Bulk-Boundary Correspondence in Disordered Topological Insulators and Superconductors

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Topics

1. Construction of the C*-algebra of observables.
2. Classification of gapped bulk systems in Van Daele KR-theory.
3. Systematic pseudo-symmetry picture for the corresponding boundary classes in Kasparov's KR-theory.

Controlled Lattice Operators

Tight-binding model over the lattice L.
bulk lattice : $L = |\mathbb{Z}^d|$, half-space: $L = |\mathbb{Z}^{d-1} \times \mathbb{N}|$
Localized lattice states: $\ell^2(L) := \ell^2(L, \mathbb{C}) \rightarrow$ Complex Hilbert space with real structure $\mathfrak{c}$ of point-wise complex conjugation.

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Definition (Controlled operators)

$T \in \mathbb{B}(\ell^2(L))$ is *controlled* or has *finite propagation*, if there is some $R > 0$ such that $\langle x|T|y\rangle = 0$ for all $x, y \in L$ with $|x - y| > R$.

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Definition (Real C*-algebra)

A *complex* C^**-algebra* A is a complex Banach algebra with an anti-linear anti-involution $* : A \to A$ s.th. $\|a^* a\| = \|a\|^2 \quad \forall a \in A$.
A *Real* C*-algebra is a complex C*-algebra A with a real involution, i.e. a $*$-isometric anti-linear involution $\bar{\cdot} : A \to A$.

Definition (Uniform Roe C*-algebra)

$C^*_u(L) := \overline{\{T \in \mathbb{B}(\ell^2(L)) \mid T \text{ controlled}\}}^{\|\cdot\|}$ defines a Real C*-algebra with real involution $\mathrm{Ad}_{\mathfrak{c}}$.

Nambu Space of Internal Degrees of Freedom

- **Single particle picture**
  V: complex vector space of internal d.o.f.; inner product $\langle \cdot, \cdot \rangle$.

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- **Many particle space without interactions:**
  Nambu space of fields:
  $$W = V \oplus V^*$$
  Choice of basis $e_1, \dots, e_n$ of V:
  $$V \oplus V^* \cong \mathrm{span}_{\mathbb{C}}(c_1^\dagger, \dots, c_n^\dagger, c_1, \dots, c_n)$$

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  $$V \oplus V^* \cong \mathrm{span}_{\mathbb{C}}(c_1^\dagger, \ldots, c_n^\dagger, c_1, \ldots, c_n)$$
  Anti-linear Riesz isomorphism $R : V \to V^*$: $R(v) = \langle v, \cdot \rangle$.
  - Real structure on W: $\gamma_W = \begin{pmatrix} 0 & R^* \\ R & 0 \end{pmatrix}$, $\gamma_W^2 = 1, \gamma_W(\lambda w) = \bar{\lambda}\gamma_W(w)$.
  - γ_W induced by fermionic anti-commutation relations $\{\cdot, \cdot\}$ and the inner product $\langle \cdot, \cdot \rangle$ on V and V^*.
  - $\bar{M} = \mathrm{Ad}_{\gamma_W}(M) = \gamma_W M \gamma_W$ real structure on $\mathrm{End}(W)$.

  Hamiltonian without interaction:
  $$H = \sum_{i,j} c_i^\dagger A_{ij} c_j + c_i^\dagger B_{ij} c_j^\dagger + c_i C_{ij} c_j + c_i D_{ij} c_j^\dagger$$
  $$\to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{End}(W), \overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}} = -\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Homogeneous Disorder

Definition (Dynamical system of disorder)

A dynamical system $(\Omega, \tau, \mathbb{Z}^d)$ describing homogeneous disorder is given by

- a compact Hausdorff space $\Omega = (\Omega_0)^{\mathbb{Z}^d}$,
- the $\mathbb{Z}^d$-action on Ω: $\tau : \mathbb{Z}^d \to \mathrm{Homeo}(\Omega)$, $\tau_x(\omega_y) = \omega_{y-x}$.

