Duality Methods for Topological Phases

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Recent Progress in Mathematics of Topological Insulators

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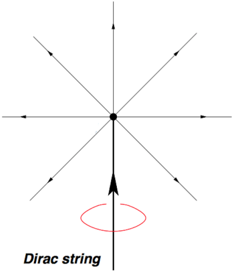
Outline

- I will discuss how Poincaré duality and T-duality can be used to understand topological phases in new ways.
- Bulk-boundary correspondence is an "index-theoretic" idea — that boundary zero modes (analysis) detect bulk topology.
- So it is natural to Poincaré dualise, and this even simplifies understanding of topological semimetals and Kane–Mele invariants by passing to Dirac-stringy picture.
- A lattice $\mathbb{Z}^d$ gives (A) unit cell and (B) Brillouin zone. These are T-dual d-tori. T-duality "mixes but preserves topology", e.g. exchanges index maps with geometric restriction maps.
- New notions of crystallographic T-duality and bulk-boundary correspondence allow new index theorems to be deduced.

Magnetic monopole and the Hopf bundle

On $\mathbb{R}^3 \setminus \{0\}$, there is a non-trivial $\mathrm{U}(1)$ bundle $\mathcal{L}_{\mathrm{Hopf}}$: $H^2(\mathbb{R}^3 \setminus \{0\}) \cong H^2(S^2) \cong \mathbb{Z}$.

Magnetic field (curvature 2-form $\mathcal{F}$) has no global vector potential $\mathcal{A}$. Chern number $\int_{S^2} \mathcal{F} \in \mathbb{Z} \leftrightarrow$ monopole charge.



Dirac string is the Poincaré dual description:

$$H^2(\mathbb{R}^3 \setminus \{0\}) \cong H^2(S^3 \setminus \{0, \infty\}) \overset{PD}{\cong} H_1(S^3, \{0, \infty\}).$$

Roughly: 1-submanifold (H_1) Poincaré $\xleftrightarrow{PD}$ $d-1$ form (H^{d-1}).

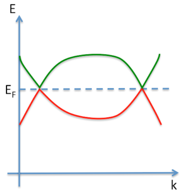
Chern pairing $\int_{S^2}$ with $\mathcal{F} \overset{PD}{\leftrightarrow}$ intersection pairing Dirac string $\#$ S^2.

2-band crossings and monopoles

Unit 3-vector $\widehat{\boldsymbol{x}} \in S^2 \cong \mathbb{CP}^1$ via the -1 eigenspace of spin operator $\widehat{\boldsymbol{x}} \cdot \boldsymbol{\sigma}$. Over S^2, these eigenspaces assemble into $\mathcal{L}_{\mathrm{Hopf}} \to S^2$.

A 3-vector field $\boldsymbol{h}$ specifies a family of 2×2 Hamiltonians:

$$H(k) = \boldsymbol{h}(k) \cdot \boldsymbol{\sigma}, \qquad k \in T.$$



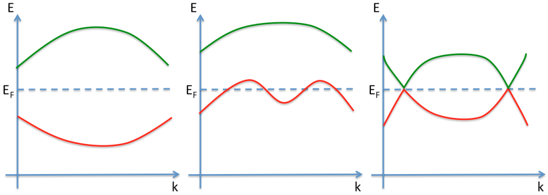
Spectrum of $H(k)$ is $\pm|\boldsymbol{h}(k)|$, so bands cross at zero set W of $\boldsymbol{h}$, generically a set of Weyl points in 3D. For $k \in T \setminus W$, Negative eigenspace of $H(k)$ is just unit vector $\widehat{\boldsymbol{h}}(k)$ thought of as a $\mathbb{CP}^1$ element.

