Hierarchical Majoranas in a Programmable Nanowire Network

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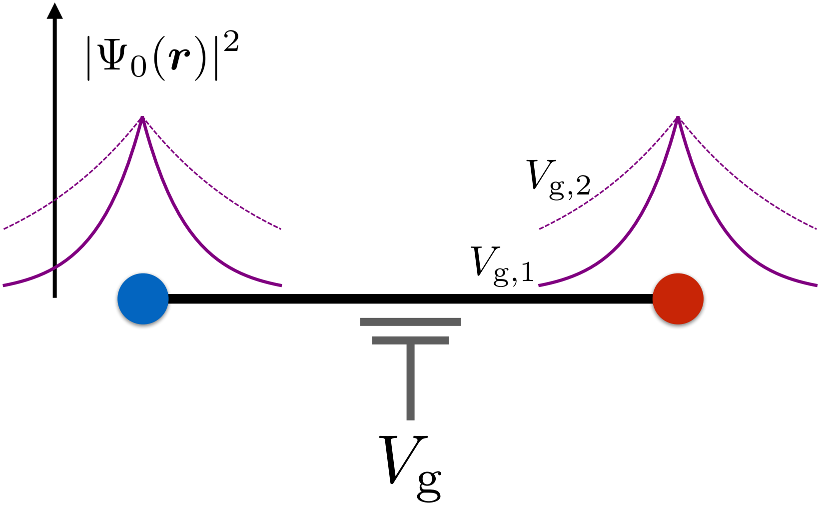
Section

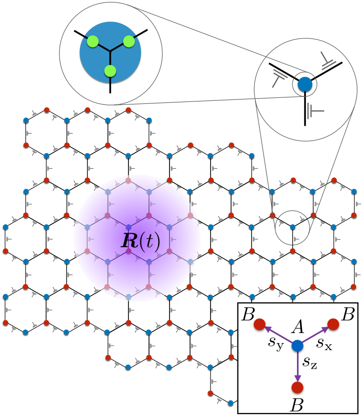
First hierarchy of quasi Majorana zero modes (QMZMs)

There are physical QMZMs:



Second hierarchy of Majorana zero modes (MZMs)

There are logical (emergent) MZMs $V_{\mathrm{g}} + \delta V_{\mathrm{g}\,\boldsymbol{r},\alpha}$:

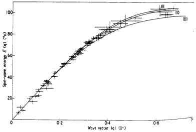


$$\delta V_{\mathrm{g}\,\boldsymbol{r},\alpha}(t) := V_0 \cos\left(\boldsymbol{K}_+ \cdot \boldsymbol{s}_\alpha + (\boldsymbol{K}_+ - \boldsymbol{K}_-) \cdot \boldsymbol{r} + q \arg\left(\boldsymbol{r} - \boldsymbol{R}(t)\right)\right), \qquad q = \pm 1.$$

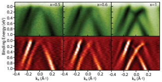
Section

Exotic excitations in many-body quantum physics

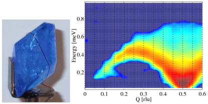
- When interactions are not too strong, excitations are “simple”:
  - smooth fluctuations about the magnetization (magnons, Bloch 1930),
  - dressed electrons (Fermi-Liquid quasiparticles, Landau 1957).
- When interactions are strong, excitations can be “exotic”:
  - spinons carrying spin-1/2 quantum numbers in antiferromagnetic spin-1/2 chains (Faddeev and Takhtajan 1981),
  - fractionally charged electrons in polyacetylene (Jackiw and Rebbi 1976; Su, Schrieffer, and Heeger 1979).



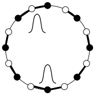
RbMnFs, Windsor 1966



Bi_xPb_{1-x}/Ag(111), Meier 2009



$CuSO_4$:$5D_2O$, Ronnow 2007



Majoranas as “exotic” excitations: I

- Starting from the pair of non-Hermitean operators $\widehat{c}$ and $\widehat{c}^{\dagger}$ obeying the fermion algebra $\{\widehat{c},\widehat{c}^{\dagger}\} := 1$ we may identify the pair of Majorana operators $\widehat{\gamma}_1 := (\widehat{c}+\widehat{c}^{\dagger})$ and $\widehat{\gamma}_2 := (\widehat{c}-\widehat{c}^{\dagger})/\mathrm{i}$ obeying the Majorana algebra $\{\widehat{\gamma}_a,\widehat{\gamma}_b\} = 2\delta_{ab}$.
- We can **always** interpret an electron as being a bound state of two Majoranas. However, this interpreation is **pertinent only if** there are fermionic Hamiltonians whose “exotic” excitations are Majoranas!
- The Majoranas making up the electron are **physically meaningfull** iff they are:
  - **(fully) deconfined due to electron-electron interactions**
  - **or (partially) deconfined due to defectuous backgrounds.**

Majoranas as “exotic” excitations: II

- Moore and Read in 1991 propose that this is so in the fractional quantum Hall effect at the filling fraction 5/2.

