A bird's-eye view on $\mathbb{Z}_2$ topology

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Kitaev's periodic table

Symmetry				Dimension							
AZ	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

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AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
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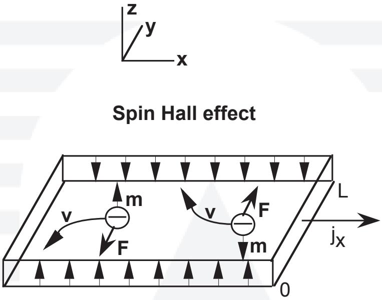
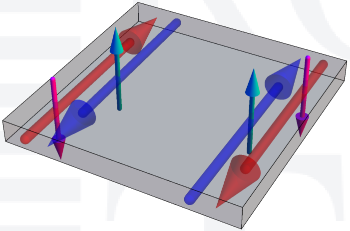
2D AII: quantum spin Hall insulator





$\mathbb{Z}_2$ classification [Fu–Kane–Mele 2005–07]

normal insulator (trivial phase) vs topological insulator (QSH phase)

$$\mathrm{FKM} := \frac{1}{2\pi}\int_{\mathrm{EBZ}} \mathcal{F} - \frac{1}{2\pi}\oint_{\partial\mathrm{EBZ}} \mathcal{A} \mod 2 \quad \in \mathbb{Z}_2$$

Outline of the presentation

Time-reversal symmetric topological insulators

d-dimensional TRS topological insulator (class AII)

A map $P \colon \mathbb{R}^d \to \mathcal{B}(\mathbb{C}^M)$ (possibly $M = \infty$) such that

- $P(k) = P(k)^* = P(k)^2$ is a rank-m orthogonal projection, $m = 2n$
- $k \mapsto P(k)$ is smooth (at least C^1)
- $k \mapsto P(k)$ is $\mathbb{Z}^d$-periodic: $P(k+\lambda) = P(k)$ for $\lambda \in \mathbb{Z}^d$ $\rightsquigarrow k \in \mathrm{BZ} \simeq \mathbb{T}^d$
- odd/fermionic time-reversal symmetry (TRS): $M = 2N$ and $\exists$ antiunitary $\Theta \colon \mathbb{C}^M \to \mathbb{C}^M$, $\Theta^2 = -\mathbf{1}$, such that $\Theta\, P(k)\, \Theta^{-1} = P(-k)$ $\rightsquigarrow k \in \mathrm{EBZ} \simeq$ "half a $\mathbb{T}^d$"

Example

$P(k)$ = family of Fermi projections of a gapped, periodic, TRS Hamiltonian, in the Bloch–Floquet representation:

$$P(k) = \mathbf{1}_{(-\infty, E_{\mathrm{F}}]}(H(k))$$

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Bloch frames

Bloch frame

A collection $\Phi(k) = \{\phi_1(k), \dots, \phi_m(k)\} \subset \mathbb{C}^M$, $k \in \mathbb{R}^d$, of orthonormal vectors such that

$$P(k) = \sum_{a=1}^{m} |\phi_a(k)\rangle \langle \phi_a(k)|$$

Φ is called

- smooth if each $k \mapsto \phi_a(k)$ is smooth
- periodic if each $k \mapsto \phi_a(k)$ is $\mathbb{Z}^d$-periodic
- TRS if $\Phi(-k) = [\Theta\Phi(k)]\,\varepsilon$, with $\varepsilon := \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$

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$$\langle \phi_a(0), \phi_a(0) \rangle = 1 \quad \text{but} \quad \begin{aligned} \langle \phi_a(0), \Theta\phi_a(0) \rangle &= \overline{\langle \Theta\phi_a(0), \Theta^2\phi_a(0) \rangle} \\ &= -\langle \phi_a(0), \Theta\phi_a(0) \rangle = 0 \end{aligned}$$

Berry connection, Berry curvature

Berry connection

$$\mathcal{A} := -\mathrm{i} \sum_{a=1}^{m} \langle \phi_a, \mathrm{d}\phi_a \rangle$$

