L) & \text{for} \quad x \in [-\tfrac{3}{2}L, -\tfrac{1}{2}L] \end{cases}$$



- Consider for $G_L = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \beta(x)\, \mathcal{E}(0,x)\, dx \;=\; v \int_{-\frac{3}{2}L}^{\frac{1}{2}L} \beta_L(x)\, T(x)\, dx$

  the finite-box nonequilibrium state

$$\omega_L^{\text{neq}}(A) \;=\; \frac{\mathrm{Tr}\big(\mathrm{e}^{-G_L} A\big)}{\mathrm{Tr}\big(\mathrm{e}^{-G_L}\big)}$$

- Let $f = f_L \in Diff_+^{\sim} S^1$ be such that $f'_L(x) = \frac{\beta_{0,L}}{\beta_L(x)}$ with $\beta_{0,L}$ fixed by the requirement that $f_L(x + 2L) = x + 2L$. Then

$$\boxed{U_{f_L}\, G_L\, U_{f_L}^{-1}} = v \int_{-\frac{3}{2}L}^{\frac{1}{2}L} \beta_L(x)\, U_{f_L} T(x)\, U_{f_L}^{-1}\, dx$$

$$= v \int_{-\frac{3}{2}L}^{\frac{1}{2}L} \beta_L(x)\, f'_L(x)^2\, T(f_L(x))\, dx - \frac{cv}{24\pi} \int_{-\frac{3}{2}L}^{\frac{1}{2}L} \beta_L(x)\, \{f_L(x), x\}\, dx$$

$$\overset{y=f_L(x)}{=} v\,\beta_{0,L} \int_{-\frac{3}{2}L}^{\frac{1}{2}L} T(y)\, dy - \underbrace{\frac{cv}{24\pi} \int_{-\frac{3}{2}L}^{\frac{1}{2}L} \beta_L(x) \{f_L(x), x\}\, dx}_{\textbf{c}-\textbf{number}}$$

$$= \boxed{\beta_{0,L} H_L + \text{const.}}$$

$\Rightarrow$ **the conjugation by U_{f_L} flattens the temperature profile !!!**

- This allows to compare the non-equilibrium and equilibrium finite-volume states:

$$\boxed{\omega_L^{\mathrm{neq}}(A) = \omega^{\mathrm{eq}}_{\beta_{0,L},L}\big(U_{f_L} A\, U_{f_L}^{-1}\big)}$$

- That relation may be applied to $A = \prod_i T_{--}(x_i^-) \prod_j T_{++}(x_j^+)$ for which one has the identity

$$U_{f_L}\, T_{\substack{--\\++}}(x^\mp)\, U_{f_L}^{-1} = \Big(\frac{\beta_{0,L}}{\beta_L(x^\mp)}\Big)^2 T_{\substack{--\\++}}\big(f_L(x^\mp)\big) - \frac{c}{24\pi}\{f_L(x^\mp), x^\mp\}$$

- The thermodynamic limit $L \to \infty$ is easily controlled using standard **CFT** techniques leading to the infinite-volume relations

$$\begin{aligned}
&\omega^{\mathrm{neq}}\Big(\prod_i T_{--}(x_i) \prod_j T_{++}(x_j)\Big) \\
&= \omega^{\mathrm{eq}}_{\beta_0}\bigg(\prod_i \Big(\frac{\beta_0^2}{\beta(x_i)^2} T_{--}(f_\beta(x_i)) - \frac{c}{24\pi}\{f_\beta(x_i), x_i\}\Big) \\
&\qquad \times \prod_j \Big(\frac{\beta_0^2}{\beta(x_j)^2} T_{++}(f_\beta(x_i)) - \frac{c}{24\pi}\{f_\beta(x_j), x_j\}\Big)\bigg)
\end{aligned}$$

where $f_\beta(x) = \int_0^x \frac{\beta_0}{\beta(x')}\, dx'$ with arbitrary β_0

- In the infinite-volume **CFT** equilibrium state $\omega^{\mathrm{eq}}_{\beta_0}\big(T_{\substack{--\\++}}(x^{\mp})\big) = \frac{\pi c}{12(v\beta_0)^2}$ leading to

$$\boxed{\omega^{\mathrm{neq}}\Big(T_{\substack{--\\++}}(x^{\mp})\Big) = \frac{\pi c}{12(v\beta(x_i)^2)} - \frac{c}{24\pi}\{f_\beta(x^{\mp}), x^{\mp}\}}$$

  which extends the result of **LLMM** about the nonequilibrium expectations of the energy density and the heat current to any unitary **CFT**