- Read and Green in 2000 give a simpler interpretation for the Majoranas of Moore and Read as the zero modes bound to vortices of a two-dimensional type II chiral *p*-wave superconductor.

- Kitaev in 2001 give a one-dimensional version of Read and Green where the the role of the vortices is taken by domain walls (like in polyacetylene), each of which shall be called a Majorana nanowire.

- Fu and Kane in 2008 show that the vortices of an *s*-wave superconductor in contact with the surface of a three-dimensional toological insulator bound Majorana zero modes.

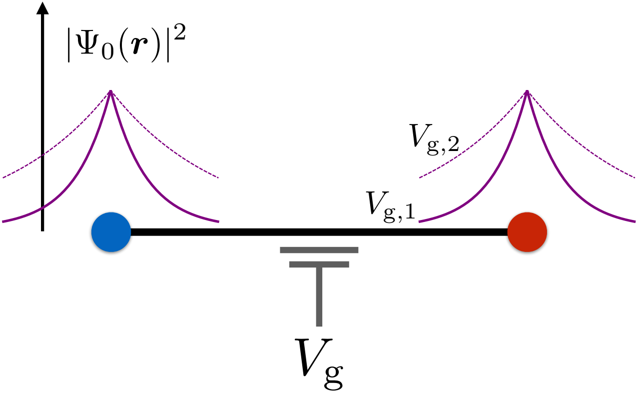
Majoranas as “exotic” excitations: III

- The Majorana zero modes (MZMs) bound to a vortex in a two-dimensional type II chiral *p*-wave superconductor obey non-Abelian statistics (Fröhlich 1988) when braided (Ivanov 2001).
- However, how does one braid a pair of superconducting vortices?
- An alternative physical platform for realizing MZMs that could be braided was proposed by several groups – Sau 2010, Alicea 2010, Lutchyn 2010, and Oreg 2010 – by building two-dimensional networks of Majorana nanowires.
- Hereto, there is a difficulty in that the braiding of these (physical) MZMs often violate adiabacity.

Section

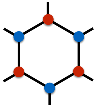
Elementary building block

The elementary building block is a nanowire which at low temperatures supports a topological superconducting gap Δ_{nw}, i.e., the nanowire hosts a pair of QMZMs at its endpoints when superconducting. We shall call such a nanowire a "Majorana nanowire."

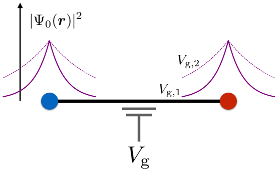


Honeycomb lattice made of Majorana nanowires

Imagine that all nearest-neighbor bonds of the honeycomb lattice



are realized by identical Majorana nanowires

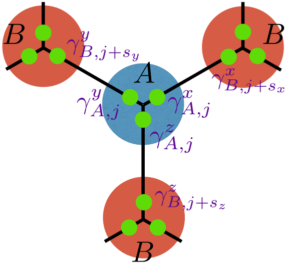


There are then two energy scales in the problem:

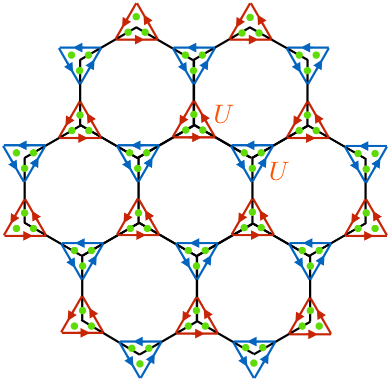
- a hybridization U
- and a hopping amplitude t.

Vertices of the honeycomb lattice are Y junctions of Majorana nanowires

The QMZMs are depicted as green dots. Effectively, there are three flavors of QMZMs on each lattice site. We label the operators creating QMZMs by $\widehat{\gamma}^{\alpha}_{S,j}$, where $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$ denotes the bond to which the QMZM belongs, while $S = A, B$ denotes the sublattices, and j is the label for the lattice sites.