Berry curvature

$$\mathcal{F} := \mathrm{d}\mathcal{A} = -\mathrm{i}\,\mathrm{Tr}(P\,\mathrm{d}P \wedge \mathrm{d}P)$$

Gauge dependence

$$\Phi^{G} := \Phi(k)\,G(k),\quad G(k) \in U(m) \quad \Longrightarrow \quad \begin{array}{c} \mathcal{A}^{G} = \mathcal{A} - \mathrm{i}\,\mathrm{Tr}\left(G^{-1}\,\mathrm{d}G\right) \\ \mathcal{F}^{G} = \mathcal{F} \end{array}$$

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Existence of Bloch frames

Theorem ([Panati AHP'07; M.–Panati AAP'15])

Assume $d = 2$.

- *The existence of smooth, periodic Bloch frames is topologically obstructed by the Chern number:*

$$c_1(P) := \frac{1}{2\pi} \int_{\mathrm{BZ}} \mathcal{F} \quad \in \mathbb{Z}.$$

- *In TRS topological insulators, $c_1(P) = 0$.*

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Existence of Bloch frames

Theorem ([Fiorenza–M.–Panati CMP'16; Cornean–M.–Teufel RMP'17; M. AQM'17])

Assume $d = 2$.

- *The existence of smooth, periodic, and TRS Bloch frames is topologically obstructed by a $\mathbb{Z}_2$ obstruction:* $\mathrm{FMP} \in \mathbb{Z}_2$.
- $\mathrm{FMP} = \mathrm{FKM} \in \mathbb{Z}_2$.
- $\mathrm{FMP} = \mathrm{GP} \in \mathbb{Z}_2$, *the Graf–Porta index [Graf–Porta CMP'13].*

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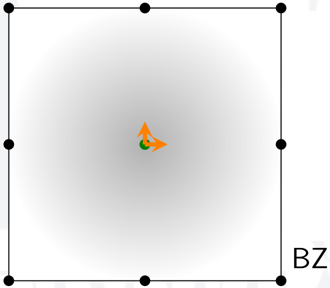
Step-by-step extension of Bloch frames

Step 1

Pick a symplectic orthonormal basis Ψ for Ran $P(0,0)$:

$$\Theta\, P(0,0)\, \Theta^{-1} = P(0,0) \quad \Longrightarrow \quad \Psi = [\Theta\Psi]\, \varepsilon$$

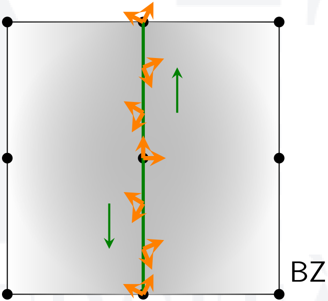


Step-by-step extension of Bloch frames

Step 2

Modified parallel transport along k_2 (preserves TRS & k_2-periodicity):

$$\Psi(0,k_2) := e^{-ik_2 X}\, T(k_2)\, \Psi \text{ with } \begin{cases} i\,\partial_{k_2} T(k_2) = i[\partial_{k_2} P(0,k_2), P(0,k_2)]\, T(k_2) \\ T(0) = \mathbf{1}_{\mathbb{C}^M}; \quad T(1) =: e^{iX} \end{cases}$$

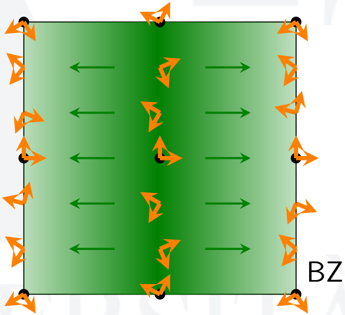


Step-by-step extension of Bloch frames

Step 3

Parallel transport along k_1 (preserves TRS & k_2-periodicity):

$$\Psi(k_1, k_2) := T_{k_2}(k_1)\,\Psi(0, k_2) \text{ with } \begin{cases} \mathrm{i}\,\partial_{k_1} T_{k_2}(k_1) = \mathrm{i}[\partial_{k_1} P(k), P(k)]\, T_{k_2}(k_1) \\ T_{k_2}(0) = \mathbf{1}_{\mathbb{C}^M}; \quad \boxed{T_{k_2}(1) =: \mathcal{T}(k_2)} \end{cases}$$