- In **CFT** the transport is ballistic and the conductivities are proportional to the δ-function in the frequency space with the coefficient called the **Dude weight**

- The result about the nonequilibrium heat current simplifies the calculation of the thermal **Drude weight** that may be given by

$$D_{\mathrm{th}} = \beta_0^2 \lim_{\beta(\cdot)\to\beta_0} \frac{1}{\beta_r - \beta_\ell} \lim_{t\to\infty} \frac{1}{t} \int \omega^{\mathrm{neq}}\big(\mathcal{J}(t,x)\big)\, dx$$

  One obtains the universal result $D_{\mathrm{th}} = \frac{\pi c v}{3\beta_0}$ that agrees with the one based on the **Green-Kubo** formula

- The 1-point expressions are an example of the general relations for the nonequilibrium expectations in any **CFT** model. E.g. for the connected 2-point function, one gets

$$\omega^{\mathrm{neq,\,c}}(T_{--}(x_1^-)\,;T_{--}(x_2^-))$$

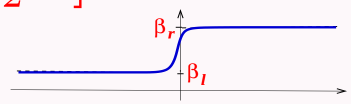
$$= \Big(\frac{\beta_0}{\beta(x_1^-)}\Big)^2\Big(\frac{\beta_0}{\beta(x_2^-)}\Big)^2\,\omega^{\mathrm{eq,\,c}}_{\beta_0}(T_{--}(f_\beta(x_1^-))\,;T_{--}(f_\beta(x_2^-)))$$
$$= \frac{\pi^2 c}{8v^4}\,\frac{1}{\beta(x_1^-)^2\beta(x_2^-)^2\,\sinh^4\big(\frac{\pi}{v\beta_0}(f_\beta(x_1^-)-f_\beta(x_2^-))\big)}$$

- One may deal in the same way the nonequilibrium expectations with insertions of primary fields

- Analogous arguments work for states with temperature and chemical potential profiles in **CFT**'s with $u(1)$-current algebra symmetries by combining conformal and gauge transformations

• Full counting statistics for the heat transfer

- For the profile states, one may obtain exact formulae for the full counting statistics (**FCS**) of the heat transfers across the kink in a $\beta(x)$-profile

- Consider a **CFT** on $[-\frac{1}{2}L, \frac{1}{2}L]$ with the boundary conditions as before. If the kink in $\beta(x)$  is narrow then

$$G_L = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \beta(x)\, \mathcal{E}(0,x)\, dx \;=\; \beta_\ell E_\ell + \beta_r E_r$$

where E_ℓ and E_r are the energies to the left and to the right of the kink, respectively

- One accesses the **FCS** of the heat transfers by performing two measurement of G_L in the nonequilibrium state ω_L^{neq} separated by time t

- By spectral decomposition

$$G_L = \sum_i g_i P_i \,, \qquad G_L(t) \equiv \mathrm{e}^{\mathrm{i}tH_L} G_L \mathrm{e}^{-\mathrm{i}tH_L} = \sum_i g_i P_i(t)$$

  If the 1^{st} measurement gives the value g_i and the 2^{nd} one g_j then the transfer of the energy across the kink in time t is

$$\Delta e = E_r(t) - E_r(0) = -(E_\ell(t) - E_\ell(0)) = \frac{g_j - g_i}{\Delta\beta}$$

  where $\Delta\beta = \beta_r - \beta_\ell$

- By the **QM** rules the probability of getting the results (g_i, g_j) is

$$p_{ij} = \omega_L^{\mathrm{neq}}\Big(P_i P_j(t)\Big)$$

  giving for the **PDF** of the energy transfers

$$p_{t,L}(\Delta e) = \sum_{ij} \delta\Big(\Delta e - \tfrac{g_j - g_i}{\Delta\beta}\Big)\, \omega^{\mathrm{neq}}\Big(P_i P_j(t)\Big)$$