The trimer limit ($U \neq 0$, $t = 0$) is defined by



$$\widehat{H}_{\text{trimer}} := \sum_{S=A,B} \sum_{j\in\Lambda_S} \mathrm{i}U\left(\widehat{\gamma}^{\mathrm{x}}_{S,j}\,\widehat{\gamma}^{\mathrm{y}}_{S,j} + \widehat{\gamma}^{\mathrm{y}}_{S,j}\,\widehat{\gamma}^{\mathrm{z}}_{S,j} + \widehat{\gamma}^{\mathrm{z}}_{S,j}\,\widehat{\gamma}^{\mathrm{x}}_{S,j}\right)$$

where the honeycomb lattice Λ is made of two interpenetrating triangular sublattices Λ_A and Λ_B, while we impose the Majorana algebra

$$\left\{\widehat{\gamma}^{\alpha}_{S,j}, \widehat{\gamma}^{\alpha'}_{S',j'}\right\} = 2\delta_{\alpha,\alpha'}\delta_{S,S'}\delta_{j,j'}, \qquad \widehat{\gamma}^{\alpha\dagger}_{S,j} = \widehat{\gamma}^{\alpha}_{S,j}.$$

The trimer Hamiltonian is the sum over $S = A, B$ and $j \in \Lambda_S$ of the pairwise commuting operators

$$\mathrm{i}U\left(\widehat{\gamma}^{\mathrm{x}}_{S,j}\,\widehat{\gamma}^{\mathrm{y}}_{S,j}+\widehat{\gamma}^{\mathrm{y}}_{S,j}\,\widehat{\gamma}^{\mathrm{z}}_{S,j}+\widehat{\gamma}^{\mathrm{z}}_{S,j}\,\widehat{\gamma}^{\mathrm{x}}_{S,j}\right).$$

As each one of these operators has the three single-particle eigenvalues

$$-\sqrt{3}\,U, \qquad 0, \qquad +\sqrt{3}\,U,$$

with the Majorana zero mode

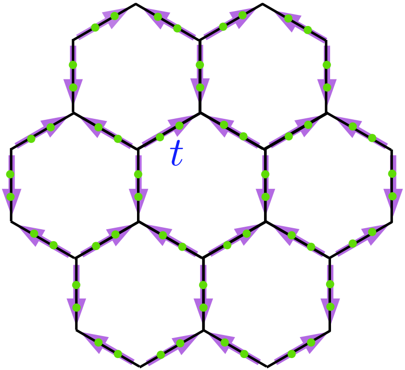
$$\hat{\eta} := \frac{1}{\sqrt{3}}\left(\widehat{\gamma}^{\mathrm{x}}_{S,j}+\widehat{\gamma}^{\mathrm{y}}_{S,j}+\widehat{\gamma}^{\mathrm{z}}_{S,j}\right),$$

it supports three doubly-degenerate flat bands with the single-particle energies

$$-\sqrt{3}\,U, \qquad 0, \qquad +\sqrt{3}\,U,$$

respectively.

The dimer limit ($U = 0$, $t \neq 0$) is defined by



$$\widehat{H}_{\text{dimer}} := \sum_{j\in\Lambda_A} \sum_{\alpha=\mathrm{x,y,z}} \mathrm{i}\,t\,\widehat{\gamma}^{\alpha}_{A,j}\,\widehat{\gamma}^{\alpha}_{B,j+\boldsymbol{s}_\alpha},$$

where $\boldsymbol{s}_\alpha$ are the unit vectors connecting the three sites in Λ_B that are nearest-neighbor to a site in Λ_A. The dimer Hamiltonian supports two triply-degenerate flat bands with the single-particle energies $-|t|$ and $+|t|$, respectively.

These single-particle energies correspond to the fermionic state

$$\hat{c}^{\alpha\dagger}_j|0\rangle := \frac{1}{2}\left(\widehat{\gamma}^{\alpha}_{A,j} - \mathrm{i}\,\widehat{\gamma}^{\alpha}_{B,j+\boldsymbol{s}_\alpha}\right)|0\rangle, \qquad \hat{c}^{\alpha}_j|0\rangle := 0,$$

being empty or occupied, respectively. There is no zero mode in the dimer limit.