Valence line bundle $\mathcal{E} \to T \setminus W$ is just the pullback

$$\mathcal{E} = \widehat{\boldsymbol{h}}^*(\mathcal{L}_{\mathrm{Hopf}}), \qquad \widehat{\boldsymbol{h}} : T \setminus W \to S^2 \ \text{(classifying map)}$$

Toy model of topological insulator and semimetal

2D Chern insulator: $T = \mathbb{T}^2, W = \emptyset$ (gap condition).
$\deg(\widehat{\boldsymbol{h}} : \mathbb{T}^2 \to S^2) = c_1(\mathcal{E}) \in H^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}$ gives Chern number.



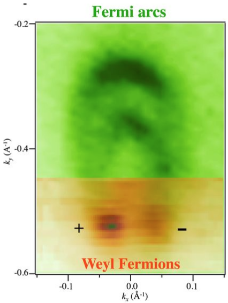
3D Weyl semimetal (WSM): $T = \mathbb{T}^3$, W a finite set. For each $w \in W$, take a local enclosing sphere S^2_w. Local obstruction to opening a gap at w is $\mathrm{Ind}_{\boldsymbol{h}}(w) := \deg(\widehat{\boldsymbol{h}}|_{S^2_w}) = c_1(\mathcal{E}|_{S^2_w})$.

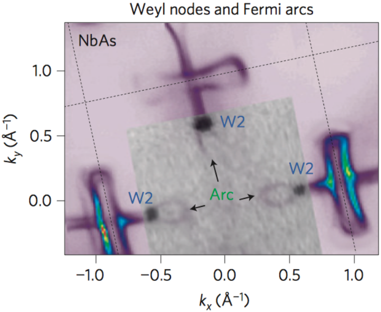
Poincaré–Hopf theorem imposes global constraint:

$$\sum_{w \in W} \mathrm{Ind}_{\boldsymbol{h}}(w) = \chi(\mathbb{T}^3) = 0, \quad \forall \text{ vector fields } \boldsymbol{h} \text{ over } \mathbb{T}^3.$$

So Weyl points occur in cancelling pairs (cf. Nielsen–Ninomiya).

Weyl semimetal

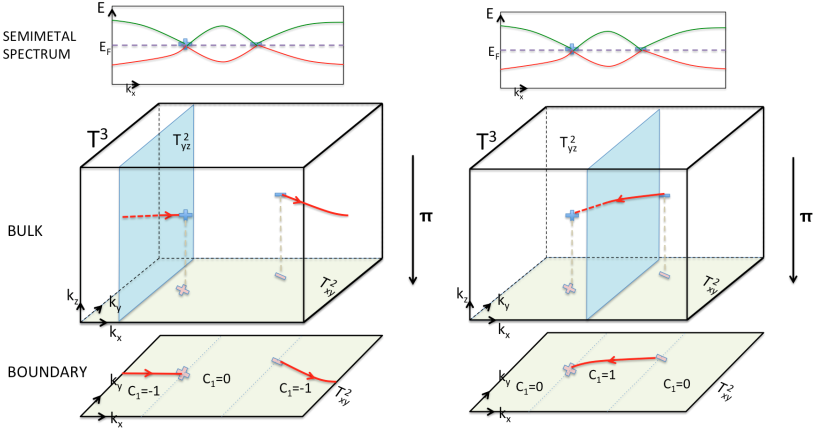




(L) S.-Y. Xu et al, Discovery of a Weyl Fermion semimetal and topological Fermi arcs, Science **349** 613 (2015);
(R) [—] Discovery of a Weyl fermion state with Fermi arcs in niobium arsenide, Nature Phys. **11** 748 (2015).

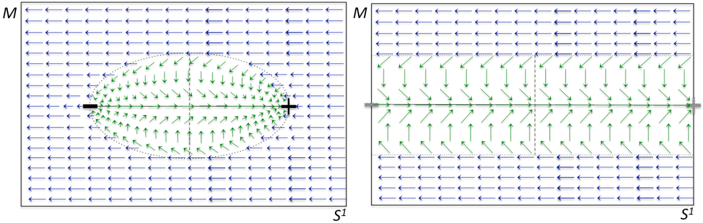
Dirac string is "invisible", but there must be one. In solid state physics, 3D Weyl semimetals are characterised by bulk Dirac strings, which are "holographically" detected on a boundary.