- The characteristic function of the probability distribution of Δe is

$$\mathcal{F}_{t,L}(\lambda) \equiv \int \mathrm{e}^{\mathrm{i}\lambda\Delta e}\, p_{t,L}(\Delta e) = \sum_{i,j} \mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}(g_j - g_i)}\, \omega_L^{\mathrm{neq}}\Big(P_i P_j(t)\Big)$$

$$= \omega_L^{\mathrm{neq}}\Big(\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}G_L}\, \mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}G_L(t)}\Big) = \omega^{\mathrm{eq}}_{\beta_{0,L},L}\Big(U_{f_L}\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}G_L}\, \mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}G_L(t)} U_{f_L}^{-1}\Big)$$

using our relation between the nonequilibrium and equilibrium states

$$U_{f_L} G_L\, U_{f_L}^{-1} = \beta_{0,L} H_L - \underbrace{\frac{cv}{24\pi}\int_{-\frac{3}{2}L}^{\frac{1}{2}L} \beta_L(x)\{f_L(x), x\}\, dx}_{\textbf{c}-\textbf{number}}$$

$$U_{f_L} G_L(t)\, U_{f_L}^{-1} = \int_{-\frac{3}{2}L}^{\frac{1}{2}L} \zeta_{L,t}(y)\, T(y)\, dy - \underbrace{\frac{cv}{24\pi}\int_{-\frac{3}{2}L}^{\frac{1}{2}L} \beta_L(x^+)\{f_L(x), x\}\, dx}_{\textbf{c}-\textbf{number}}$$

where $\quad \zeta_{L,t}(y) = v\beta_{0,L}\,\frac{\beta_L(f_L^{-1}(y)+vt)}{\beta_L(f_L^{-1}(y))}$

Since $H_L = \frac{\pi}{L}(L_0 - \frac{c}{24})$,

$$\mathcal{F}_{L,t}(\lambda) = \omega_L^{\mathrm{neq}}\Big(\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}G_L}\,\mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}G_L(t)}\Big)$$

$$= \boxed{\frac{\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau_s(L_0-\frac{c}{24})}\,U_{f_s}\Big)}{\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau_0(L_0-\frac{c}{24})}\Big)}}\;\frac{\mathrm{e}^{\mathrm{i}s\,\frac{cv}{24\pi}\int_{-\frac{3}{2}L}^{\frac{1}{2}L}\big(\beta_L(x)-\beta_L(x^+)\big)\{f_L(x),x\}\,dx}}{c_{s,\zeta_{L,t}}}$$

for $s = \frac{\lambda}{\Delta\beta}$, $\tau_s = \frac{(\mathrm{i}-s)v\beta_{0,L}}{2L}$, and $f_s \in Diff_+^{\sim}S^1$ solving the flow equation $\partial_s f_s(y) = -\zeta_{L,t}(f_s(y))\,,\ f_0(y) = y$

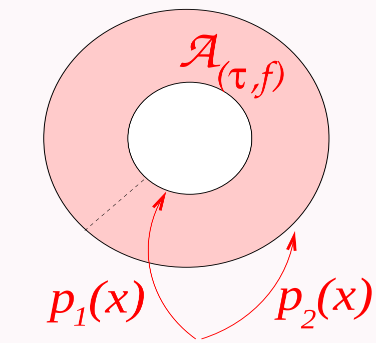
- One usually views the denominator $\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{c}{24})}\Big)$ as the character of the **Virasoro** algebra representation in the space of states of **CFT**

- Similarly, the numerator $\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{c}{24})}\,U_f\Big)$ may be viewed as the character of the corresponding representation of $Diff_+^{\sim}S^1$