Reversal of time

Time reversal is defined by the rules

$$\mathrm{i} \mapsto -\mathrm{i}, \qquad \widehat{\gamma}^{\alpha}_{A,j} \mapsto +\widehat{\gamma}^{\alpha}_{A,j}, \qquad \widehat{\gamma}^{\alpha}_{B,j+\boldsymbol{s}_\alpha} \mapsto -\widehat{\gamma}^{\alpha}_{B,j+\boldsymbol{s}_\alpha}.$$

The motivation for this definition is that we would like to interpret

$$\hat{c}^{\alpha}_{A,j} := \frac{1}{2}\left(\widehat{\gamma}^{\alpha}_{A,j} + \mathrm{i}\,\widehat{\gamma}^{\alpha}_{B,j+\boldsymbol{s}_\alpha}\right)$$

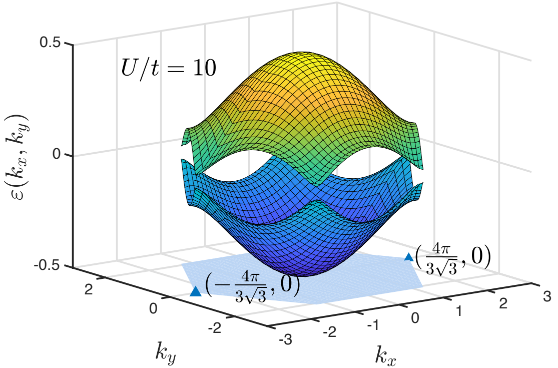
as a fermion operator localized on the directed bond $\langle j \in \Lambda_A, j + \boldsymbol{s}_\alpha \in \Lambda_B\rangle$ of the honeycomb lattices that is left invariant by the operation of time reversal. One verifies that

$$\widehat{H}_{\mathrm{dimer}} \mapsto +\widehat{H}_{\mathrm{dimer}}, \qquad \widehat{H}_{\mathrm{trimer}} \mapsto -\widehat{H}_{\mathrm{trimer}}.$$

Although $\widehat{H}_{\mathrm{trimer}}$ is odd under time reversal, the zero-energy flat band transforms trivially, whereas the finite-energy bands are interchanged.

Hamiltonian for the network of nanowires

The pair of **particle-hole symmetric** bands with the lowest energy in magnitudes for $(\widehat{H}_{\text{trimer}} + \widehat{H}_{\text{dimer}})/t$ when $U/t = 10$ with $U > t > 0$ displays a **Haldane gap** at the corners of the Brillouin zone Ω_{BZ} (depicted in light blue) [the magnitude of the Haldane gap follows from $\varepsilon_{\pm}(\boldsymbol{K}_{+}) = \varepsilon_{\pm}(\boldsymbol{K}_{-}) \approx \pm\frac{t^2}{2\sqrt{3}\,U} + \mathcal{O}(t^4/U^3)$]:

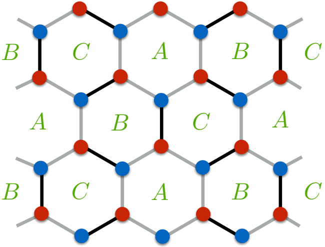


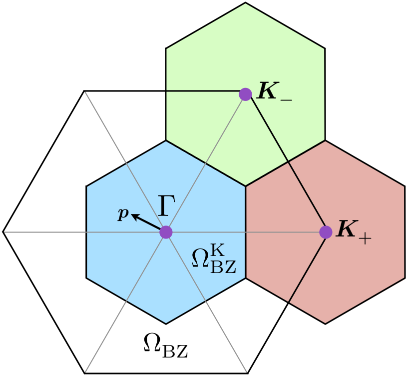
A Kekulé dimerization competes with the Haldane gap:

If we add the perturbation

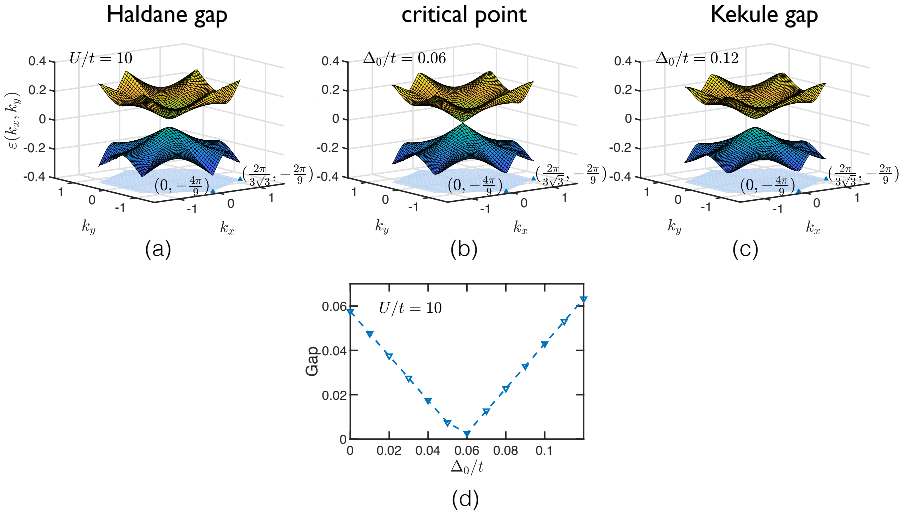
$$\delta\widehat{H}_{\text{dimer}} := \mathrm{i}\sum_{j\in\Lambda_A}\sum_{\alpha=\mathrm{x,y,z}} \delta t_{j,\alpha}\,\widehat{\gamma}^{\alpha}_{A,j}\,\widehat{\gamma}^{\alpha}_{B,j+\boldsymbol{s}_\alpha}, \qquad \delta t_{j,\alpha} := \Delta\, e^{\mathrm{i}\boldsymbol{K}_+\cdot\boldsymbol{s}_\alpha}\, e^{\mathrm{i}\boldsymbol{G}\cdot\boldsymbol{r}_j} + \text{c.c.}, \qquad \boldsymbol{G} := \boldsymbol{K}_+ - \boldsymbol{K}_-,$$

where the Kekulé amplitude is defined by $\Delta := \Delta_0\, e^{\mathrm{i}\varphi}$, $\Delta_0 := |\Delta|$, and $\varphi \in [0, 2\pi)$, we lower the space-group symmetry to



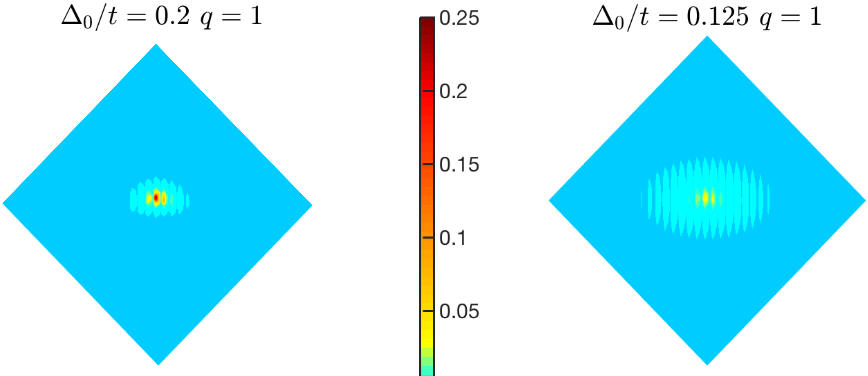


If so, the two lowest particle-symmetric single-particle bands are (Hou 2007; Ryu 2009)



A logical Majorana zero mode (MZM) is bound to the Kekulé vortex (Hou 2007)

$\delta t_{\boldsymbol{r},\alpha} := \Delta_0 \cos\left(\boldsymbol{K}_+ \cdot \boldsymbol{s}_\alpha + \boldsymbol{G}\cdot\boldsymbol{r} + q\arg(\boldsymbol{r})\right),\; q = \pm 1$:

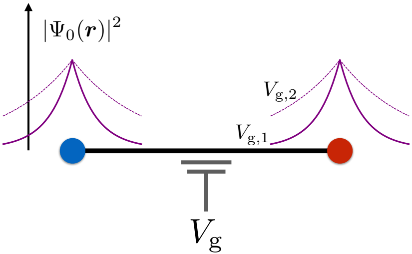


(a) (b)

Section

Estimate of *U*

The relevant energy scales in a Majorana nanowire of length $\mathfrak{a}$ are:



$$\Delta_{nw} := \frac{g\,\mu_B\,|B|}{2} - \sqrt{\Delta_{sc}^2 + V_g^2} \approx U > 0$$

- the external magnetic field $|B|$ needed to break time-reversal symmetry,
- the proximity-induced superconducting gap $|\Delta_{sc}|$ needed to break charge conservation,
- the chemical potential $|V_g|$ of the wire in proximity of the *s*-wave superconducting substrate,
- the Rashba energy scale $\hbar\, v_{F,nw}$ needed to break spin-rotation symmetry that enters through the Fermi velocity $v_{F,nw}$ of the nanowire.