Surface Fermi arcs detect global WSM topology



[L] and [R] have topologically distinct Dirac strings in “dual picture”. In “Berry curvature picture”, their valence bundles have different distributions of Chern numbers on 2D slices. Boundary state appears for slices with nonzero Chern number $\leftrightarrow$ Dirac string intersects the slice. Fermi arc is projected Dirac string.

Dirac string indicates "topological phase transition"

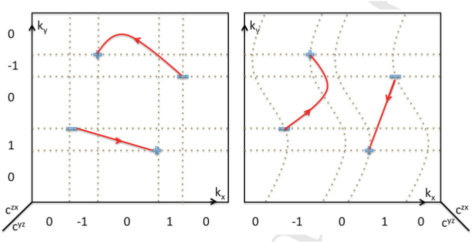


Create a $\pm$ pair locally, stretch Dirac string around a non-trivial cycle and annihilate $\pm$. This produces a transition from trivial insulator to weak Chern insulator, recorded by residual Dirac string (a loop).

"Singular homotopy" classes of nonsingular vector fields on T are classified by $H_1(T)$, i.e. Dirac strings. These were called *Euler structures* by Turaev '89.

Differential topology of semimetals [Mathai+T, CMP '17]

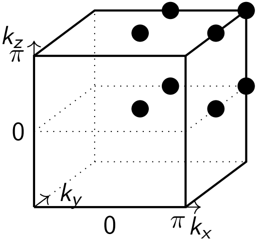
$$
\begin{tikzcd}[column sep=large, row sep=huge]
\cdots \to 0 \arrow[r] & \overbrace{H^2(\mathbb{T}^3)}^{\substack{\text{TI}\\ \text{Chern}\\ \text{classes}}} \arrow[r, "\text{restrict}", "\text{bands}"'] \arrow[d, "\mathsf{PD}"'] & \overbrace{H^2(\mathbb{T}^3 \setminus W)}^{\substack{\text{WSM}\\ \text{invariants}}} \arrow[r, "\text{restrict}", "\text{bands}"'] \arrow[d, "\mathsf{PD}"'] & \overbrace{H^2(S^2_W)}^{\substack{\text{Weyl pt}\\ \text{charges}}} \arrow[r, "\text{sum}"] \arrow[d, "\mathsf{PD}"'] & \overbrace{H^3(\mathbb{T}^3)}^{\substack{\text{total}\\ \text{charge}}} \arrow[r] \arrow[d, "\mathsf{PD}"'] & 0 \to \cdots \\
\cdots \to 0 \arrow[r] & \underbrace{H_1(\mathbb{T}^3)}_{\substack{\text{closed}\\ \text{Dirac strings}}} \arrow[r] & \underbrace{H_1(\mathbb{T}^3, W)}_{\text{Dirac strings}} \arrow[r, "\partial"] & \underbrace{H_0(W)}_{\substack{\text{Weyl pt}\\ \text{charges}}} \arrow[r] & \underbrace{H_0(\mathbb{T}^3)}_{\substack{\text{total}\\ \text{charge}}} \arrow[r] & 0 \to \cdots
\end{tikzcd}
$$



Dirac strings keep track of Weyl point "history". Projection onto Fermi arcs is Poincaré dual to "integrating out transverse momenta" (a Gysin map), which is also a slice-wise analytic index.

Time-reversal

Actually nature is more subtle — good experimental examples of TI and WSM have *time-reversal symmetry* Θ with $\Theta^2 = -1$.



Time-reversal also implements momentum reversal $\theta : k \mapsto -k$. If $\mathbb{T}$ is unit complex numbers e^{ik} with complex conjugation fixing $k = 0, \pi$, then $\mathbb{T}^d$ has 2^d fixed points.