• Characters of $Diff^{\sim}_{+}S^1$

- The characters of $Diff^{\sim}_{+}S^1$ may be reduced to those of the respective **Virasoro** representation (this did not seem to exist in the literature)
- According to **G. Segal**, the operator $e^{2\pi i\tau(L_0-\frac{c}{24})}U_f$ is proportional to the chiral **Euclidian CFT** amplitude of the complex annulus

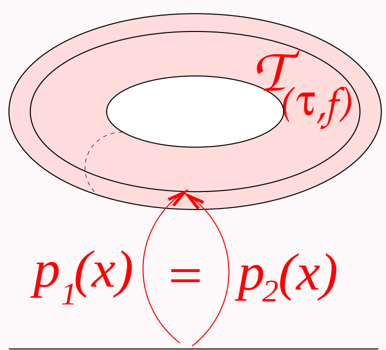
$$\mathcal{A}_{\tau,f} = \{ z \,|\, |e^{2\pi i\tau}| \le |z| \le 1 \}$$



with the boundary components parameterized by

$$p_1(x) = e^{2\pi i\tau} e^{-\frac{\pi i}{L} f(x)}, \qquad p_2(x) = e^{-\frac{\pi i}{L} x}$$

- Usually, group characters are **class functions** invariant under the adjoint action



What it means here is that (up to a scalar factor) $\mathrm{Tr}\left(e^{2\pi i\tau(L_0-\frac{c}{24})}U_f\right)$ depends only on the torus $\mathcal{T}_{\beta,f}$ obtained from $\mathcal{A}_{\tau,f}$ by sewing togeather its parameterized boundary components

- Indeed, $\mathrm{Tr}\left(\mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{c}{24})}\,U_f\right)$ is proportional to the **CFT** amplitude of the torus $\mathcal{T}_{\beta,f}$ with its natural complex structure

- The complex torus $\mathcal{T}_{\beta,f}$ is isomorphic to $\mathcal{T}_{\widehat{\tau},f_0}$ for $f_0(x)\equiv x$ and some $\widehat{\tau}$ in the upper half plane. This implies the relation

$$\mathrm{Tr}\left(\mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{c}{24})}\,U_f\right) \,=\, C_{\tau,f}\ \mathrm{Tr}\left(\mathrm{e}^{2\pi\mathrm{i}\widehat{\tau}(L_0-\frac{c}{24})}\right)$$

  where the trace on the right-hand-side is the **CFT** amplitude of the annulus $\mathcal{A}_{\widehat{\tau},f_0}$ and $C_{\tau,f}$ is a complex number due to the projective character of the chiral **CFT** amplitudes

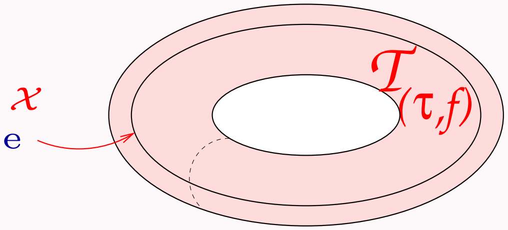
- The constant $C_{\tau,f}$ may be expressed in terms of determinants of **Fredholm** operators on $L^2(\mathbb{R}/(2L\mathbb{Z}))\equiv\mathcal{H}$ that appear in the context of a **Riemann**-**Hilbert**-type problem on the torus $\mathcal{T}_{\tau,f}$

- $\widehat{\tau}$ may be obtained by solving a related **Fredholm** equation

- **The Riemann-Hilbert problem on $\mathcal{T}_{\tau,f}$**

- Given a function $X \in \mathcal{H}$ one searches for a holomorphic function $\mathcal{X}$ on $\mathcal{A}_{\tau,f}$ such that

$$X = X_1 - X_2 \qquad \text{for} \qquad X_i = \mathcal{X} \circ p_i$$



- Let $P_>$ and $P_<$ be the orthogonal projectors in $\mathcal{H}$ on the subspaces spanned by functions $\mathrm{e}^{-\frac{\pi \mathrm{i}}{L} nx}$ with $n > 0$ and $n < 0$, respectively