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Disorder on the level of operators:

Definition (The disordered bulk C*-algebra)

Let $U : \mathbb{Z}^d \to \mathbb{B}(\ell^2(|\mathbb{Z}^d|))$ be the action via translations. The Real C^*-algebra of bulk observables is given by

$$A_d^W = \overline{\left\{ T \in C(\Omega, C_u^*(|\mathbb{Z}^d|) \otimes \mathrm{End}(W)) \mid T(\tau_x(\omega)) = U_x T(\omega) U_x^{-1} \, \forall x \in \mathbb{Z}^d \right\}}^{\|\cdot\|}$$
$$\subset C(\Omega) \otimes C_u^*(|\mathbb{Z}^d|) \otimes \mathrm{End}(W)$$

Real structure on A_d^W induced by real structures on $\mathrm{End}(W)$ and $C_u^*(|\mathbb{Z}^d|)$.

Bulk C*-algebra as crossed product C*-algebra

Theorem (Crossed product form of bulk C*-algebra)

$$A_d^W = (C(\Omega) \otimes \mathrm{End}(W)) \rtimes \mathbb{Z}^d.$$

The crossed product C*-algebra is the norm-closure of the non-commutative polynomials

$$\Big\{ \sum_{x \in \mathbb{Z}^d} M_x u_1^{x_1} \cdots u_d^{x_d} \mid M_x \in C(\Omega) \otimes \mathrm{End}(W), M_x = 0 \text{ for almost all } x \in \mathbb{Z}^d \Big\},$$

where

$$u_i M(\omega) u_i^* = M\big(\tau_{e_i}(\omega)\big), u_i^* = u_i^{-1}, u_i u_j = u_j u_i$$

for all $i, j \in \{1, \ldots, d\}$ and $M \in C(\Omega) \otimes \mathrm{End}(W) = C(\Omega, \mathrm{End}(W))$.

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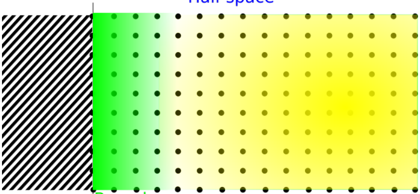
for all $i, j \in \{1, \ldots, d\}$ and $M \in C(\Omega) \otimes \mathrm{End}(W) = C(\Omega, \mathrm{End}(W))$.
Clean system: $\Omega_0 = \{pt\}$.
Trivial action of $\mathbb{Z}^d$ on $\Omega \to$ translational invariance, $A_d^W = \mathrm{End}(W) \otimes C^*(\mathbb{Z}^d)$.

Half-space and boundary C*-algebra



Half-space C*-algebra: $\hat{A}_d^W \cong \overline{\left\{\sum_{n_1,n_2\in\mathbb{N}} p_{n_1,n_2}(\hat{u}_d)^{n_1}(\hat{u}_d^*)^{n_2}\right\}}^{\|\cdot\|}$, for $p_{n_1,n_2} \in A_{d-1}^W$ and

$$\hat{u}_d^*\hat{u}_d = 1,\ \hat{u}_d\hat{u}_d^* = 1 - P_0,$$
$$\hat{u}_d M(\omega) = M(\tau_{e_d}(\omega))\hat{u}_d,\ \hat{u}_d^* M(\omega) = M(\tau_{-e_d}(\omega))\hat{u}_d^*,$$

where P_0 is a 1-dim. projection ($P_0 \hat{=} |0\rangle\langle 0|$).

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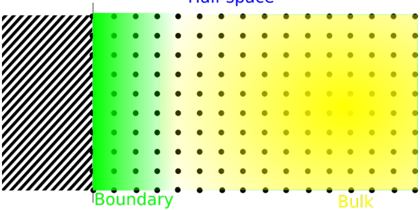

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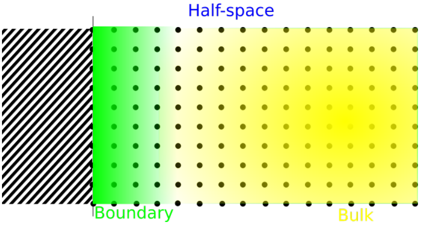
Boundary C*-algebra: $B_d^W := \hat{A}_d^W P_0 \hat{A}_d^W \cong A_{d-1}^W \otimes \mathbb{K}(\ell^2(\mathbb{N})) \to$ ideal in $\hat{A}_d^W$

Bulk-boundary short exact sequence



Short exact sequence of Real C*-algebras: $\left(\mathbb{K} = \mathbb{K}(\ell^2(\mathbb{N}))\right)$

$$0 \to A_{d-1}^{W} \otimes \mathbb{K} \overset{\iota}{\hookrightarrow} \hat{A}_d^{W} \xrightarrow{\pi} A_d^{W} \to 0,$$

where π is the bulk-projection (Real $*$-homomorphism) defined by $\pi(\hat{u}_d) = u_d$ and $\pi(a) = a$ for $a \in A_{d-1}^{W}$.