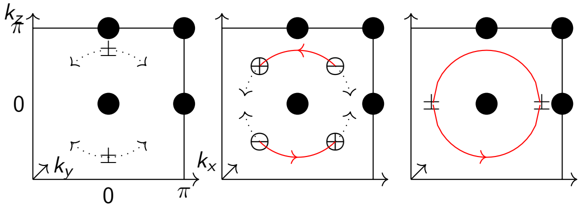
Fu–Kane–Mele used "Berry curvature picture" to derive three weak $\mathbb{Z}_2$ invariants ν_i and one strong $\mathbb{Z}_2$ invariant ν_0 in 3D.

There is an easy derivation using θ-symmetric Dirac strings, which furthermore clarifies the "phase transitions"!

θ-symmetric Dirac strings

With Θ symmetry, a pair w_+, w_- must have a partner pair at $\theta(w_+), \theta(w_-)$.

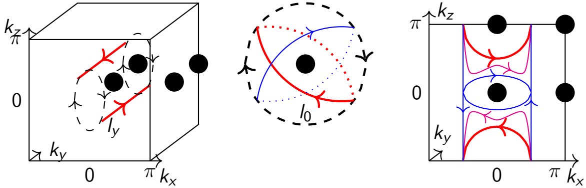
A strong FKM invariant is generated by circular θ-symmetric Weyl point creation-annihilation process [Halasz–Balents '12, PRB].



This suggests a *homology* classification of 3D TI, in terms of closed "θ-symmetric Dirac strings" avoiding the fixed points.

θ symmetric Dirac strings

Here are some θ-symmetric Dirac strings.



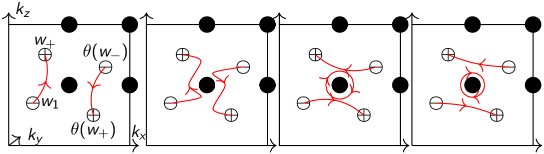
Can θ-symmetrically rotate l_x, l_y, l_z, and l_0 onto their oppositely-oriented versions $\Rightarrow$ 2-torsion cycles! These are the only independent generators since, e.g. $l_{(0,0,0)} + l_z = l_{(0,0,\pi)}$.

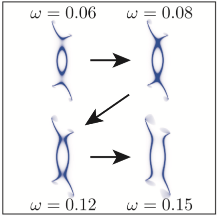
Technically, $\mathbb{Z}_2^4 \cong H_1(\mathbb{T}^3 \setminus F) \overset{PD}{\longleftrightarrow} H^2_{\mathbb{Z}_2}(\mathbb{T}^3, F; \mathbb{Z}(1))$

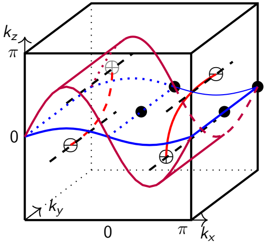
RHS is the cohomological meaning of FKM invariants [De Nittis–Gomi '16 CMP].

θ-symmetric Dirac strings and $\mathbb{Z}_2$-monopoles

Fermi arcs and Dirac cones (ν_0) can transmute!



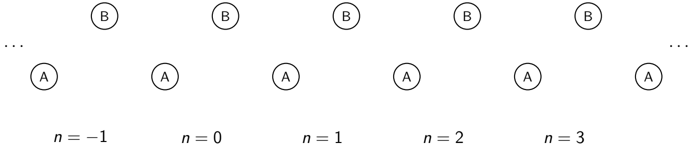




Trap $w_+, \theta(w_+)$ between two θ-symmetric 2-tori (purple/blue). Dirac string only pierces blue torus: $\nu_{\text{blue}} = -\nu_{\text{purple}}$.

