Strongly Disordered Floquet Topological Systems

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based on joint work with Clément Tauber
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Recent progress in mathematics of topological insulators

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Floquet systems

- Periodically time-dep. Hamiltonian $H : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ induces a unitary map $U : [0, 1] \to \mathcal{U}(\mathcal{H})$ via the Schrödinger equation $\mathrm{i}\,\dot{U} = HU$ with $U(0) \equiv \mathbb{1}$.

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- Long time dynamics of the system determined by $U(1)$ because $U(n + t) = U(1)^n U(t)$ for $t \in (0, 1)$, $n \in \mathbb{N}$.
- Main object however is U, not H, and all the questions (such as existence of a gap) are asked w.r.t. $U(1)$.

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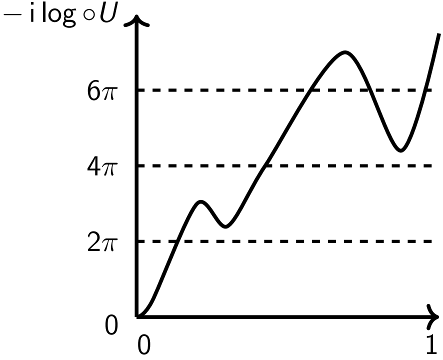

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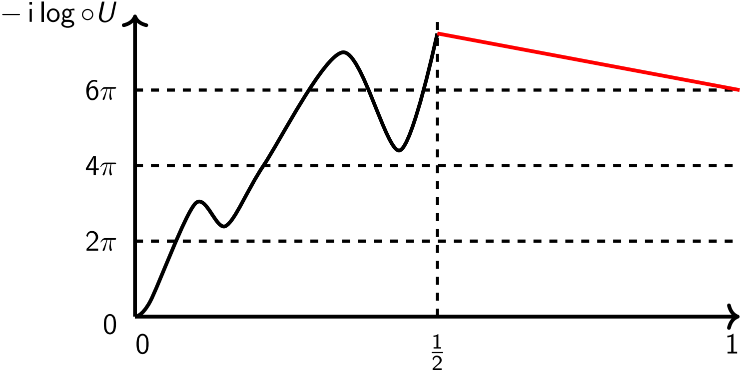

Higher dimensions

- In $d > 1$, $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ with N the internal levels; We ask that $H : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ be piecewise continuous in time and *local* in the sense that $\|\langle \delta_x, H(t)\delta_y \rangle\|$ is exp. decaying in $\|x - y\|$ (uniformly in $t \in \mathbb{S}^1$).

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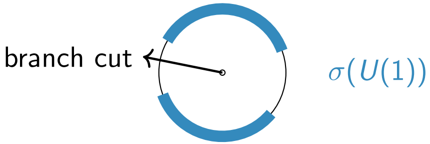

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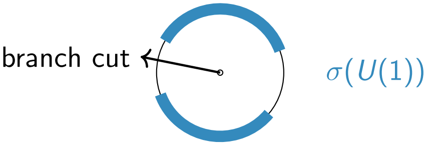

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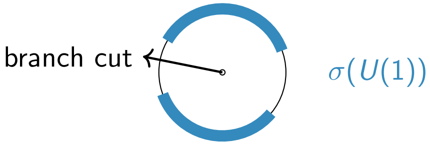

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- Gap condition is not related to insulator property (unlike static case)!

Higher dimensions (cont.)

- In transl. invar. case we get a cont. loop $U^{\mathrm{rel}} : \mathbb{S}^1 \times \mathbb{T}^d \to \mathcal{U}(N)$ based at $\mathbb{1}$, i.e. an element in suspension of C-star algebra $C(\mathbb{T}^d)$. Hence such unitary loops are classified by $K_1(SC(\mathbb{T}^d)) \cong K_0(C(\mathbb{T}^d))$; get same classification as static top. insulators of class A in d dim.

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- Can consider also other symmetry classes, but need to decide how symmetry operations should interact with time variable. Can Get analogous periodic table (see Roy, Harper (2017)).
- As in static case, $\exists$ bulk picture (on $\mathcal{H} \equiv \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$) and edge picture on half-space $\mathcal{H}_E := \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \otimes \mathbb{C}^N$ obtained by truncating a given bulk Hamiltonian with some B.C. (truncation always on H, not U!).

What we studied and previous results

We study the $2D$ no-symmetries case in the bulk and on the edge. The input is a bulk $H : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ (piecewise) cont. in time and local in space. It induces a bulk evolution $U : [0,1] \to \mathcal{U}(\mathcal{H})$ via Schödinger, an edge Hamiltonian $H_E : \mathbb{S}^1 \to \mathcal{H}_E$ (via truncation to half-space with Dirichlet) and an edge evolution $U_E : [0,1] \to \mathcal{U}(\mathcal{H}_E)$ via Schrödinger from H_E.

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- Define H_E^{rel} as the concatenation of H_E and the truncation of $-\mathrm{i}\log(U(1))$. Induces evol. $U_E^{\mathrm{rel}} : [0,1] \to \mathcal{U}(\mathcal{H}_E)$ (*not* a loop). Edge invar. is charge pumped along 1 direction after one period of U_E^{rel}: depends only on endpoint $U_E^{\mathrm{rel}}(1)$!