Classification of gapped free-fermion bulk groundstates with symmetries

Van Daele KR-theory of graded C*-algebras → KR-classes for gapped bulk systems

Physical Input → KR-classes for gapped bulk systems

KR-classes for gapped bulk systems

Reference: Kellendonk (2015), arxiv: 1509.06271

Van Daele KR-theory

Definition (Graded, Real C*-algebra)

Let A be a Real C*-algebra. A *grading* on A is a decomposition

$$A = A^{(0)} \oplus A^{(1)} \text{ with } a_i \in A^{(i)}, a_j \in A^{(j)} \Rightarrow a_i a_j \in A^{(i+j)},\ \bar{a}_i \in A^{(i)}\, \forall i,j \in \mathbb{Z}_2.$$

$A^{(0)}$: 'even' elements, $A^{(1)}$: 'odd' elements.

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Example

$Cl_{a,b}$: Clifford algebra generated by the positive generators $K_1, \ldots, K_a$ and the negative generators $I_1, \ldots, I_b$, s.th. for all $m, n \in \{1, \ldots, a\}$, $i, j \in \{1, \ldots, b\}$:

$$K_m K_n + K_n K_m = 2\delta_{m,n},\ K_m^* = K_m,\ \bar{K}_m = K_m,$$
$$I_i I_j + I_j I_i = -2\delta_{i,j},\ I_i^* = -I_i, \bar{I}_i = I_i,$$
$$K_m I_i + I_i K_m = 0.$$

Standard grading: K_n, I_i odd $\forall\, n, i$.

Van Daele KR-theory

Let A be a graded, Real C*-algebra.

$$\mathcal{F}(A) := \{a \in A^{(1)} \mid a^* = a,\ a^2 = 1, \bar{a} = a\},$$
$$F(A) := \mathcal{F}(A)/\text{homotopy}.$$

For $[x] \in F(M_n(A)), [y] \in F(M_m(A))$ let $[x] + [y] := \left[\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right] \in F(M_{n+m}(A))$.

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Definition

Choose a reference element $e \in \mathcal{F}(A)$.
Van Daele KR-theory for A w.r.t. e is defined as the inductive limit

$$\mathrm{DKR}_e(A) := \lim_{\rightarrow n} F(M_n(A)),$$

where $F(M_n(A)) \ni [x] \mapsto \left[\begin{pmatrix} x & 0 \\ 0 & e \end{pmatrix}\right] \in F(M_{n+1}(A))$.

Bulk classification in Van Daele KR-theory

Theorem (Stability)

$\mathrm{DKR}_e(A) \cong \mathrm{DKR}_e(A \otimes M_n(\mathbb{C}))$ *for all* $n \in \mathbb{N}$.

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Theorem

If $e \in \mathcal{F}(A)$ *with* $e \sim_{hom} -e$, *then* $\mathrm{DKR}_e(A)$ *is a group that is, up to isomorphism, independent of the choice of* e*:*

- $[x] + [y] = \left[\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\right]$,
- *Neutral element:* $[e] = 0$,
- *Inverse for* $[x] \in F(M_n(A))$*:* $[-e_n x e_n] \in F(M_n(A))$, *where* $e_n = e \oplus e \cdots \oplus e$.

$\to \mathrm{DKR}(A)$

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$\rightarrow \mathrm{DKR}(A)$

Theorem

If A *is a Real,* **ungraded** C^**-algebra, then* $\mathrm{DKR}(A \otimes Cl_{a+1,b}) \cong \mathrm{KR}_{b-a}(A)$.

Bulk Classification - Physical Input

symmetry-class	# of pseudo-symmetries	physical symmetries
D	0	none,
DIII	1	time reversal T,
AII	2	T, charge conj. Q,
CII	3	T, Q, twisted particle-hole conj. C,
C	4	spin rotations j_1, j_2, j_3,
CI	5	j_1, j_2, j_3, T,
AI	6	j_1, j_2, j_3, T, Q,
BDI	7	j_1, j_2, j_3, T, Q, C.

