Band topology and submanifolds of matrices

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Theme of my talk

Submanifold in the space of matrices

I will relate:

- **cohomology classes on such submanifolds;**
- **topological invariants of quantum systems on lattices such as insulators and semimetals.**

What I will talk about is something like a basic idea for future possible works, and it stemmed from discussion with collaborators:

Ken Shiozaki, Masatoshi Sato, Guo Chuan Thiang, ...

Plan of my talk

1. **Introduction**
2. **Submanifolds in the space of matrices**
3. **Change of topological numbers**
4. **Semimetal**

Quantum system on lattice

- I would like to consider a certain quantum mechanical system on a lattice $\mathbb{Z}^d \subset \mathbb{R}^d$.
- The Hilbert space in this system is

$$L^2(\mathbb{Z}^d, \mathbb{C}^n) = \left\{ \psi = (\psi(j))_{j \in \mathbb{Z}^d} \middle| \sum \|\psi(j)\|^2 < \infty \right\},$$

  where n is the internal freedom at each site.
- The Hamiltonian $H : L^2(\mathbb{Z}^d, \mathbb{C}^n) \to L^2(\mathbb{Z}^d, \mathbb{C}^n)$ is a self-adjoint operator which is
  - commuting with the translation operator on the lattice,
  - describing a "short range interaction".

Fourier transformed Hamiltonian

- Under the assumptions on the Hamiltonian H, its information is completely encoded into the Fourier transformed Hamiltonian

$$\hat{H} : T^d \to \mathcal{H}(\mathbb{C}^n).$$

This is a continuous map from the d-dimensional torus (BZ torus) to the space of $n \times n$ Hermitian matrices

$$\mathcal{H}(\mathbb{C}^n) = \{H \in M(n, \mathbb{C}) | \; H^\dagger = H\} \cong \mathbb{R}^{n^2}.$$

- Let us say that the quantum system is
  - insulator if $\det \hat{H}(k) \neq 0$ for all $k \in T^d$,
  - semimetal if $\det \hat{H}(k) = 0$ at some $k \in T^d$.

(The definition may be physically insufficient.)

Classification of topological phase by homotopy

- Now, the classification of the topological phase described by $\hat{H}$ can be done by the classification of homotopy class of $\hat{H}$ within a fixed type of phases.
- For example, if $\hat{H}$ is an insulator $(\det \hat{H} \neq 0)$, then we consider the homotopy within insulators. In other words, we consider the homotopy classes of

$$T^d \to \mathcal{H}_0(\mathbb{C}^n) = \{H \in \mathcal{H}(\mathbb{C}^n) |\ \det H \neq 0\}.$$

- A more equivalence relation (such as "stability" under an addition of uninteresting phases) can be introduced , but I will not consider it today for simplicity.

Topological invariant: the case of insulator

- Then, for the classification of the homotopy class of $\hat{H}$, it is useful to consider a topological invariant of $\hat{H}$, a quantity invariant under the homotopy deformation.
- For example, in the case of an insulator

$$\hat{H} : T^d \to \mathcal{H}_0(\mathbb{C}^n) = \{H \in \mathcal{H}(\mathbb{C}^n) | \ \det H \neq 0\},$$

one has the Chern classes of the Bloch vector bundle

$$E = \bigcup_{k \in T^d} \bigoplus_{\lambda < 0} \mathrm{Ker}(\hat{H}(k) - \lambda).$$

- In 2D case, the integration of the Chern class $c_1(E)$ is a topological invariant of the insulator.

Topological invariant: the case of insulator

- In the example of an insulator, the Chern classes of Bloch bundles are closely related to the subspace

$$\mathcal{H}_0(\mathbb{C}^n) = \{H \in \mathcal{H}(\mathbb{C}^n) |\ \det H \neq 0\}$$

in the space of Hermitian matrices $\mathcal{H}(\mathbb{C}^n)$.