- Let $Q_{\tau,f} : \mathcal{H}_0 \longrightarrow \mathcal{H}_0$, for $\mathcal{H}_0 \subset \mathcal{H}$ composed of functions with vanishing integral, be the operator

$$(P_> + P_<)(X_1 - X_2) \xrightarrow{Q_{\tau,f}} P_> X_1 - P_< X_2$$

- $Q_{\tau,f}$ is **traceclass**. Explicitly

$$Q_{\tau,f} = (K_{11}+K_{12}-K_{21})(I-K_{11}-K_{12}-K_{21})^{-1}(P_{<}-K_{12})-K_{12}$$

where $K_{ij} : \mathcal{H} \longrightarrow \mathcal{H}$ have smooth kernels (on the circle $\mathbb{R}/(2L\mathbb{Z})$)

$$(K_{11}X)(x) = \frac{1}{2L}\int_{-\frac{3}{2}L}^{\frac{1}{2}L}\left(\frac{f'(y)}{\mathrm{e}^{\frac{\pi\mathrm{i}}{L}(f(y)-f(x))}-1} - \frac{1}{\mathrm{e}^{\frac{\pi\mathrm{i}}{L}(y-x)}-1}\right)X(y)\,dy$$

$$(K_{12}X)(x) = -\frac{1}{2L}\int_{-\frac{3}{2}L}^{\frac{1}{2}L}\frac{1}{\mathrm{e}^{2\pi\mathrm{i}\tau}\mathrm{e}^{\frac{\pi\mathrm{i}}{L}(y-f(x))}-1}\,X(y)\,dy$$

$$(K_{21}X)(x) = \frac{1}{2L}\int_{-\frac{3}{2}L}^{\frac{1}{2}L}\frac{f'(y)}{\mathrm{e}^{-2\pi\mathrm{i}\tau}\mathrm{e}^{\frac{\pi\mathrm{i}}{L}(f(y)-x)}-1}\,X(y)\,dy$$

and as such are traceclass

- The theory of determinant bundles of **Quillen** and **Segal** implies that

$$\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau L_0}\,U_f\Big) \;=\; \underbrace{\Big(\frac{\det(I-Q_{\tau,f})}{\det(I-Q_{\widehat{\tau},f_0})}\Big)^{\frac{c}{2}}\,\langle 0|U_f\,|0\rangle}_{\mathrm{e}^{\frac{\pi c}{12}\mathrm{i}(\tau-\widehat{\tau})}C_{\tau,f}}\;\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\widehat{\tau}L_0}\Big)$$

- This reduces the characters of $Diff^{\sim}_{+}S^1$ to the more standard ones of the **Virasoro** algebra and to the vacuum expectation value of U_f

- The effective modular parameter $\hat{\tau}$ solves the equation

$$\hat{\tau}-\tau=\frac{1}{(2L)^2}\int_{-\frac{3}{2}L}^{\frac{1}{2}L}(f-f_0)(dX_1-df)$$

where X_1 is the inner boundary value $\mathcal{X}\circ p_1$ of a holomorphic function on $\mathcal{T}_{\tau,f}$ with the jump $f-f_0+2L(\hat{\tau}-\tau)$ across the sewing line

- One obtains this way a reduced formula for the characteristic function $\mathcal{F}_{L,t}(\lambda)$ of **FCS** in which $c_{s,\zeta}^{-1}\,\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau_s(L_0-\frac{c}{24})}\,U_{f_s}\Big)$ was the only non-explicit entry

- **The vacuum matrix element of** U_{f_s}

  - For the flow $f_s(x)$ of the vector field $-\zeta(x)\partial_x$

$$c_{s,\zeta}^{-1}\,\langle 0\big|U_{f_s}\,\big|0\rangle \,=\, \langle 0\big|\mathrm{e}^{\mathrm{i}s\int\zeta(x)\,T(x)\,dx}\big|0\rangle \,=\, \mathrm{e}^{-\frac{c}{2}\int_0^s \mathrm{Tr}\big(V_{\zeta,f_{s'}}\big)\,ds'}$$

    where $V_{\zeta,f}=P_>F_fP_<(\zeta\partial)P_>(P_>F_fP_>)^{-1}$ for $(F_fX)(x)=X(f^{-1}(x))$ is a traceclass operator