$$\left\{\begin{array}{l}\text{Quasi-particle vacuum (QPV)} \\ J \in A_d^W : \ J^2 = -1, \bar{J} = J, J^* = -J \\ H := -iJ : \ \text{flattened Hamiltonian}\end{array}\right\} \overset{1:1}{\leftrightarrow} \left\{\begin{array}{l}\text{Free-fermion groundstate} \\ \text{projection } P_+ := \frac{1}{2}(1 + iJ)\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{pseudo-symmetries} \\ J_1, \ldots, J_s \in \mathrm{End}(W) \subset A_d^W : \\ J_i J_j + J_j J_i = -2\delta_{i,j}, \ \bar{J_i} = J_i, J_i^* = -J_i, \\ J_i J + J J_i = 0 \, \forall \, i, j \in \{1, \ldots, s\}\end{array}\right\} \overset{1:1}{\leftrightarrow} \left\{\begin{array}{l}\text{physical symmetries of} \\ \text{the free fermion groundstate}\end{array}\right\}$$

Ref.: Zirnbauer, Kennedy (2014), arxiv: 1412.4808

Bulk classification: class D, $s = 0$

Consider $J \in A_d^W$, $J^2 = -1, J^* = -J, \bar{J} = J$.

<u>$s = 0$:</u>

- No restriction on $J \to$ classify all $x \in A_d^W$ s.th. $x^2 = -1, x^* = -x, \bar{x} = x$.

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- Bijection:

$$A_d^W \ni x \mapsto x \otimes I_1 \in \mathcal{F}(A_d^W \otimes Cl_{0,1})$$

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- KR-class for QPV in class D:

$$\left[\begin{pmatrix} J \otimes I_1 & 0 \\ 0 & -J_0 \otimes I_1 \end{pmatrix}\right] \in \mathrm{DKR}_e(M_2(A_d^W \otimes Cl_{0,1})) \cong \mathrm{KR}_2(A_d^W).$$

where $e = \begin{pmatrix} J_0 \otimes I_1 & 0 \\ 0 & -J_0 \otimes I_1 \end{pmatrix}$, $J_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{End}(W) \subset A_d^W$.

$\cos(t)\begin{pmatrix} J_0 \otimes I_1 & 0 \\ 0 & -J_0 \otimes I_1 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 & J_0 \otimes I_1 \\ J_0 \otimes I_1 & 0 \end{pmatrix}$ connects e and $-e$.

Bulk classification: class DIII, $s = 1$

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CII	3	T, Q, twisted particle-hole conj. C,
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- $T : V \to V$, $T^2 = -1$, $T^* = -T$
  $\to$ pseudo-symmetry: $J_1 = \begin{pmatrix} 0 & TR^* \\ RT & 0 \end{pmatrix} \in \mathrm{End}(W) \subset A_d^W$.

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  $\to$ pseudo-symmetry: $J_1 = \begin{pmatrix} 0 & TR^* \\ RT & 0 \end{pmatrix} \in \mathrm{End}(W) \subset A_d^W$.
- Use TR^* to split $\mathrm{End}(W) = \mathrm{End}(V) \otimes \mathbb{H}^{\mathbb{C}}$, where $\mathbb{H}^{\mathbb{C}}$ denotes the complexification of the quaternions $\mathbb{H} = \mathrm{span}_{\mathbb{R}} \left(\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) \right)$.

$$\boxed{A_d^W \cong A_d^V \otimes \mathbb{H}^{\mathbb{C}}}$$

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$$A_d^W \cong A_d^V \otimes \mathbb{H}^{\mathbb{C}}: \quad J_1 = \begin{pmatrix} 0 & TR^* \\ RT & 0 \end{pmatrix} \mapsto 1 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in A_d^V \otimes \mathbb{H}^{\mathbb{C}}$$

$$\gamma = \begin{pmatrix} 0 & R^* \\ R & 0 \end{pmatrix} \mapsto T \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

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- QPV $J \in A_d^W$ in class *DIII* commutes with γ and anti-commutes with J_1:

$$J \mapsto x_1 \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + x_2 \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in A_d^V \otimes \mathbb{H}^{\mathbb{C}}, \tag{1}$$

with $\bar{x}_i := T^* x_i T = x_i$, $x_i^* = x_i, x_i^2 = 1$ for $i = 1, 2$ and $x_1 x_2 = x_2 x_1, x_1^2 + x_2^2 = 1$.

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$$(x_1, x_2) \mapsto x_1 \otimes K_1 + x_2 \otimes K_2 \in \mathcal{F}(A_d^V \otimes Cl_{2,0})$$

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T defines a **quaternionic** ($T^* = -T, T^2 = -1$) structure on A_d^V.