- Actually, $\mathcal{H}_0(\mathbb{C}^n) \subset \mathcal{H}(\mathbb{C}^n)$ is an open submanifold homotopy equivalent to the Grassmannian of $\mathbb{C}^n$.
- To consider semimetals, we allow singularities for the Hamiltonian $\hat{H} : T^d \to \mathcal{H}(\mathbb{C}^n)$, namely, points $k \in T^d$ at which $\det \hat{H}(k) = 0$.
- This leads us to consider subspaces in $\mathcal{H}(\mathbb{C}^n)$ other than $\mathcal{H}_0(\mathbb{C}^n)$.

The things I would like to talk today:

1. **Some subspaces of matrices give rise to submanifolds.**
2. **Some cohomology classes on such submanifolds provide invariants of insulators and semimetals.**
3. **An application is a formula of the change of topological numbers under a deformation allowing singularities.**

The things I would like to talk today:

1. **Some subspaces of matrices give rise to submanifolds.**
2. **Some cohomology classes on such submanifolds provide invariants of insulators and semimetals.**
3. **An application is a formula of the change of topological numbers under a deformation allowing singularities.**

- **Originally, I anticipated that a use of the submanifolds of matrices can be applied to a detection of new topological phases. But, the things I tell you today are still on the way to the goal.**

The space of complex square matrices

- **To warm up, I would like to consider submanifolds in the space of complex matrices $\mathcal{M}(\mathbb{C}^n) = M(n, \mathbb{C})$, rather than those in the space of Hermitian matrices $\mathcal{H}(\mathbb{C}^n)$.**

The space of complex square matrices

- To warm up, I would like to consider submanifolds in the space of complex matrices $\mathcal{M}(\mathbb{C}^n) = M(n, \mathbb{C})$, rather than those in the space of Hermitian matrices $\mathcal{H}(\mathbb{C}^n)$.
- If we consider a quantum system of class AIII, then the chiral symmetry arrows us to assume that a Fourier-transformed Hamiltonian is of the form

$$\hat{H}(k) = \begin{pmatrix} 0 & A(k) \\ A(k)^\dagger & 0 \end{pmatrix}.$$

Hence the Hamiltonian $\hat{H}$ amounts to

$$A : \ T^d \to \mathcal{M}(\mathbb{C}^n) = M(n, \mathbb{C}).$$

- Because of this fact, it is still meaningful to study $\mathcal{M}(\mathbb{C}^n)$ in the context of quantum systems.

Subspaces in $\mathcal{M}(\mathbb{C}^n) = M(n, \mathbb{C}) \cong \mathbb{C}^{n^2}$

Definition

For an integer k, we put

$$\mathcal{M}_{\leq k}(\mathbb{C}^n) := \{A \in \mathcal{M}(\mathbb{C}^n) | \dim \mathrm{Ker} A \leq k\},$$
$$\mathcal{M}_{k}(\mathbb{C}^n) := \{A \in \mathcal{M}(\mathbb{C}^n) | \dim \mathrm{Ker} A = k\}.$$

Here A is regarded as a linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ by the left multiplication.

$$\mathcal{M}_{\leq k}(\mathbb{C}^n) = \mathcal{M}_{\leq k-1}(\mathbb{C}^n) \cup \mathcal{M}_{k}(\mathbb{C}^n)$$

$$\cdots \subset \mathcal{M}_{\leq k-1}(\mathbb{C}^n) \subset \mathcal{M}_{\leq k}(\mathbb{C}^n) \subset \cdots$$

- **In general, $\mathcal{M}_{\leq k}(\mathbb{C}^n) \subset \mathcal{M}(\mathbb{C}^n)$ is an open complex submanifold, and $\dim_{\mathbb{C}} \mathcal{M}_{\leq k} = \dim_{\mathbb{C}} \mathcal{M} = n^2$.**

Example: n = 1

$$\mathcal{M}_{\leq k}(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n) \mid \dim \operatorname{Ker} A \leq k\} \overset{\text{open}}{\subset} \mathcal{M}(\mathbb{C}^n),$$
$$\mathcal{M}_k(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n) \mid \dim \operatorname{Ker} A = k\}.$$

$$\begin{array}{l} \mathcal{M}_{\leq 1}(\mathbb{C}) = M(1, \mathbb{C}) = \mathbb{C} \\ \quad \bigcup \\ \mathcal{M}_{\leq 0}(\mathbb{C}) = GL(1, \mathbb{C}) = \mathbb{C}^\times \end{array} \qquad \mathcal{M}_1(\mathbb{C}) = \{0\}$$