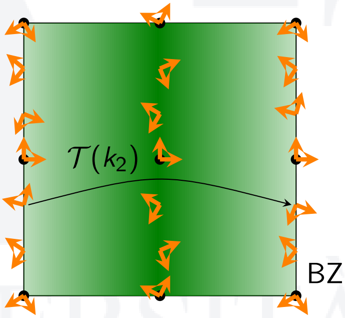


Step-by-step extension of Bloch frames

Matching matrix

$$\Psi(1/2, k_2) = \Psi(-1/2, k_2)\,\mathcal{T}(k_2), \quad \mathcal{T}(k_2) \in U(m)$$

$k_2 \mapsto \mathcal{T}(k_2)$ is smooth, $\mathbb{Z}$-periodic, and TRS: $\boxed{\varepsilon\,\mathcal{T}(k_2) = \mathcal{T}(-k_2)^{\mathsf{T}}\,\varepsilon}$



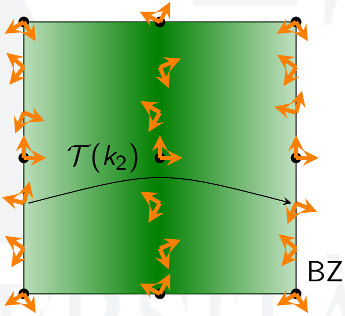
Step-by-step extension of Bloch frames

Topological obstruction

A smooth, periodic, and TRS Bloch frame exists

$$\Longleftrightarrow$$

$$\mathcal{T} \sim_{\mathbb{Z}_2 - h} \mathbf{1}$$



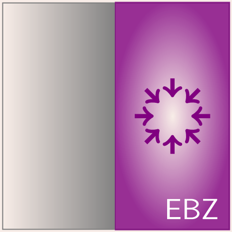
Obstruction matrix

- Ψ as above (smooth, k_2-periodic, TRS, matching matrix $\mathcal{T}(k_2)$)
- Φ fully symmetric

Obstruction matrix

$$\Phi(k) = \Psi(k) U_{\mathrm{obs}}(k), \qquad U_{\mathrm{obs}}(k) \in U(m)$$

- w.l.o.g. $U_{\mathrm{obs}}(0, k_2) \equiv \mathbf{1} \equiv U_{\mathrm{obs}}(k_1, \pm 1/2)$
- $k \mapsto U_{\mathrm{obs}}(k)$ is smooth
- $k_2 \mapsto U_{\mathrm{obs}}(k_1, k_2)$ is $\mathbb{Z}$-periodic
- $\varepsilon\, U_{\mathrm{obs}}(k)^* = U_{\mathrm{obs}}(-k)^{\mathsf{T}}\, \varepsilon$
- $\boxed{\mathcal{T}(k_2) = \varepsilon^{-1}\, \overline{U_{\mathrm{obs}}(1/2, -k_2)}\, \varepsilon\, U_{\mathrm{obs}}(1/2, k_2)^*}$



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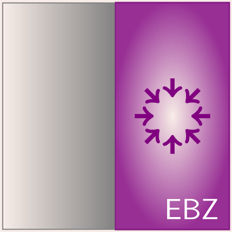

Fiorenza–Monaco–Panati index [Fiorenza–M.–Panati CMP'16]

$$\boxed{\mathrm{FMP} := \mathrm{wind}_{\partial \mathrm{EBZ}}(\det U_{\mathrm{obs}}) \bmod 2 \qquad \in \mathbb{Z}_2}$$

FKM = FMP

$$\mathcal{A}_{\mathrm{obs}} = \mathcal{A} - \mathrm{i}\,\mathrm{Tr}\left(U_{\mathrm{obs}}^{-1}\,\mathrm{d}U_{\mathrm{obs}}\right)$$