Estimate of t

Physical QMZMs are bound to the end points of this Majorana nanowire if and only if

$$\frac{g\mu_{\mathrm{B}}\,|B_z|}{2} > \sqrt{\Delta_{\mathrm{sc}}^2 + V_{\mathrm{g}}^2}.$$

The decay length for a physical QMZM bound to the end points of a Majorana nanowire is

$$\xi_{\mathrm{physical}} = \frac{\hbar\, v_{\mathrm{F,nw}}}{\Delta_{\mathrm{nw}}}.$$

It follows that the overlap between two physical QMZMs is then approximately given by

$$t \sim \frac{\hbar\, v_{\mathrm{F,nw}}}{\mathfrak{a}}\,\kappa\, e^{-\kappa}, \qquad \kappa := \frac{\mathfrak{a}\,\Delta_{\mathrm{nw}}}{\hbar\, v_{\mathrm{F,nw}}},$$

when measured in units of energy.

Estimate of Δ_0 and $\xi_{\rm logical}$

If we compute the leading order change $\delta\Delta_{\rm nw}$ resulting from $V_{\rm g} \to V_{\rm g} + \delta V_{\rm g}$, we find the estimate

$$\frac{\Delta_0}{t} \sim \frac{\delta t}{t} \approx \frac{\kappa - 1}{\kappa} \frac{\mathfrak{a}}{\xi_{\rm sc}} \frac{V_{\rm g}^2/\Delta_{\rm sc}^2}{\sqrt{1 + V_{\rm g}^2/\Delta_{\rm sc}^2}} \frac{\delta V_{\rm g}}{V_{\rm g}}$$

for the Kekulé gap measured in units of t. The decay length of a logical MZM is then

$$\xi_{\rm logical} := \frac{t}{\delta t}\,\mathfrak{a}.$$

For an InSb/Al Majorana wire (Lutchyn 2018),

$$\Delta_{\rm sc} \sim 0.2\,{\rm meV}, \qquad v_{\rm F,nw} \sim 0.2 - 1.0\ {\rm eV} \times\ , \qquad \xi_{\rm sc} \sim 100 - 500\,{\rm nm}$$

so that for a Majorana nanowires of length $\mathfrak{a} \sim 1\mu{\rm m}$ and with $\kappa \approx 2$,

$$\frac{\mathfrak{a}}{\xi_{\rm sc}} \sim 2 - 10, \qquad \frac{\Delta_0}{t} \approx \frac{1}{2} \frac{\mathfrak{a}}{\xi_{\rm sc}} \frac{V_{\rm g}^2/\Delta_{\rm sc}^2}{\sqrt{1 + V_{\rm g}^2/\Delta_{\rm sc}^2}} \frac{\delta V_{\rm g}}{V_{\rm g}}.$$

The prefactor in front of $\delta V_{\mathrm{g}}/V_{\mathrm{g}}$ on the right-hand side can be chosen to be of order one by choosing the ratio $V_{\mathrm{g}}^2/\Delta_{\mathrm{sc}}^2$ so as to compensate the factor $\mathfrak{a}/(2\xi_{\mathrm{sc}}) \sim 1.0 - 5.0$. (The corresponding bias V_{g} should thus be of roughly the same order as Δ_{sc}.)

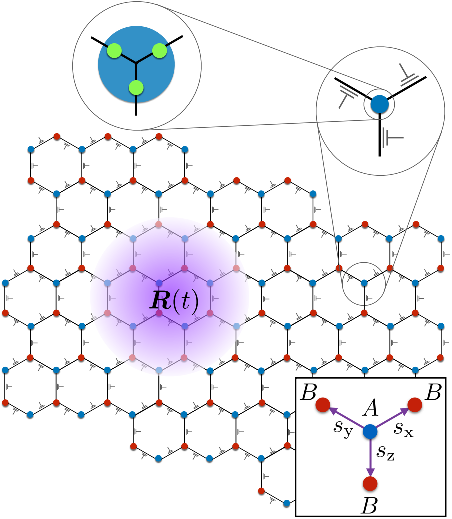
If so, the ratio $\Delta_0/t \approx \delta V_{\mathrm{g}}/V_{\mathrm{g}}$.

Consequently, by using modulations with δV_{g} of the same order as V_{g},

one can make the Kekulé gap of the order of t, and hence the size of the logical MZMs as small as the length scale of the wire size $\mathfrak{a}$.

Section

There follows the support of the logical MZM in



What could be measured?