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4th result: equality

All invariants are equal, including bulk-edge correspondence. Uses continuity argument.

The mobility gap regime

- Via Combes-Thomas, $\mathbb{S}^1 \neq \sigma(U(1))$ implies that $\|h(U(1))_{xy}\|$ decays in $\|x - y\|$ for h holomorphic. This off-diagonal decay is apparently all we need for a well-defined topological phase.

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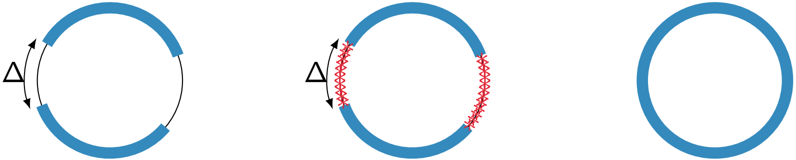
- Hamza, Joye, Stolz (2009) e.g. prove that certain random unitary ops. have dyn. loc. We assume the a.-s. results of loc. deterministically, i.e. we assume that $\exists \mu > 0$ s.t. for any $\varepsilon > 0$ $\exists C_\varepsilon < \infty$ with

$$\sup_{g \in B_1(\Delta)} \|g(U(1))_{xy}\| \leq C_\varepsilon \, \mathrm{e}^{-\mu\|x-y\|+\varepsilon\|x\|}$$

with $B_1(\Delta)$ the set of Borel bdd. maps $|g| \leq 1$ constant outside of $\Delta \subseteq \mathbb{S}^1$, which is called the mobility gap. Implies spectral localization in Δ via RAGE.

The mobility gap regime (cont.)



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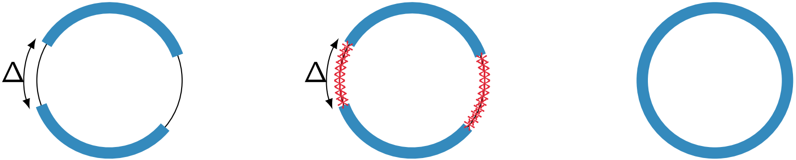

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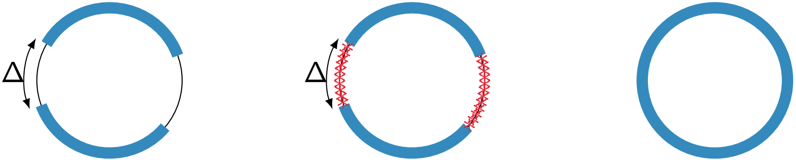

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Main point over [GT18]: Use loc. instead of Combes-Thomas to get (weak) locality of $\log(U(1))$; then generalize all notions from uniform decay in $\|x-y\|$ to allow possible explosion in $\|x\|$ simultaneously, which we call *weakly-local* operators:

$$\|A_{xy}\| \leq C_\varepsilon \, \mathrm{e}^{-\mu\|x-y\|+\varepsilon\|x\|}.$$

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  where $A_{,i} \equiv \mathrm{i}[\Lambda_i, A]$ with Λ_i a switch function. We have $W(U^{\mathrm{rel}}) = W(U) - W(\mathrm{e}^{\cdot \log_\lambda(U(1))})$, so that some winding of $\mathrm{e}^{\cdot \log_\lambda(U(1))}$ is removed, but what does it mean physically? (non-top. transport contributions?)

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- Edge invariant contains significant information from the bulk, namely, it depends on U^{rel}_E which is the evolution of H^{rel}_E, which is the concatenation of H_E and the truncation of $-\mathrm{i}\log(U(1))$. The latter is a bulk object. Want bulk-edge correspondence where bulk and edge invariants depend on H and H_E alone, without intertwining their evolutions during the proof.

The stretch function construction

- The stretch function (used by Sadel, Schulz-Baldes (2017) only for the edge in spec. gap case) smooth map $F_\Delta : \mathbb{C} \setminus \{ 0 \} \to \mathbb{C}$; restricted to $\mathbb{S}^1$: constant 1 outside Δ, has winding number 1.

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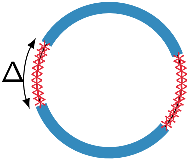

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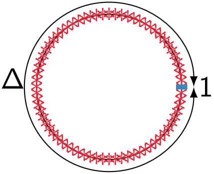

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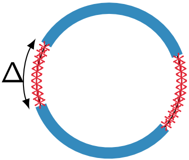

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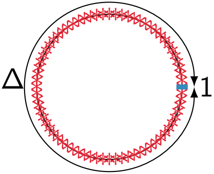

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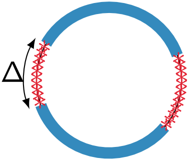

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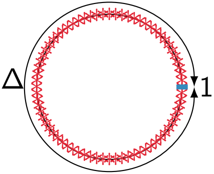

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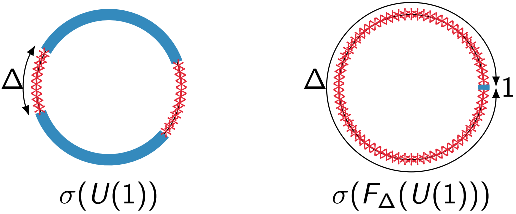

- $F_\Delta(U(1))$ is *dynamically localized* on $\mathbb{S}^1 \setminus \{ 1 \}$.
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- F_Δ chooses the gap for Floquet just like $\chi_{(-\infty, E_F)}$ chooses the gap for the IQHE, so F_Δ is like the Floquet's Fermi projection.