Hence by Stokes

$$\frac{1}{2\pi}\int_{\mathrm{EBZ}} \mathcal{F} = \frac{1}{2\pi}\oint_{\partial\mathrm{EBZ}} \mathcal{A} = \frac{1}{2\pi}\oint_{\partial\mathrm{EBZ}} \mathcal{A}_{\mathrm{obs}} + \frac{\mathrm{i}}{2\pi}\oint_{\partial\mathrm{EBZ}} \mathrm{Tr}\left(U_{\mathrm{obs}}^{-1}\,\mathrm{d}U_{\mathrm{obs}}\right)$$

or

$$\mathrm{wind}_{\partial\mathrm{EBZ}}(\det U_{\mathrm{obs}}) = \frac{1}{2\pi}\int_{\mathrm{EBZ}} \mathcal{F} - \frac{1}{2\pi}\oint_{\partial\mathrm{EBZ}} \mathcal{A}_{\mathrm{obs}}$$

$$\mathrm{FMP} = \mathrm{FKM} \quad \in \mathbb{Z}_2$$

$\mathbb{Z}_2$-homotopy theory of matching matrices

$\alpha\colon S^1 \to U(m)$ smooth, $\mathbb{Z}$-periodic, and TRS, i.e. $\varepsilon\, \alpha(k_2) = \alpha(-k_2)^{\mathsf{T}}\, \varepsilon$

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Kramers degeneracy

Eigenvalues of $\alpha(0)$, $\alpha(1/2)$ are even-degenerate

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Proposition ([Graf–Porta CMP'13; Cornean–M.–Teufel RMP'17])

The following are equivalent:

- $\alpha \sim_{\mathbb{Z}_2-h} \mathbf{1}$
- $\alpha(k_2) = \mathrm{e}^{\mathrm{i}h_1(k_2)}\,\mathrm{e}^{\mathrm{i}h_2(k_2)}$ *with* $h_i = h_i^*$ *smooth, periodic, and TRS*
- $\mathrm{Rueda}(\alpha) \equiv 0 \bmod 2$, *where*

$$\mathrm{Rueda}(\alpha) := \frac{1}{2\pi\mathrm{i}}\left(\int_0^{1/2} \mathrm{Tr}\left(\alpha^{-1}\,\mathrm{d}\alpha\right) - 2\log\frac{\sqrt{\det\alpha(1/2)}}{\sqrt{\det\alpha(0)}}\right) \in \mathbb{Z}$$

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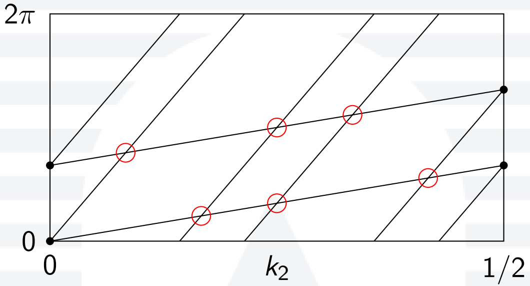
Graf–Porta index

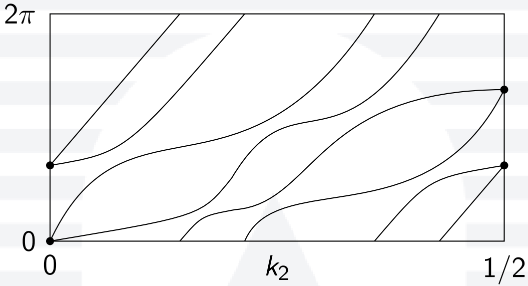
$$\boxed{\mathrm{GP} := \mathrm{Rueda}(\mathcal{T}) \bmod 2 \qquad \in \mathbb{Z}_2}$$