Weyl points are "$\mathbb{Z}_2$-FKM monopoles" [T+Sato+Gomi, Nucl.Phys.B '17].

Topological phase in Su–Schrieffer–Heeger model



$\mathbb{Z}$-translations and sublattice operator $\mathsf{S} = 1_A \oplus -1_B$.

A chiral/super-symmetric Hamiltonian $H = H^\dagger$ commutes with $\mathbb{Z}$, but $H\mathsf{S} = -\mathsf{S}H$. So H exchanges $A \leftrightarrow B$.

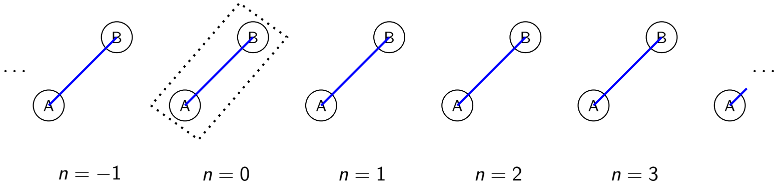
After Fourier transform to $L^2(S^1) \oplus L^2(S^1)$,

$$H\mathsf{S} = -\mathsf{S}H \quad \Longleftrightarrow \quad H(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}, \quad U(k) \in \mathbb{C}$$

"Gap condition": $0 \notin \mathrm{spec}(H) \Leftrightarrow U(k) \in \mathbb{C}^*$.

SSH model

$\mathrm{Wind}(U : S^1 \to \mathbb{C}^*)$ distinguishes topological phases of gapped, $\mathbb{Z}$-invariant, supersymmetric 1D Hamiltonians!



Intracell $H_{\mathrm{blue}}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has winding number 0.

Intercell[1] $H_{\mathrm{red}}(k) = \begin{pmatrix} 0 & e^{\mathrm{i}k} \\ e^{-\mathrm{i}k} & 0 \end{pmatrix}$ has winding number 1.

[1] Recall that translation becomes multiplication by $e^{\mathrm{i}k}$ under Fourier.

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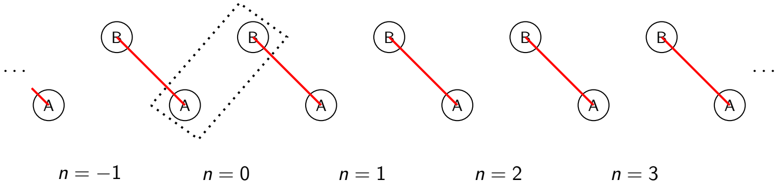

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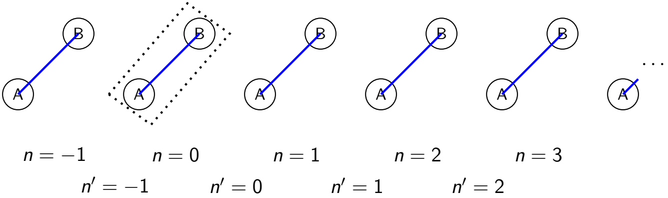
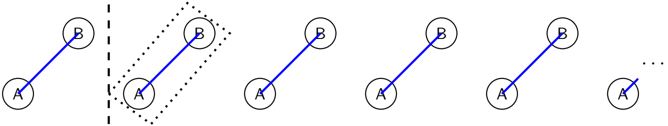
SSH model

Puzzle: $H_{\mathrm{blue}} \sim_{\mathrm{unitary}} H_{\mathrm{red}}$, so how can Wind($U$) be seen??



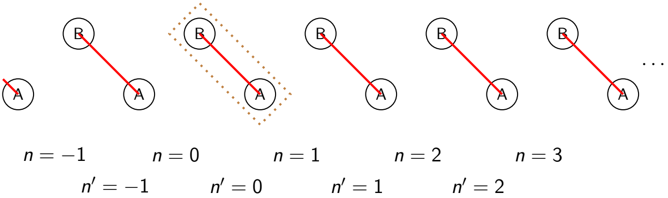
Fourier trans. required origin choice for each sublattice $\mathbb{Z}$.
A boundary "fixes the gauge", and also cuts a red link.