  - The above formula may be obtained from the result for massless free field, giving a **Fredholm** determinant (**Bruneau-Dereziński** 2007), raised to power c

  - It represents the characteristic function of the distribution of coarse-grained energy density in the vacuum state

  - In recent paper by **Fewster-Hollands** arXiv:1805.0428 it was related to a conformal welding problem, a cousin of the **Riemann-Hilbert** one, and a differential equation was proposed for $\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau_0(L_0-\frac{c}{24})}\,U_{f_s}\Big)$

- **Summarizing:** the generating function $\mathcal{F}_{L,t}(\lambda)$ of **FCS** is expressed by the **Virasoro** character of the space of states and explicit **Fredholm** determinants

• **FCS** for the heat transfer in the thermodynamic limit

- The formula for characteristic function of the **FCS** heat transfer simplifies in the limit $L \to \infty$ giving

$$\mathcal{F}_t(\lambda) = \lim_{L\to\infty} \mathcal{F}_{L,t}(\lambda) = \mathrm{e}^{-\frac{c}{2}\sum\limits_{+,-}\int\limits_0^s \mathrm{Tr}\Big((I-\mathcal{Q}^{\pm}_{t,s'})^{-1}\partial_{s'}\mathcal{Q}^{\pm}_{t,s'} - \mathcal{V}^{\pm}_{t,s'}\Big)ds'}$$
$$\times\ \mathrm{e}^{\mathrm{i}s\,\frac{cv}{24\pi}\int\Big(2\beta(x)-\beta(x^+)+\beta(x^-)\Big)\{f_\beta(x),x\}\,dx}$$

where operators $\mathcal{Q}$ in $L^2(\mathbb{R})$ are related to the integral operators $\mathcal{K}_{ij}$ obtained in the $L \to \infty$ limit from K_{ij}

$$\mathcal{Q} = (\mathcal{K}_{11} + \mathcal{K}_{12} - \mathcal{K}_{21})(I - \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21})^{-1}(P_< - \mathcal{K}_{12}) - \mathcal{K}_{12}$$

$$(\mathcal{K}_{11}X)(x) = \tfrac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \Big(\frac{f'(y)}{f(y) - f(x)} - \frac{1}{y - x} \Big) X(y)\, dy$$

$$(\mathcal{K}_{12}X)(x) = -\tfrac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{1}{y - f(x) + (\mathrm{i} - s)v\beta_0} X(y)\, dy$$

$$(\mathcal{K}_{21}X)(x) = \tfrac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{f'(y)}{f(y) - x - (\mathrm{i} - s)v\beta_0} X(y)\, dy$$

$\mathcal{Q}^{\pm}_{t,s}$ correspond to $f(y) = f_{\pm s}(\pm y)$ for $\partial_s f_{\pm s}(\pm y) = \mp\zeta_{\pm t}(f_{\pm s}(\mp y))$ (right- and left-movers contributions) with $\zeta_{\pm t}(y) = v\beta_0 \frac{\beta(f_\beta^{-1}(y) \pm vt)}{\beta(f_\beta^{-1}(y))}$, $f_\beta(x) = \int_0^x \frac{\beta_0}{\beta(x')}\, dx'$, $\beta_0^{-1} = \frac{1}{2}(\beta_{\mathcal{L}}^{-1} + \beta_{\mathcal{R}}^{-1})$