Rueda and logarithm



Extra degeneracies in $\sigma(\alpha(k_2))$, $k_2 \in (0, 1/2)$, can be lifted

Rueda and logarithm

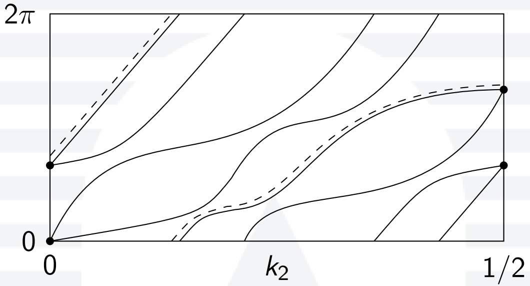


Extra degeneracies in $\sigma(\alpha(k_2))$, $k_2 \in (0, 1/2)$, can be lifted $\Longrightarrow$

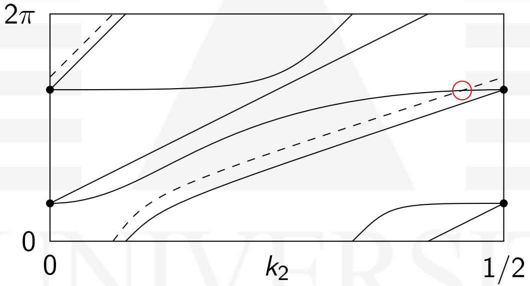
$$\boxed{\alpha(k_2) = \alpha_{\mathrm{gen}}(k_2)\, \mathrm{e}^{\mathrm{i} h_2(k_2)}}$$

h_2 smooth, periodic, TRS

Rueda and logarithm



Rueda(α) = 0 $\Longrightarrow$
$\alpha_{\mathrm{gen}}(k_2) = \mathrm{e}^{\mathrm{i}h_1(k_2)}$,
h_1 smooth, periodic, TRS



Rueda(α) = 1 $\Longrightarrow$
no smooth, periodic, TRS log

FMP = GP

Proposition ([Cornean–M.–Teufel RMP'17])

If $\boxed{\alpha(k_2) = \varepsilon^{-1}\,\gamma(-k_2)^{\mathsf{T}}\,\varepsilon\,\gamma(k_2)}$ *with* $\gamma \colon S^1 \to U(m)$ *smooth and* $\mathbb{Z}$-*periodic, then*

$$\mathrm{Rueda}(\alpha) = \mathrm{wind}_{S^1}(\det \gamma).$$

$$\mathcal{T}(k_2) = \varepsilon^{-1}\,\overline{U_{\mathrm{obs}}(1/2,-k_2)}\,\varepsilon\,U_{\mathrm{obs}}(1/2,k_2)^* \rightsquigarrow \gamma(k_2) = U_{\mathrm{obs}}(1/2,k_2)^{-1}$$

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An index from field theory

Carpentier–Delplace–Fruchart–Gawędzki–Tauber index [CDFGT NPB'15]

$$(-1)^{\mathrm{GT}+} := \sqrt{\exp\left(\mathrm{i}\, S_{\mathrm{WZW}}[\mathbf{1} - 2P]\right)} \quad \in \mathbb{Z}_2$$

- TQFT
- Defined as a holonomy over an equivariant bundle gerbe (not today!)
- Applies to periodically-driven systems as well (Floquet insulators)

WZW action

Field

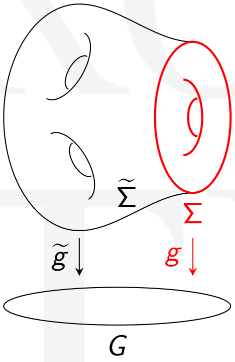
$$g\colon \Sigma \to G \text{ smooth}$$

Σ = 2D compact, closed surface (later $\Sigma = \mathbb{T}^2$)
G = compact matrix Lie group (later $G = U(M)$)

Field extension

$$\widetilde{g}\colon \widetilde{\Sigma} \to G \text{ smooth}$$

with $\partial\widetilde{\Sigma} = \Sigma$ (later $\widetilde{\Sigma}$ = solid torus) and $\widetilde{g}\big|_{\partial\widetilde{\Sigma}} = g$



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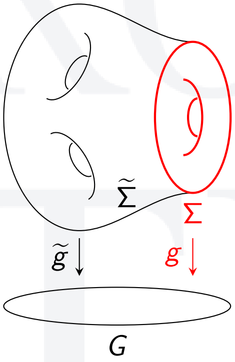