The boundary "detects" the winding invariant of H_{red} analytically as a "dangling zero A-mode"!

SSH model

Puzzle: $H_{\text{blue}} \sim_{\text{unitary}} H_{\text{red}}$, so how can Wind$(U)$ be seen??



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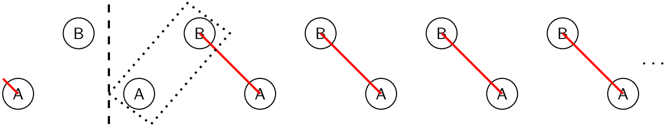

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SSH model and Toeplitz operators

For general Hamiltonian, $\#A - \#B$ zero modes is topological because of an index theorem!

Truncation to $n \geq 0 \Leftrightarrow$ restrict from $L^2(S^1)$ to *Hardy space* $\mathcal{H}^2$. Symbol U is quantised to Toeplitz operator T_U.

$$\begin{tikzcd} \mathcal{H}^2 \arrow[r, "\iota"] \arrow[rrr, bend right, "T_U"'] & L^2(S^1) \arrow[r, "U"] & L^2(S^1) \arrow[r, "p"] & \mathcal{H}^2. \end{tikzcd}$$

Göhberg–Krein index theorem: T_U is Fredholm iff U is invertible in $C(S^1)$, and $\mathrm{Ind}(T_U) = -\mathrm{Wind}(U)$.

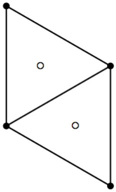
$\#B - \#A \equiv \mathrm{Ind}(T_U) = \int_{S^1} \mathrm{ch}(U) \equiv \mathrm{Wind}(U)$.

K-theoretic index and bulk-boundary correspondence

K-theory: $[U] \in K^{-1}(S^1) \cong \mathbb{Z}$, has an index pairing with K-homology of S^1, via the K-theory connecting map ∂ for Toeplitz algebra extension of $C(S^1)$, which is actually a topological Gysin/integration map:

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0.$$

$$\partial : K^{-1}(S^1) \xrightarrow{\sim} K^0(\star) = \mathbb{Z}$$



Toeplitz extensions contain half-space operators and capture a very specific type of geometric bulk-boundary relation. Other bulk-boundary geometries are possible.

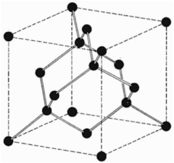
Expect a dependence of bulk-edge correspondence (analytic zero modes) on the geometrical bulk-edge relation.

Crystallographic groups

A crystallographic space group $\mathscr{G}$ is a discrete cocompact subgroup of isometries of affine Euclidean space R^d.

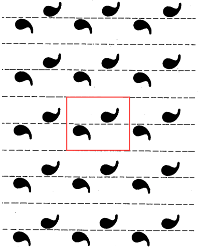
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^d & \longrightarrow & \mathrm{Euc}(d) & \longrightarrow & \mathrm{O}(d) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{Z}^d & \longrightarrow & \mathscr{G} & \longrightarrow & F & \longrightarrow & 1 \end{array}$$

$\mathscr{G}$ is an extension of finite point group F by lattice subgroup $\mathbb{Z}^d$.



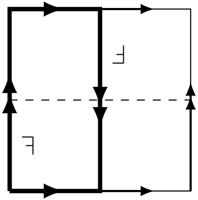
Classification of $\mathscr{G}$-symmetric Hamiltonians $\leftrightarrow$ *twisted* F-equivariant K-theory of Brillouin torus [Freed–Moore '13, T'16, AHP].