- The existence of the “logical" MZMs can be probed via scanning tunneling microscopy (STM), where they manifest themselves as zero-bias peaks in the tunneling differential conductance.
- For a system with $2N$ “logical" MZMs, each pair of MZMs constitutes a fermionic state that can be either empty or filled.
- The fermion parity (even or odd, respectively) of each pair then specifies the state of a qubit. Thus, the dimension of the Hilbert space spanned by the quantum states of these qubits grows as 2^{N-1} once the total fermion parity of the $2N$ MZMs has been fixed.
- To verify that braiding the “logical" MZMs acts in the desired way, one needs a means of measuring the fermion parity of any pair of MZMs.
- If we exploit the fact that the “logical" MZMs can be moved adiabatically by adjusting the array of gate voltages, bringing a pair of “logical" MZMs together by merging two Kekulé vortices effectively “fuses" the two MZMs.
- To determine whether the pair of MZMs were in an even- or odd-fermion-parity state, one can measure – with scanning single-electron transistor microscopy (SSETM) – the local charge distribution in the vicinity of the fused pair.

Section

Logical MZM as zero mode of an effective Dirac Hamiltonian with a Kekulé vortex $\Delta_{\mathrm{vtx}}(\boldsymbol{r}) := \Delta_0(\boldsymbol{r})\, e^{\mathrm{i}(\varphi+n\theta)}$ with $\boldsymbol{r} = |\boldsymbol{r}| \begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}^{\mathsf{T}}$ and $\varphi \in [0, 2\pi)$:

The pair of particle-hole symmetric bands with the lowest energies is encoded by

$$\widetilde{\mathcal{H}}_{\mathrm{Kek}}(\boldsymbol{r}) := \begin{pmatrix} 0 & 2\mathrm{i}\partial_z & \Delta_{\mathrm{vtx}}(\boldsymbol{r}) & 0 \\ 2\mathrm{i}\partial_{\bar{z}} & 0 & 0 & \Delta_{\mathrm{vtx}}(\boldsymbol{r}) \\ \overline{\Delta_{\mathrm{vtx}}}(\boldsymbol{r}) & 0 & 0 & -2\mathrm{i}\partial_z \\ 0 & \overline{\Delta_{\mathrm{vtx}}}(\boldsymbol{r}) & -2\mathrm{i}\partial_{\bar{z}} & 0 \end{pmatrix}, \qquad z = x + \mathrm{i}y.$$

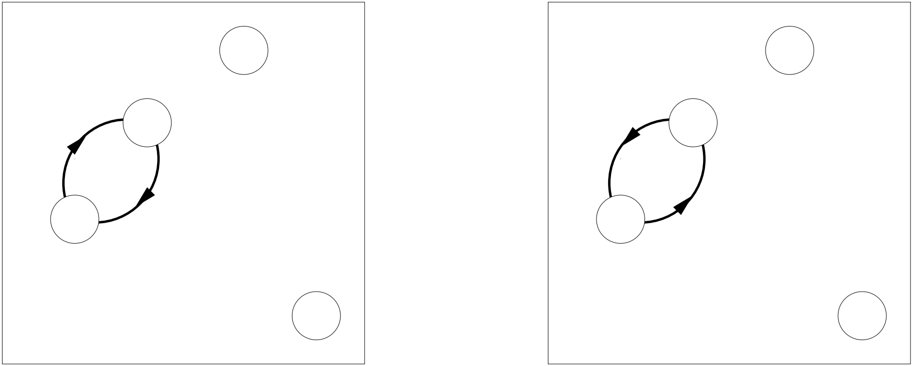
When $n = -1$, a normalizable zero mode is supported on sublattice A and given by

$$\Psi_{A,0}(\boldsymbol{r}) = \mathcal{N} \begin{pmatrix} e^{+\mathrm{i}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}\, e^{-\int_0^r \mathrm{d}r'\, \Delta_0(r')} \\ 0 \\ 0 \\ e^{-\mathrm{i}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}\, e^{-\int_0^r \mathrm{d}r'\, \Delta_0(r')} \end{pmatrix}, \qquad \varphi \in [0, 2\pi).$$