- In general, we have obvious identifications

$$\mathcal{M}_{\leq n}(\mathbb{C}^n) = \mathcal{M}(\mathbb{C}^n) \cong \mathbb{C}^{n^2}, \qquad \mathcal{M}_n(\mathbb{C}^n) = \{0\},$$
$$\mathcal{M}_{\leq n-1}(\mathbb{C}^n) \cong \mathbb{C}^{n^2} \backslash \{0\},$$
$$\mathcal{M}_{\leq 0}(\mathbb{C}^n) = \mathcal{M}_0(\mathbb{C}^n) = GL(n, \mathbb{C}).$$

Example: $n = 2$

$$\mathcal{M}_{\leq k}(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n) |\ \dim \mathrm{Ker} A \leq k\} \overset{\text{open}}{\subset} \mathcal{M}(\mathbb{C}^n),$$
$$\mathcal{M}_k(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n) |\ \dim \mathrm{Ker} A = k\}.$$

$$\mathcal{M}_{\leq 2}(\mathbb{C}^2) = M(2, \mathbb{C}) \cong \mathbb{C}^4$$
$$\bigcup \qquad \mathcal{M}_2(\mathbb{C}^2) = \{0\}$$
$$\mathcal{M}_{\leq 1}(\mathbb{C}^2) \cong \mathbb{C}^4 \backslash \{0\}$$
$$\bigcup \qquad \mathcal{M}_1(\mathbb{C}^2) = \{A \neq 0, \det A = 0\}$$
$$\mathcal{M}_{\leq 0}(\mathbb{C}^2) = GL(2, \mathbb{C})$$

Subspace $\mathcal{M}_k(\mathbb{C}^n) \subset \mathcal{M}_{\leq k}(\mathbb{C}^n)$

$$\mathcal{M}_{\leq k}(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n) \mid \dim \mathrm{Ker} A \leq k\} \overset{\text{open}}{\subset} \mathcal{M}(\mathbb{C}^n),$$
$$\mathcal{M}_k(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n) \mid \dim \mathrm{Ker} A = k\}.$$

Fact

$\mathcal{M}_k(\mathbb{C}^n) \subset \mathcal{M}_{\leq k}(\mathbb{C}^n)$ is a closed complex submanifold of

$$\dim_{\mathbb{C}} \mathcal{M}_k(\mathbb{C}^n) = n^2 - k^2.$$

$$\begin{array}{c} \mathcal{M}_{\leq 1}(\mathbb{C}) = M(1, \mathbb{C}) = \mathbb{C} \\ \bigcup \\ \mathcal{M}_{\leq 0}(\mathbb{C}) = GL(1, \mathbb{C}) = \mathbb{C}^{\times} \end{array} \qquad \mathcal{M}_1(\mathbb{C}) = \{0\}$$

Example: $n = 2$

$$\mathcal{M}_{\leq k}(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n)|\ \dim \mathrm{Ker} A \leq k\} \overset{\text{op}}{\subset} \mathcal{M}(\mathbb{C}^n),$$
$$\mathcal{M}_{k}(\mathbb{C}^n) = \{A \in \mathcal{M}(\mathbb{C}^n)|\ \dim \mathrm{Ker} A = k\} \underset{\text{cl}}{\subset} \mathcal{M}_{\leq k}(\mathbb{C}^n).$$

$$\mathcal{M}_{\leq 2}(\mathbb{C}^2) = M(2, \mathbb{C}) \cong \mathbb{C}^4$$
$$\bigcup$$
$$\mathcal{M}_{\leq 1}(\mathbb{C}^2) \cong \mathbb{C}^4 \backslash \{0\}$$
$$\bigcup$$
$$\mathcal{M}_{\leq 0}(\mathbb{C}^2) = GL(2, \mathbb{C})$$

$$\mathcal{M}_{2}(\mathbb{C}^2) = \{0\}$$

$$\mathcal{M}_{1}(\mathbb{C}^2) = \{A \neq 0, \det A = 0\}$$

Cohomology

$$\begin{array}{ccc} \mathcal{M}_{\leq k}(\mathbb{C}^n) = \mathcal{M}_{\leq k-1}(\mathbb{C}^n) \cup \mathcal{M}_k(\mathbb{C}^n) & \supset & \mathcal{M}_k(\mathbb{C}^n) \\ \bigcup & & \text{closed} \\ \mathcal{M}_{\leq k-1}(\mathbb{C}^n) \text{ open} & & \end{array}$$