Wess–Zumino–Witten (WZW) action

$$S_{\mathrm{WZW}}[g] := \frac{1}{12\pi} \int_{\widetilde{\Sigma}} \mathrm{Tr} \left\{ \left(\widetilde{g}^{-1}\, \mathrm{d}\widetilde{g} \right)^{\wedge 3} \right\}$$

WZW amplitude

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$S_{\mathrm{WZW}}[g]$ depends *a priori* from extension $\widetilde{g}$, but if $\widetilde{g}_1\big|_{\partial\widetilde{\Sigma}} = \widetilde{g}_2\big|_{\partial\widetilde{\Sigma}}$

$$\frac{1}{12\pi} \int_{\widetilde{\Sigma}} \mathrm{Tr} \left\{ \left(\widetilde{g}_1^{-1} \, \mathrm{d}\widetilde{g}_1 \right)^{\wedge 3} \right\} - \frac{1}{12\pi} \int_{\widetilde{\Sigma}} \mathrm{Tr} \left\{ \left(\widetilde{g}_2^{-1} \, \mathrm{d}\widetilde{g}_2 \right)^{\wedge 3} \right\} \in 2\pi\mathbb{Z}$$

WZW amplitude

$$\boxed{\mathrm{WZW}[g] := \exp\left(\mathrm{i}\, S_{\mathrm{WZW}}[g] \right) \in U(1)}$$

The Chern number as a WZW amplitude

Proposition

$P\colon \mathbb{T}^2 \to \mathcal{B}(\mathbb{C}^M)$ smooth, $P(k) = P(k)^* = P(k)^2$. Set

$$\boxed{u_P(k) := \mathbf{1} - 2P(k) \in U(M)}.$$

Then $\mathrm{WZW}\,[u_P] = (-1)^{c_1(P)}$.

Proof.

Extension to $\widetilde{\Sigma} := [0,1] \times \mathbb{T}^2$

$$\boxed{\widetilde{u}_P(t,k) := \exp(\mathrm{i}\pi t P(k)) = \mathbf{1} - P(k) + \mathrm{e}^{\mathrm{i}\pi t} P(k)}$$

- $\widetilde{u}_P(t=0,k) \equiv \mathbf{1}$, $\widetilde{u}_P(t=1,k) = u_P(k) \rightsquigarrow \widetilde{\Sigma} = \mathbb{D} \times \mathbb{T}$
- $\mathrm{Tr}\{(\widetilde{u}_P^{-1}\,\mathrm{d}\widetilde{u}_P)^{\wedge 3}\} = 6\pi(1-\cos(\pi t))\,\mathrm{d}t \wedge \mathcal{F} \Rightarrow S_{\mathrm{WZW}}[u_P] = \pi c_1(P)$. □

Equivariant $U(M)$-valued fields and extensions

TRS $\Theta\colon \mathbb{C}^M \to \mathbb{C}^M$ induces $g \mapsto \Theta\, g\, \Theta^{-1}$, $g \in U(M)$
Assume Σ has involution $\vartheta\colon \Sigma \to \Sigma$, $\vartheta \circ \vartheta = \mathbf{1}_\Sigma$ (later $\vartheta(k) = -k$ on $\mathbb{T}^2$)

Equivariant field

$g\colon \Sigma \to U(M)$ such that $\boxed{g(\vartheta(k)) = \Theta\, g(k)\, \Theta^{-1}}$

Equivariant field extension

$\widetilde{g}\colon \widetilde{\Sigma} \to G$ extension of g such that

- $\widetilde{\Sigma}$ has involution $\widetilde{\vartheta}$ and $\widetilde{\vartheta}\big|_{\partial\widetilde{\Sigma}} = \vartheta$
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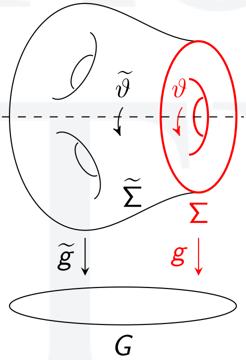