There follows the "logical" MZM operator

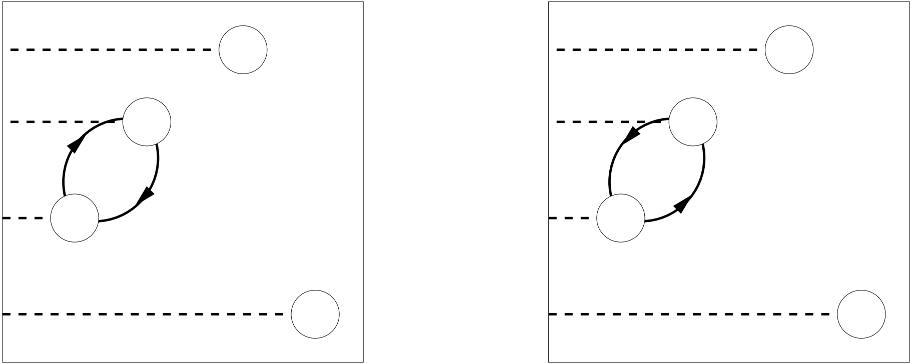
$$\widehat{\gamma}_A := \int \mathrm{d}^2\boldsymbol{r}\, \left[u_A(\boldsymbol{r})\, \widehat{a}_{A,+}(\boldsymbol{r}) + \overline{u_A}(\boldsymbol{r})\, \widehat{a}_{A,-}(\boldsymbol{r})\right] = \widehat{\gamma}_A^{\dagger} \quad \text{as } \widehat{a}_{A,+}(\boldsymbol{r}) = \widehat{a}^{\dagger}_{A,-}(\boldsymbol{r}).$$

Braiding of vortices



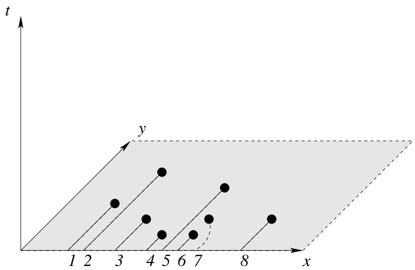
Interchanging two vortices is a commutative operation.

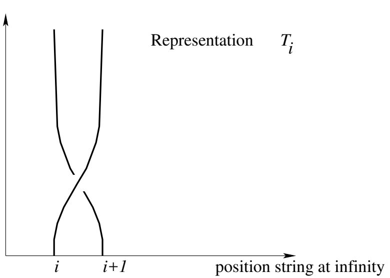
Braiding of logical MZMs



- Interchanging two logical MZMs always implies that one and only one string is crossed by one and only one MZM.
- The single-particle wave function of the logical MZM that crosses the string changes by the phase factor $\exp(\mathrm{i}2\pi/2) = -1$.

Braiding of logical MZMs: definition





Interchange T_i of two logical MZMs is

$$T_i : \left\{\widehat{\gamma}_j\right\} \to \left\{\widehat{\gamma}_j\right\},$$

$$\widehat{\gamma}_j \mapsto T_i(\widehat{\gamma}_j) := \begin{cases} +\gamma_{i+1}, & j = i, \\ -\gamma_i, & j = i+1, \\ +\widehat{\gamma}_j, & j \neq i, i+1. \end{cases}$$

Braiding of logical MZMs: representation

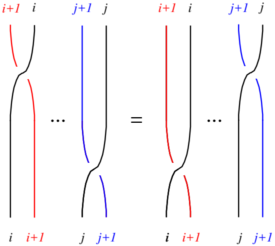
Let $i = 1, \cdots, 2n$ index $2n$ Kekulé vortices. The interchange of two Kekulé vortices – such that their strings never cross the remaining $2(n-1)$ Kekulé vortices – is represented through conjugation by

$$\widehat{\tau}(T_i) := \exp\left(\frac{\pi}{4}\,\widehat{\gamma}_{i+1}\,\widehat{\gamma}_i\right) = \frac{1}{\sqrt{2}}\left(1 + \widehat{\gamma}_{i+1}\,\widehat{\gamma}_i\right).$$

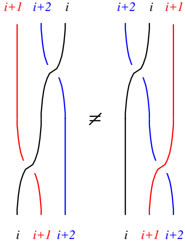
They realize a 2^n-dimensional representation of the braiding group B_{2n}, the group generated by the interchanges T_i with $i, j = 1, \cdots, 2n$ modulo the relations

$$T_i\,T_j = T_j\,T_i \text{ with } |i-j| > 1 \qquad T_i\,T_{i+1}\,T_i = T_{i+1}\,T_i\,T_{i+1}$$

$T_i\ T_j = T_j\ T_i$ if $|i-j|>1$



$T_i\ T_{i+1} \neq T_{i+1}\ T_i$



$T_i\ T_{i+1}\ T_i = T_{i+1}\ T_i\ T_{i+1}$