- To apply the submanifolds of matrices, we need to know their cohomology groups.
- A basic tool is the long exact sequence associated to the pair $(\mathcal{M}_{\leq k}(\mathbb{C}^n), \mathcal{M}_{\leq k-1}(\mathbb{C}^n))$:

$$\cdots \to H^m(\mathcal{M}_{\leq k}, \mathcal{M}_{\leq k-1}) \to H^m(\mathcal{M}_{\leq k}) \to H^m(\mathcal{M}_{\leq k-1}) \to$$
$$\cdots \to H^{m+1}(\mathcal{M}_{\leq k}, \mathcal{M}_{\leq k-1}) \to \cdots$$

Cohomology

$$\begin{array}{ccc} \mathcal{M}_{\leq k}(\mathbb{C}^n) = \mathcal{M}_{\leq k-1}(\mathbb{C}^n) \cup \mathcal{M}_k(\mathbb{C}^n) & \supset & \mathcal{M}_k(\mathbb{C}^n) \\ \bigcup & & \text{closed} \\ \mathcal{M}_{\leq k-1}(\mathbb{C}^n) \text{ open} & & \end{array}$$

- **Another basic tool is an isomorphism** ("Alexander-Poincaré duality")

$$H^m(\mathcal{M}_{\leq k}(\mathbb{C}^n), \mathcal{M}_{\leq k-1}(\mathbb{C}^n)) \overset{\alpha}{\cong} H^{m-2k^2}(\mathcal{M}_k(\mathbb{C}^n)).$$

- $n = 1$ **and** $k = 1$:

$$H^m(\mathcal{M}_{\leq 1}, \mathcal{M}_{\leq 0}) = H^m(\mathbb{C}, \mathbb{C}^\times) \cong H^m(D^2, \partial D^2),$$
$$H^{m-2}(\mathcal{M}_1) = H^{m-2}(\{0\}).$$

Remark

- **Next, I will apply the submanifolds and their cohomology classes to quantum systems of class AIII.**
- **Before that, I remark that what I presented here is a finite-dimensional analogue of Koschorke's work:**

  U. Koschorke, Infinite dimensional K-theory and characteristic classes of Fredholm bundle maps. 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) pp. 95–133 Amer. Math. Soc., Providence, R.I.

- **In this paper, characteristic classes for K-theory are constructed by using submanifolds in the space of Fredholm operators.**

Remark

- Replacing $\mathbb{C}^n$ by an infinite-dimensional Hilbert space $\mathcal{H}$, we can make sense of the following generalizations of $\mathcal{M}_{\leq k}(\mathbb{C}^n)$ and $\mathcal{M}_k(\mathbb{C}^n)$:

$$\mathcal{F}_{\leq k}(\mathcal{H}) = \left\{ \begin{array}{l} A : \mathcal{H} \to \mathcal{H} \text{ Fredholm operator} \\ \mathrm{dim}\mathrm{Ker}A = \mathrm{dim}\mathrm{Coker}A \leq k \end{array} \right\},$$

$$\mathcal{F}_k(\mathcal{H}) = \left\{ \begin{array}{l} A : \mathcal{H} \to \mathcal{H} \text{ Fredholm operator} \\ \mathrm{dim}\mathrm{Ker}A = \mathrm{dim}\mathrm{Coker}A = k \end{array} \right\}.$$

- It turns out that $\mathcal{F}_k(\mathcal{H}) \subset \mathcal{F}_{\leq k}(\mathcal{H})$ is a complex submanifold of codimension k^2, although $\mathcal{F}_k(\mathcal{H})$ and $\mathcal{F}_{\leq k}(\mathcal{H})$ are infinite-dimensional.
- There is also an isomorphism

$$H^m(\mathcal{F}_{\leq k}, \mathcal{F}_{\leq k-1}) \cong H^{m-2k^2}(\mathcal{F}_k).$$