- Operators $\mathcal{V}^{\pm}_{t,s}$ are obtained from $V_{\zeta,f}$ in the $L\to\infty$ limit by setting $\zeta(y)=\zeta_{\pm t}(\pm y)$ and $f(y)=f_{\pm s}(\pm y)$
- It follows that $\mathcal{F}_t(\lambda)$ is universal depending only on the profile $\beta(x)$ and the central charge c of the **CFT**
- $1^{\rm st}$ moment $\big\langle(\Delta e)(t)\big\rangle=\frac{1}{\rm i}\partial_\lambda\big|_{\lambda=0}\mathcal{F}_t(\lambda)$

$$=\frac{\pi c}{12(v\beta_0)^2\Delta\beta}\sum_{\pm}\hat{\xi}^{\pm}_t(0)+\frac{cv}{24\pi\Delta\beta}\sum_{\pm}\int(\beta(x)-\beta(x^{\pm}))\{f_\beta(x),x\}dx$$

where $\hat{\xi}^{\pm}_t(p)=\int {\rm e}^{{\rm i}px}\big(\zeta_{\pm t}(\pm x)-v\beta_0\big)dx$

- $2^{\rm nd}$ moment $\big\langle(\Delta e)(t)\,;(\Delta e)(t)\big\rangle^c=-\partial^2_\lambda\big|_{\lambda=0}\ln\mathcal{F}_t(\lambda)$

$$=\frac{c}{48\pi^2(\Delta\beta)^2}\sum_{\pm}\int\frac{p(p^2+\frac{4\pi^2}{(v\beta_0)^2})}{1-{\rm e}^{-v\beta_0p}}\,\hat{\xi}^{\pm}_t(p)\,\hat{\xi}^{\pm}_t(-p)\,dp$$

- **Fano** factor $\dfrac{\big\langle(\Delta e)(t)\,;(\Delta e)(t)\big\rangle^c}{\big\langle(\Delta e)(t)\big\rangle}$?

- One should be able to extract the large deviations asymptotics of **Bernard-Doyon** (2012)

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathcal{F}_t(\lambda) \;=\; \frac{\pi c}{12}\Big(\frac{1}{\beta_\ell - \mathrm{i}\lambda} - \frac{1}{\beta_\ell} + \frac{1}{\beta_r + \mathrm{i}\lambda} - \frac{1}{\beta_r}\Big) \equiv \Phi(\lambda)$$

  from our exact formula for $\mathcal{F}_t(\lambda)$

- The above formula means that at large t the energy transfers $\Delta e(t)$ become a **Lévy** process with the jump rates

$$w(x,y) = \frac{\pi c}{12}\Big(\mathrm{e}^{-\beta_\ell (y-x)}\theta(y-x) + \mathrm{e}^{-\beta_r (x-y)}\theta(x-y)\Big)$$

  where $\theta(\,\cdot\,)$ is the **Heaviside** step function

- For large t, $p_t(\Delta e) \propto \mathrm{e}^{\,tI(\frac{\Delta e}{t})}$ with the rate function

$$I(\sigma) = \min_{\nu\in[-\beta_r,\beta_\ell]}\Big(\Phi(-\mathrm{i}\nu) - \nu\sigma\Big) \;=\; \frac{\pi c}{12}\begin{cases} -\beta_\ell\sigma + o(\sigma) & \text{for} \quad \sigma\to\infty \\ \;\;\beta_r\sigma + o(\sigma) & \text{for} \quad \sigma\to-\infty \end{cases}$$

  possessing the **Gallavotti-Cohen** symmetry $I(-\sigma) = I(\sigma) - \sigma\Delta\beta$

Conclusions

- In a **CFT** conformal symmetries may be used to map inhomogeneous situations to homogeneous ones
- That allowed to express nonequilibrium expectations in states with temperature profile in terms of equilibrium ones
- Profile states were e.g. shown to describe on mesoscopic scales dense cold gases in $1d$ traps
- States where one imposes also the profiles of chemical potential can be treated similarly in theories with current-algebra symmetries (e.g. in the local **Luttinger** model)
- The general results confirmed and extended the particular ones obtained by **LLMM** for the **Luttinger** model through perturbative calculations
- The **FCS** statistics of energy transfers in such states was reduced to the character of $Diff_+^{\sim} S^1$ and shown to become a universal functional of the temperature profile in the thermodynamic limit