Glide reflections, pg, Klein bottle



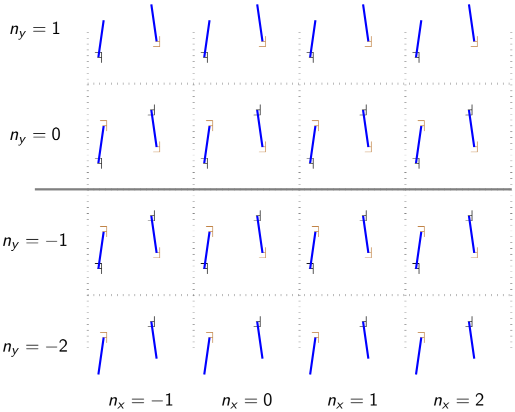
$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \text{pg} \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

$\mathbb{Z}_2$ lifts to *glide reflection* of infinite order. Fundamental domain is a Klein bottle.



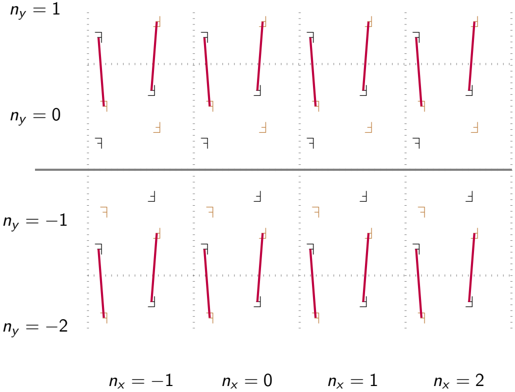
K-theory calculation predicts a $\mathbb{Z}_2$ chiral-pg-symmetric "Klein bottle" phase. How to detect this?

Trivial pg-symmetric phase



H_{blue} has no zero modes when cut along a glide axis.

The $\mathbb{Z}/2$ "Klein bottle" phase



H_{purple} has "glide" zero modes when cut along a glide axis.
Q: Why are zero modes 2-torsion?

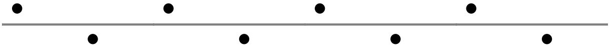
Mod 2 Super-index theorem [Gomi+T, 1804.03945]

H_{purple} is indeed non-trivial in $K^{1+\tau}_{\mathbb{Z}_2}(\mathbb{T}^2)$, and detected by

(1) topological Gysin map $K^{1+\tau}_{\mathbb{Z}_2}(\mathbb{T}^2) \to K^{0+\tau+c}_{\mathbb{Z}_2}(\mathbb{T}_x) \cong \mathbb{Z}/2$.

(2) analytic $\mathbb{Z}/2$ index for a "twisted Toeplitz family" over $\mathbb{T}_x$.

Zero modes have glide symmetry, i.e. frieze group p11g $\cong \mathbb{Z}$.



Not pure 1D: glide reflection reverses "upper/lower".

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \text{p11g} \cong \mathbb{Z} \xrightarrow{c = \mathrm{mod}\, 2} \mathbb{Z}_2 \longrightarrow 1.$$

Zero mode space is $\mathbb{Z}_2$-graded into "upper/lower" subspaces, and has super-representation p11g. So our result is a super-index theorem.

Crystallographic T-duality [Gomi+T 1806.11385]

A lattice $\mathbb{Z}^d$ naturally provides two *different* d-tori:
(1) Position space $T^d = R^d/\mathbb{Z}^d$; (2) Momentum space $\mathbb{T}^d = \widehat{\mathbb{Z}}^d$.

T^d (unit cell) and $\mathbb{T}^d$ (Brillouin torus) are T-dual. Topological invariants of one are mapped bijectively (but not identically) onto those of the other ("topological Fourier transform").

For any crystallographic $\mathscr{G}$, the point group F acts affinely on T^d. In fact, space group $\leftrightarrow$ affine torus action!

Dually, F acts on $\mathbb{T}^d$ with a possible twist nonsymmorphicity.