Change of topological numbers

- Recall that a certain quantum system on a d-dimensional lattice is described by a map $H : T^d \to \mathcal{H}(\mathbb{C}^n)$.
- An insulator is characterized by the gap condition; $\det H(k) \neq 0$ for all $k \in T^d$.
- A quantum system is class AIII, if there is a linear symmetry Γ (called a chiral symmetry) such that

$$\Gamma^2 = 1, \qquad \Gamma H(k) = -H(k)\Gamma.$$

- Without loss of generality, we can assume

$$\Gamma = \begin{pmatrix} 1_{\mathbb{C}^n} & 0 \\ 0 & -1_{\mathbb{C}^n} \end{pmatrix}, \quad H(k) = \begin{pmatrix} 0 & A(k)^\dagger \\ A(k) & 0 \end{pmatrix}.$$

Class AIII insulator

- Then a d-dim insulator of class AIII is described by

$$A : T^d \to \mathcal{M}_{\leq 0}(\mathbb{C}^n) = \mathcal{M}_0(\mathbb{C}^n) = GL(n, \mathbb{C}).$$

- It is well-known that $GL(n, \mathbb{C}) \simeq U(n)$, and

$$H^\bullet(U(n); \mathbb{Z}) = \bigwedge(\nu_1, \ldots, \nu_{2n-1}),$$

where $\nu_{2j-1} \in H^{2j-1}(U(n); \mathbb{Z})$.

- In the case that $d = 2n - 1$, we can define the d-dimensional winding number by

$$\mathrm{top}(A) = \int_{T^{2n-1}} A^* \nu_{2n-1} \in \mathbb{Z}.$$

Change of topological phases

- Now, let us consider a one-parameter family of $(2n-1)$-dimensional class AIII quantum systems

$$\tilde{A} : T^{2n-1} \times [0,1] \to \mathcal{M}(\mathbb{C}^n) = \mathbb{C}^{n^2}.$$

Assumption

1. $\tilde{A}$ is a smooth map.
2. $\tilde{A}(k,t) \in \mathcal{M}_{\leq 1}(\mathbb{C}^n) \subset \mathcal{M}(\mathbb{C}^n)$ for all k, t.
3. $\tilde{A}(k,i) \in \mathcal{M}_0(\mathbb{C}^n)$ for all k and $i = 0, 1$.
4. a condition which ensures that

$$\Sigma := \{(k,t) \in T^{2n-1} \times [0,1] \mid \det \tilde{A}(k,t) = 0\}$$

is a $(2n-2)$-dimensional submanifold of $T^{2n-1} \times [0,1]$.

Change of topological numbers

$$(T^{2n-1} \times [0,1], T^{2n-1} \times \{0,1\}) \xrightarrow{\tilde{A}} (\mathcal{M}_{\leq 1}(\mathbb{C}^n), \mathcal{M}_{\leq 0}(\mathbb{C}^n))$$

$$\Sigma = \{(k,t) \in T^{2n-1} \times [0,1] |\ \det \tilde{A}(k,t) = 0\} \xrightarrow{\tilde{A}|_\Sigma} \mathcal{M}_1(\mathbb{C}^n)$$

Theorem

Under the assumption, we have

$$\mathrm{top}(\tilde{A}|_{T^{2n-1}\times\{0\}}) - \mathrm{top}(\tilde{A}|_{T^{2n-1}\times\{1\}}) = \int_\Sigma \tilde{A}|_\Sigma^* \gamma_{2n-2},$$

where $\gamma_{2n-2} \in H^{2n-2}(\mathcal{M}_1(\mathbb{C}^n))$ **is the class induced from** $\nu_{2n-1} \in H^{2n-1}(\mathcal{M}_{\leq 0}(\mathbb{C}^n))$ **through**

$$H^{2n-1}(\mathcal{M}_{\leq 0}) \xrightarrow{\delta} H^{2n}(\mathcal{M}_{\leq 1}, \mathcal{M}_{\leq 0}) \overset{\alpha}{\cong} H^{2n-2}(\mathcal{M}_1).$$