The completely localization case

- Let $V : [0,1] \to \mathcal{U}(\mathcal{H})$ be some bulk evolution s.t. $V(1)$ is completely localized, in the sense that it obeys a det. dyn. loc. estimate on $\mathbb{S}^1$ except some finitely many special points; we ask that the Chern $\sharp$ assoc. to each such point vanish.

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- Define the *bulk magnetization operator* $M(V) := \int_{[0,1]} \mathbb{Im}\, V^* \Lambda_1 \,\mathrm{i}\, \dot{V} V^* \Lambda_2 V$ and the total (orbital) magnetization $\mathcal{M}(V) := \int_{z\in\mathbb{S}^1} \operatorname{tr} M(V) \,\mathrm{d}\, P(z)$ with P the proj. valued spectral measure of $V(1)$. Related to magnetization studied by Rudner, Lindner et al (2017). If $\Lambda_i \sim x_i$ then like orbital angular momentum $\frac{1}{2}\mathbf{r}(t) \times \dot{\mathbf{r}}(t)$.

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- Define the *edge time-avg. charge pumping* assoc. to $V_E(1)$, the evolution of the truncated Hamiltonian assoc. to V: $\mathcal{P}_E(V_E(1)) := \lim_{n\to\infty} \lim_{r\to\infty} \frac{1}{n} \operatorname{tr}(V_E(1)^*)^n [\Lambda_1, V_E(1)^n] \Lambda_{2,r}^{\perp}$ where $\Lambda_{2,r}^{\perp}$ restricts to a vertical band from zero to r.

Connecting everything

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If $U : [0,1] \to \mathcal{U}(\mathcal{H})$ is s.t. $U(1)$ is completely loc. as above, then $\mathcal{M}(U) = W(U^{\mathrm{rel}})$.

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Theorem

If $U : [0,1] \to \mathcal{U}(\mathcal{H})$ has a mobility gap at Δ, and $U^{\mathrm{rel}} : \mathbb{S}^1 \to \mathcal{U}(\mathcal{H})$ is the rel. construction w.r.t. a cut in Δ then

$$W(U^{\mathrm{rel}}) = W((F_\Delta \circ U)^{\mathrm{rel}}) = \mathcal{M}(F_\Delta \circ U) = \mathcal{P}_E(F_\Delta(U_E(1)))\,.$$

Idea for proof

- We start with

$$W(U^{\text{rel}}) = W(U) - W(\mathrm{e}^{\cdot \log_\lambda(U(1))})$$

$$(\delta_\alpha := -\mathrm{i}\, U^* U_{,\alpha})$$

$$= \frac{1}{2} \operatorname{tr} \int_{[0,1]} \varepsilon_{\alpha\beta}(\delta_\alpha \dot{\delta}_\beta - \delta_\alpha^\lambda \dot{\delta}_\beta^\lambda)$$

$$(U_{,\alpha} \equiv \mathrm{i}[\Lambda_\alpha, U] \wedge \delta_\alpha(t) = \delta_\alpha^\lambda(t) \forall t \in \{ 0, 1 \})$$

$$= \operatorname{tr} M(U) - M(\mathrm{e}^{\cdot \log_\lambda(U(1))})$$

Now use localization to prove (the regularized) trace of $M(\mathrm{e}^{\cdot \log(U(1))})$ is finite and actually zero.

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$$\begin{aligned} W(U^{\mathrm{rel}}) &= W(U) - W(\mathrm{e}^{\cdot \log_\lambda(U(1))}) \\ &\quad (\delta_\alpha := -\mathrm{i}\, U^* U_{,\alpha}) \\ &= \frac{1}{2} \mathrm{tr} \int_{[0,1]} \varepsilon_{\alpha\beta} (\delta_\alpha \dot{\delta}_\beta - \delta_\alpha^\lambda \dot{\delta}_\beta^\lambda) \\ &\quad (U_{,\alpha} \equiv \mathrm{i}[\Lambda_\alpha, U] \wedge \delta_\alpha(t) = \delta_\alpha^\lambda(t) \forall t \in \{0,1\}) \\ &= \mathrm{tr}\, M(U) - M(\mathrm{e}^{\cdot \log_\lambda(U(1))}) \end{aligned}$$

  Now use localization to prove (the regularized) trace of $M(\mathrm{e}^{\cdot \log(U(1))})$ is finite and actually zero.

- For $W(U^{\mathrm{rel}}) = W((F_\Delta \circ U)^{\mathrm{rel}})$ we use continuity of W under interpolation from the smooth F_Δ to the identity map, *in the mobility gap regime*.