Theorem: There is a zoo of "crystallographic T-dualities"

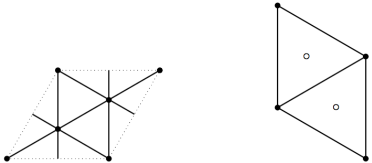
$$K_F^{d-\bullet+c'}(T^d_{\mathrm{affine}}) \cong K_F^{-\bullet+\tau+c}(\mathbb{T}^d_{\mathrm{dual}}).$$

Technical subtlety: *graded* twists are needed (physics gave a clue).

Crystallographic T-duality [Gomi+T 1806.11385]

$$K_F^{d-\bullet+c'}(T^d_{\mathrm{affine}}) \cong K_F^{-\bullet+\tau+c}(\mathbb{T}^d_{\mathrm{dual}}).$$

Can be formulated as a Fourier–Mukai transform, or using Poincaré duality and a super-Baum–Connes assembly map.



(a) p3m1 (b) p31m

Application: Topological phases for p3m1 are dual to those for p31m. Similarly for FCC $\leftrightarrow$ BCC (many non-self-dual pairs in 3D!).

Application: AHSS computations of K^1 has extension problems. Simply inspect the (known) K^0 on the T-dual side!

T-duality and bulk-boundary correspondence

Momentum space analysis of bulk-boundary correspondence should be describing something geometrically obvious in position space.

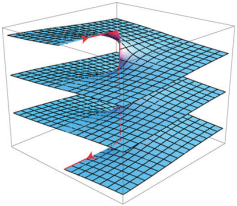
This does not say that the bulk/boundary topological invariants are themselves easy to "picture" in position space.

Rather, the bulk-to-boundary index transfer *map* in momentum space "has to be" a geometrically natural *map* of the corresponding position space invariants.

Exactly parallel to Fourier transforms — translational invariance makes things look easy in momentum space, but integrating (momenta) is generally hard. Yet we know $\int_{S^1}$ simply effects restriction-to-zeroth-Fourier-coefficients (in position space).

T-duality and bulk-boundary correspondence [Mathai+T '16, CMP]

$$\begin{tikzcd}
\boxed{\text{Position space bulk invariant}} \arrow[r, "\sim", "\mathrm{T-duality}"'] \arrow[d, "\text{Restriction to boundary}"'] & \boxed{\text{Momentum space bulk invariant}} \arrow[d, "\text{bulk-boundary transfer homomorphism}"] \\
\boxed{\text{Position space boundary invariant}} \arrow[r, "\sim", "\mathrm{T-duality}"'] & \boxed{\text{Momentum space boundary invariant}}
\end{tikzcd}$$

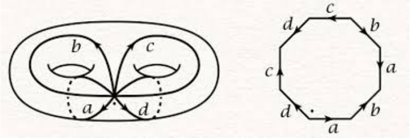


This allows us to reason about bulk-boundary correspondence even if "momentum space" in the naïve sense is unavailable, because T-duality still makes sense!

[Hannabuss-Mathai+T '16, ATMP]

Fractional bulk-boundary correspondence [Mathai+T, 1712.02952]

In hyperbolic plane $\mathbb{H}$, there is a notion of "space group" $\Gamma_{g,\boldsymbol{\nu}}$ with torsion-free "translation lattice" Γ_g. "Unit cell" is genus g Riemann surface $\mathbb{H}/\Gamma_g$.



$\mathbb{H}/\Gamma_{g,\boldsymbol{\nu}}$ has *fractional* orbifold Euler characteristic

$$\phi = 2(g-1) + \sum_{j=1}^{r}(1 - \frac{1}{\nu_j}) \in \mathbb{Q}.$$

Analogue of Chern numbers of valence line bundles are fractional. Hard to write down half-plane tight-binding model to formulate bulk-boundary correspondence. But geometrically easy to describe, and implicitly defines "momentum space" fractional bulk-boundary index via "Riemann surface T-duality".