Example: $d = 1$ $(n = 1)$

$$(T^{2n-1} \times [0,1], T^{2n-1} \times \{0,1\}) \xrightarrow{\tilde{A}} (\mathcal{M}_{\leq 1}(\mathbb{C}^n), \mathcal{M}_{\leq 0}(\mathbb{C}^n))$$

$$\Sigma = \{(k,t) \in T^{2n-1} \times [0,1] | \ \det \tilde{A}(k,t) = 0\}$$

$$\mathrm{top}(\tilde{A}|_{T^{2n-1}\times\{0\}}) - \mathrm{top}(\tilde{A}|_{T^{2n-1}\times\{1\}}) = \int_{\Sigma} \tilde{A}|_{\Sigma}^{*}\gamma_{2n-2}$$

- **In the case of $n = 1$, let us consider**

$$\tilde{A} : \ \mathbb{R}/2\pi\mathbb{Z} \times [0,1] \longrightarrow \mathcal{M}_{\leq 1}(\mathbb{C}) = \mathbb{C}$$

$$\tilde{A}(k,t) = (1-t) + te^{ik}$$

- **Then the change of the topological numbers**

$$0 - 1 = -1$$

agrees with the number of points in $\Sigma = \{(\pi, 1/2)\}$ counted with weight $\tilde{A}^*\gamma_0 = -1$.

Remark

- **I just explained the case of class AIII, but the formula can be generalized:**
  1. **we can consider the other Altland-Zirnbauer classes (class A, AI, BDI, D, ...)**
  2. **we can allow crystalline symmetry of the lattice**
- **In the first generalization, we need to consider a suitable equivariant cohomology, for real classes.**
- **In the second generalization, we need to consider a fiber bundle of submanifolds of matrices in addition, for nonsymmorphic crystalline symmetry.**

Semimetal

- **I would like to relate submanifolds of matrices with the "semimetal class" of V. Mathai and G. C. Thiang:**

  V. Mathai and G. C. Thiang, Global topology of Weyl semimetals and Fermi arcs. Journal of Physics A: Mathematical and Theoretical 50 (11), 11LT01

  V. Mathai and G. C. Thiang, Differential topology of semimetals. Communications in Mathematical Physics 355 (2), 561-602

Submanifolds of traceless Hermitian matrices

- For the semimetal setup, let us consider the subspaces of $n \times n$ traceless Hermitian matrices

$$\mathcal{H}^0(\mathbb{C}^n) = \{H \in \mathcal{M}(\mathbb{C}^n) |\ H^\dagger = H, \mathrm{tr}H = 0\} \cong \mathbb{R}^{n^2-1},$$
$$\mathcal{H}^0_{\leq k}(\mathbb{C}^n) = \{H \in \mathcal{H}^0(\mathbb{C}^n) |\ \mathrm{dimKer}H \leq k\},$$
$$\mathcal{H}^0_{k}(\mathbb{C}^n) = \{H \in \mathcal{H}^0(\mathbb{C}^n) |\ \mathrm{dimKer}H = k\}.$$

- The traceless condition leads to

$$\mathcal{H}^0_{n-1}(\mathbb{C}^n) = \emptyset, \qquad \mathcal{H}^0_{\leq n-2}(\mathbb{C}^n) = \mathcal{H}^0_{\leq n-1}(\mathbb{C}^n).$$

- Except for this point, we have the same results as before.

Submanifolds of traceless Hermitian matrices

$$\mathcal{H}^0_{\leq k}(\mathbb{C}^n) = \{H \in \mathcal{H}^0(\mathbb{C}^n) |\ \mathrm{dimKer} H \leq k\},$$
$$\mathcal{H}^0_{k}(\mathbb{C}^n) = \{H \in \mathcal{H}^0(\mathbb{C}^n) |\ \mathrm{dimKer} H = k\}.$$