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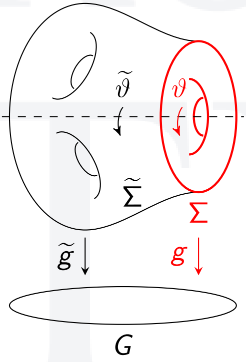

$$\frac{1}{12\pi} \int_{\widetilde{\Sigma}} \mathrm{Tr}\left\{ (\widetilde{g}_1^{-1}\, \mathrm{d}\widetilde{g}_1)^{\wedge 3} - (\widetilde{g}_2^{-1}\, \mathrm{d}\widetilde{g}_2)^{\wedge 3} \right\} \in 4\pi\mathbb{Z}$$

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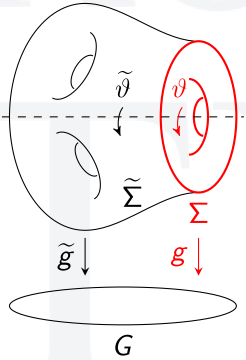

$$\boxed{\sqrt{\mathrm{WZW}[g]} := \exp\left(\mathrm{i} S_{\mathrm{WZW}}[g]/2\right)}$$

GT+ = FKM

Theorem ([M.–Tauber LMP'17])

$$(-1)^{\mathrm{GT+}} = \sqrt{\mathrm{WZW}[\mathbf{1} - 2P]} = (-1)^{\mathrm{FKM}} \in \mathbb{Z}_2.$$

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With $\widetilde{g}_P(t,k) = \exp(\mathrm{i}2\pi t\, P(k))$

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$$g_P : \underbrace{S^1 \times \mathbb{T}_*}_{\text{not BZ!}} \to U(M), \quad g_P(t, k_2) := \exp(\mathrm{i}2\pi t\, P(k_*, k_2))$$

Equivariant adjoint Polyakov–Wiegmann formula

Proof.

- By 1D discussion, $P(k_*, k_2) = W(k_2)\, P(k_*, 0)\, W(k_2)^*$, with $W(k_2) := \mathrm{e}^{-\mathrm{i} k_2 X}\, T(k_2)$ modified parallel transport.
- g_P has adjoint structure:

$$g_P(t, k_2) = W(k_2) g_P(t, 0) W(k_2)^* \equiv W(k_2) f_P(t) W(k_2)^*.$$

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$$\boxed{S_{\mathrm{WZW}}[g h g^{-1}] = S_{\mathrm{WZW}}[h] + \frac{1}{4\pi} \int_{S^1 \times \mathbb{T}_*} (g \times h)^* \beta \mod 4\pi\mathbb{Z}}$$

- For $g = W$, $h = f_P$

$$S_{\mathrm{WZW}}[f_P] = 0,$$

$$\frac{1}{4\pi} \int_{S^1 \times \mathbb{T}_*} (W \times f_P)^* \beta = \mathrm{i} \int_{\mathbb{T}_*} \mathrm{Tr}\left\{ P(k_*, 0) W^{-1}\, \mathrm{d}W \right\} = -\mathrm{i} \int_{\mathbb{T}_*} \mathcal{A}. \qquad \square$$

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$$\boxed{S_{\mathrm{WZW}}[g h g^{-1}] = S_{\mathrm{WZW}}[h] + \frac{1}{4\pi} \int_{S^1 \times \mathbb{T}_*} (g \times h)^* \beta \mod 4\pi\mathbb{Z}}$$

- For $g = W$, $h = f_P$

$$S_{\mathrm{WZW}}[f_P] = 0,$$

$$\frac{1}{4\pi} \int_{S^1 \times \mathbb{T}_*} (W \times f_P)^* \beta = \mathrm{i} \int_{\mathbb{T}_*} \mathrm{Tr} \left\{ P(k_*, 0) W^{-1}\, \mathrm{d}W \right\} = -\mathrm{i} \int_{\mathbb{T}_*} \mathcal{A}. \quad \square$$