- QPV $J \in A_d^W$ in class *DIII* commutes with γ and anti-commutes with J_1:

$$J \mapsto x_1 \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + x_2 \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in A_d^V \otimes \mathbb{H}^{\mathbb{C}}, \tag{1}$$

  with $\bar{x}_i := T^* x_i T = x_i$, $x_i^* = x_i, x_i^2 = 1$ for $i = 1, 2$ and $x_1 x_2 = x_2 x_1, x_1^2 + x_2^2 = 1$.
- Bijection:

$$(x_1, x_2) \mapsto x_1 \otimes K_1 + x_2 \otimes K_2 \in \mathcal{F}(A_d^V \otimes Cl_{2,0})$$

- DKR-class for QPV (1) in class DIII:

$$[x_1 \otimes K_1 + x_2 \otimes K_2] \in \mathrm{DKR}(A_d^V \otimes Cl_{2,0}) \cong KR_{-1}(A_d^V) \cong KR_3(A_d^W),$$

  where e.g. $e = 1 \otimes K_1$.

Bulk classification

symmetry-class	# of pseudo-symmetries	physical symmetries
D	0	none,
DIII	1	time reversal T,
AII	2	T, charge conj. Q,
CII	3	T, Q, twisted particle-hole conj. C,
C	4	spin rotations j_1, j_2, j_3,
CI	5	j_1, j_2, j_3, T,
AI	6	j_1, j_2, j_3, T, Q,
BDI	7	j_1, j_2, j_3, T, Q, C.

Observation

A bulk QPV in symmetry class s defines a class in $KR_{s+2}(A_d^W)$.

Bulk-boundary correspondence:

1. Construct a boundary morphism $\partial : \mathrm{KR}_{s+2}(A_d^W) \to \mathrm{KR}_{s+1}(A_{d-1}^W)$ inducing bulk-boundary correspondence.
2. Use the Kasparov picture of the KR-classes to get a systematic picture of the boundary classes.

$$\begin{array}{ccc}
\mathrm{KR}_{b-a+1}(A_d^W) \cong \mathrm{DKR}(A_d^W \otimes Cl_{a,b}) & \overset{\sim}{\dashleftarrow} & \mathrm{KKR}(Cl_{b,a}, A_d^W \otimes Cl_{0,1}) \cong \mathrm{KR}_{b-a+1}(A_d^W) \\
\downarrow \partial & & \downarrow \partial \\
\mathrm{KR}_{b-a}(A_{d-1}^W) \cong \mathrm{DKR}(A_{d-1}^W \otimes Cl_{a+1,b}) & \overset{\sim}{\longleftarrow} & \mathrm{KKR}(Cl_{b,a+1}, A_{d-1}^W \otimes Cl_{0,1}) \cong \mathrm{KR}_{b-a}(A_{d-1}^W)
\end{array}$$

Long exact sequence of KR-theory

Short exact sequence of Real C*-algebras:

$$0 \to A^W_{d-1} \otimes \mathbb{K} \overset{\iota}{\hookrightarrow} \hat{A}^W_d \overset{\pi}{\longrightarrow} A^W_d \to 0$$

Long exact sequence of KR-theory: (∂: connecting/boundary morphism)

$$\cdots \overset{\partial}{\dashrightarrow} \mathrm{KR}_i(A^W_{d-1} \otimes \mathbb{K}) \overset{\iota_*}{\longrightarrow} \mathrm{KR}_i(\hat{A}^W_d) \overset{\pi_*}{\longrightarrow} \mathrm{KR}_i(A^W_d)$$

$$\overset{\partial}{\longrightarrow} \mathrm{KR}_{i-1}(A^W_{d-1} \otimes \mathbb{K}) \overset{\iota_*}{\longrightarrow} \mathrm{KR}_{i-1}(\hat{A}^W_d) \overset{\pi_*}{\longrightarrow} \mathrm{KR}_{i-1}(A^W_d) \overset{\partial}{\dashrightarrow} \cdots$$

Boundary classes in KKR-theory

$\mathrm{KR}_{s+1}(A^W_{d-1}) \cong \mathrm{KKR}(Cl_{s,0}, A^W_{d-1} \otimes Cl_{0,1})$:

- The elements are (equivalence classes of) tuples $\big(\psi : Cl_{s,0} \to \mathbb{B}(\mathcal{H}), F \in \mathbb{B}(\mathcal{H})\big)$, where $\mathcal{H} = \ell^2(\mathbb{N}) \otimes A^W_{d-1} \otimes Cl_{0,1}$ and ψ is a grading preserving $*$-morphism such that

$$
\begin{aligned}
&(F^* - F)\psi(Cl_{s,0}) = 0,\\
&F\psi(K_i) + \psi(K_i)F = 0 \quad \forall\, i \in \{1, \dots, s\},\\
&(F^2 - 1)\psi(Cl_{s,0}) \in \mathbb{K}(\mathcal{H}) = \mathbb{K}(\ell^2(\mathbb{N})) \otimes A^W_{d-1} \otimes Cl_{0,1}.
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$$

- **Degenerate tuples:** $(F^2 - 1)\psi(Cl_{s,0}) = 0 \to$ Trivial KR-theory.
- **Equivalence relations:**
  Unitary equivalence: $(\psi_1, F_1) \sim_u (v^*\psi_1 v, v^* F_1 v)$ for unitary, even $v \in \mathbb{B}(\mathcal{H})$.
  Operator homotopy equivalence: Continuous path (ψ, F_t) for $t \in [0, 1]$, then $(\psi, F_0) \sim_h (\psi, F_1)$.
  Stabilization: Direct sum $(\psi_1, F_1) \oplus (\psi_2, F_2) = (\psi_1 \oplus \psi_2, F_1 \oplus F_2)$ well defined, since $\mathcal{H} \oplus \mathcal{H} \cong \mathcal{H}$.
  $\to (\psi, f) \sim_s (\psi, F) \oplus (\psi_{deg}, F_{deg})$ if (ψ_{deg}, F_{deg}) is degenerate.

Boundary classes in KKR-theory

Theorem

The boundary class for symmetry class s is given by

$$\left[(\mathcal{H}, \psi, \hat{J} \otimes I_1)\right] \in KKR(Cl_{s,0}, A^W_{d-1} \otimes Cl_{0,1}),$$

where $\mathcal{H} = \ell^2(\mathbb{N}) \otimes A^W_{d-1} \otimes Cl_{0,1}$ *and* $\psi : Cl_{s,0} \to \mathbb{B}(\mathcal{H}); \psi(K_i) = J_i \otimes I_1$ *for* $i = 1, \ldots, s$.

- $\hat{J} \in \hat{A}^W_{d-1} \subseteq \mathbb{B}(\ell^2(\mathbb{N}) \otimes A^W_{d-1})$ half-space QPV corresponding to $J \in A^W_d$, i.e. $\pi(\hat{J}) = J$ for the bulk-projection π. $\Rightarrow 1 + \hat{J}^2 \in A^W_{d-1} \otimes \mathbb{K}$.
- Pseudo symmetries $J_1, \ldots, J_s \in \mathrm{End}(W) \subset \hat{A}^W_d$ anti-commute with $\hat{J} \in \hat{A}^W_d$. $\Rightarrow [\psi(x), \hat{J} \otimes I_1] = 0 \, \forall \, x \in Cl_{s,0}$.

Results

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- Solid motivation for the crossed product algebra as observable algebra for the disordered tight-binding model.
- Canonical construction of KR-classes of gapped bulk systems.
- Systematic picture for gapless boundary classification.
- Properties of bulk-boundary correspondence $\partial : \mathrm{KR}_{s+2}(A_d^W) \to \mathrm{KR}_{s+1}(A_{d-1}^W)$:
  - $J \in A_d^W$ defines a bulk KR-class in ker(∂) if and only if $J \in A_{d-1}^W \subset A_d^W$.
  - $\mathrm{Im}(\partial) = KR_{s+1}(A_{d-1}^W)$ in clean system. In general false for disordered systems.
  - Given a fixed bulk class, the boundary classes can be different for different directions of the boundary.

- Strong topological invariants agree, up to a sign, in bulk and boundary for **any direction**.[1]
- Non-trivial strong invariant in bulk $\Rightarrow$ Gapless boundaries.

symmetry-class	# of pseudo-symmetries	Dimension d 0	1	2	3	4	5	6	7
D	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	2	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	3	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	4	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	5	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	6	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	7	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$

Table: Strong Topological Invariants

[1] Bourne, Kellendonk, Rennie (2016), arxiv: 1604.02337