- **In general, $\mathcal{H}^0_{\leq k}(\mathbb{C}^n) \subset \mathcal{H}^0(\mathbb{C}^n)$ is an open submanifold of dimension $\dim_{\mathbb{R}} \mathcal{H}^0_{\leq k}(\mathbb{C}^n) = \dim_{\mathbb{R}} \mathcal{H}^0(\mathbb{C}^n) = n^2 - 1$.**
- **$\mathcal{H}^0_{k}(\mathbb{C}^n) \subset \mathcal{H}^0_{\leq k}(\mathbb{C}^n)$ is a closed submanifold of dimension $n^2 - k^2 - 1$ for $k \leq n - 2$, with orientable normal bundle.**
- **In the case of $n = 2$:**

$$\begin{array}{c} \mathcal{H}^0_{\leq 2}(\mathbb{C}^2) \cong \mathbb{R}^3 \\ \bigcup \\ \mathcal{H}^0_{\leq 0}(\mathbb{C}^2) = \mathcal{H}^0_{\leq 1}(\mathbb{C}^2) \cong \mathbb{R}^3 \backslash \{0\} \end{array} \qquad \mathcal{H}_2(\mathbb{C}^2) = \{0\}$$

3d "Weyl" semimetal

$$\mathcal{H}^0_{\leq 2}(\mathbb{C}^2) \cong \mathbb{R}^3$$
$$\bigcup$$
$$\mathcal{H}^0_{\leq 0}(\mathbb{C}^2) = \mathcal{H}^0_{\leq 1}(\mathbb{C}^2) \cong \mathbb{R}^3 \backslash \{0\}$$

$$\mathcal{H}_2(\mathbb{C}^2) = \{0\}$$

- **Suppose that we are given a continuous map**

$$H : T^3 \to \mathcal{H}^0_{\leq 2}(\mathbb{C}^2) = \mathcal{H}^0(\mathbb{C}^2) \cong \mathbb{R}^3$$

such that the set of "Weyl points"

$$W = \{k \in T^3 | \ \det H(k) = 0\}$$

consists of a finite number of points.

3d "Weyl" semimetal

$$W = \{k \in T^3 |\ \det H(k) = 0\}$$

$$\begin{array}{ccc} W & & \mathcal{H}^0_2(\mathbb{C}^2) = \{0\} \\ \cap & & \cap \\ T^3 & \xrightarrow{H} & \mathcal{H}^0_{\leq 2}(\mathbb{C}^2) \cong \mathbb{R}^3 \\ \cup & & \cup \\ T^3 \backslash W & & \mathcal{H}^0_{\leq 0}(\mathbb{C}^2) = \mathcal{H}^0_{\leq 1}(\mathbb{C}^2) \cong \mathbb{R}^3 \backslash \{0\} \simeq S^2 \end{array}$$

- We can then have a map

$$H : T^3 \backslash W \to \mathcal{H}^0_{\leq 0}(\mathbb{C}^2).$$

- The pullback of a generator in $H^2(\mathcal{H}^0_{\leq 0}(\mathbb{C}^2)) \cong \mathbb{Z}$ is essentially the semimetal class of Mathai and Thiang:

The semimetal class of $H \in H^2(T^3 \backslash W)$.

3d "Weyl" semimetal

- The "Mayer-Vietoris sequence" of Mathai-Thiang is equivalent to

$$\overbrace{H^2(T^3)}^{\text{insulator}} \to \overbrace{H^2(T^3\backslash W)}^{\text{semimetal}} \xrightarrow{\delta} H^3(T^3, T^3\backslash W) \overset{\alpha}{\cong} \overbrace{H^0(W)}^{\text{charge}}.$$

- There is the universal counterpart

$$H^2(\mathcal{H}^0_{\leq 2}) \to H^2(\mathcal{H}^0_{\leq 0}) \xrightarrow{\delta} H^3(\mathcal{H}^0_{\leq 2}, \mathcal{H}^0_{\leq 0}) \overset{\alpha}{\cong} H^0(\mathcal{H}^0_2).$$

- A use of the submanifolds

$$\cdots \subset \mathcal{H}^0_{\leq k-1}(\mathbb{C}^n) \subset \mathcal{H}^0_{\leq k}(\mathbb{C}^n) \subset \cdots$$

would generalize the semimetal class.