Further properties

- FKM $\in \mathbb{Z}_2$ is a complete homotopy invariant of $2D$ topological insulators in class AII, hence classify TRS-isomorphism class of the Bloch bundle:

$$P_0 \sim_{\mathbb{Z}_2 - h} P_1 \iff \mathsf{FKM}(P_0) = \mathsf{FKM}(P_1) \in \mathbb{Z}_2 \iff \mathcal{E}_0 \simeq_{\mathsf{TRS}} \mathcal{E}_1$$

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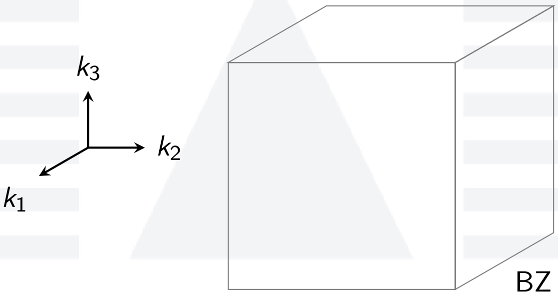

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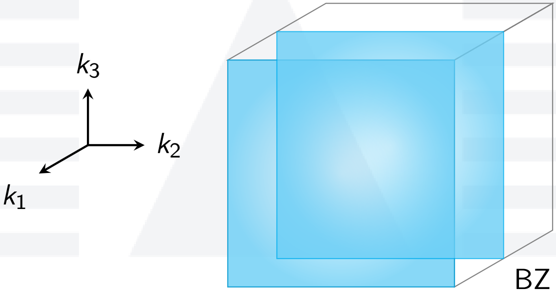

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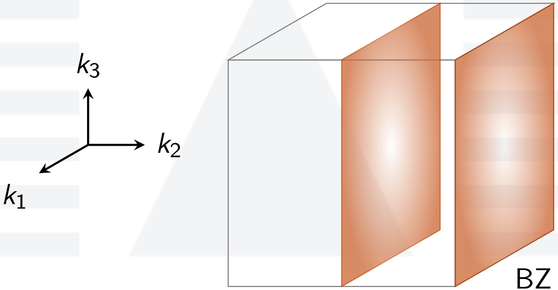

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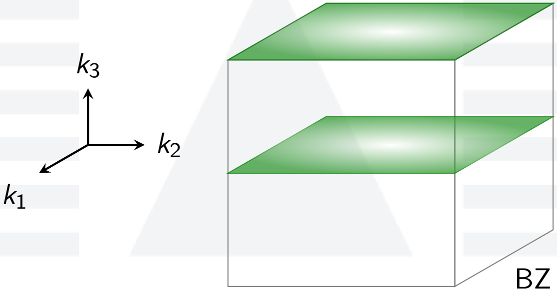

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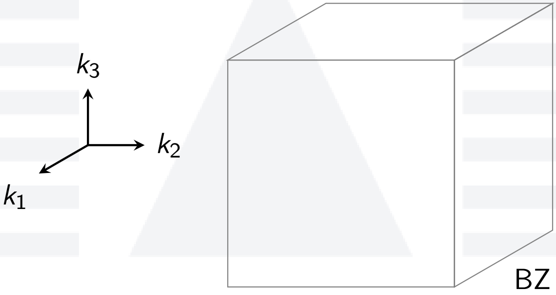

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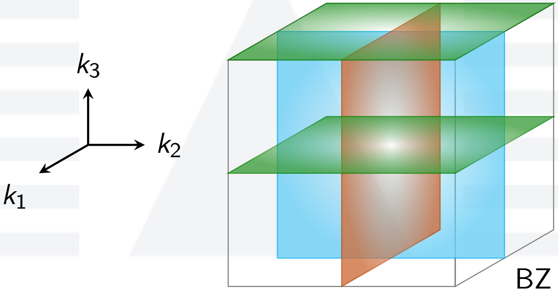

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$$(-1)^{\mathrm{FKM}} = \prod_{k \equiv -k \bmod \mathbb{Z}^2} \frac{\sqrt{\det w(k)}}{\mathrm{Pf}\, w(k)} \qquad w(k)_{ab} := \langle \psi_a(k), \Theta \psi_b(k) \rangle$$

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