Spectral gap-labelling conjecture for magnetic Schrödinger operators and recent progress

Recent progress in mathematics of topological insulators

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Collaborators and references

joint work with:

- Moulay Tahar Benameur (University of Montpellier, France);

[BM15]

M-T. Benameur and V. M.,

Gap-labelling conjecture with non-zero magnetic field.

Advances in Mathematics, **325**, (2018) 116–164.

[BM18]

M-T. Benameur and V. M.,

Proof of the magnetic gap-labelling conjecture for principal solenoidal tori.

9 pages, **[1806.06302]**.

Outline of talk

1. Magnetic Schrödinger operators
2. Motivations for the magnetic gap-labelling conjecture
3. Magnetic gap-labelling conjecture
4. What is known about the conjecture.
5. Solonoidal tori and principal solenoidal tori.
6. Recent progress in higher dimensions

Magnetic Schrodinger operators

- Consider Euclidean space $\mathbb{R}^d$ equipped with its usual Riemannian metric $\sum_{j=1}^{d} dx_j^2$.

- the magnetic field $B = \frac{1}{2}\sum_{j,k} \Theta_{jk} dx_j \wedge dx_k = \frac{1}{2} dx^t \Theta dx$, where Θ is a constant $(d \times d)$ skew-symmetric matrix, dx is the column vector with entries dx_j and dx^t is the corresponding row vector, and matrix multiplication is used. B is closed.

-Let us now pick a 1-form η such that $d\eta = B$. This is always possible since B is a closed 2-form and $\mathbb{R}^d$ is contractible. We may regard η as defining a connection $\nabla = d + i\eta$ on the trivial line bundle $\mathcal{L}$ over $\mathbb{R}^d$, whose curvature is iB. Physically we can think of η as the electromagnetic vector potential for a uniform magnetic field B normal to $\mathbb{R}^d$.

Magnetic Schrodinger operators

Using the Riemannian metric the Hamiltonian of an electron in this magnetic field is given by

$$H = \frac{1}{2}\nabla^\dagger\nabla + V = \frac{1}{2}(d + i\eta)^\dagger(d + i\eta) + V, \quad \text{acting on } L^2(\mathbb{R}^d)$$

where † denotes the adjoint and V is a smooth real-valued bounded function. H is (formally) self-adjoint & bounded below.

The restriction of H to a bounded domain Ω (with piecewise smooth boundary $\partial\Omega$) in $\mathbb{R}^d$ is denoted by H_Ω.
Imposing self-adjoint (Dirichlet/Neumann...) boundary conditions, then since H_Ω becomes a self-adjoint elliptic operator, it has an (unbounded) purely discrete real spectrum, but which is bounded below.
Moreover the eigenvalues all have finite multiplicity.

Magnetic Schrodinger operators

Define the (spectral) counting function

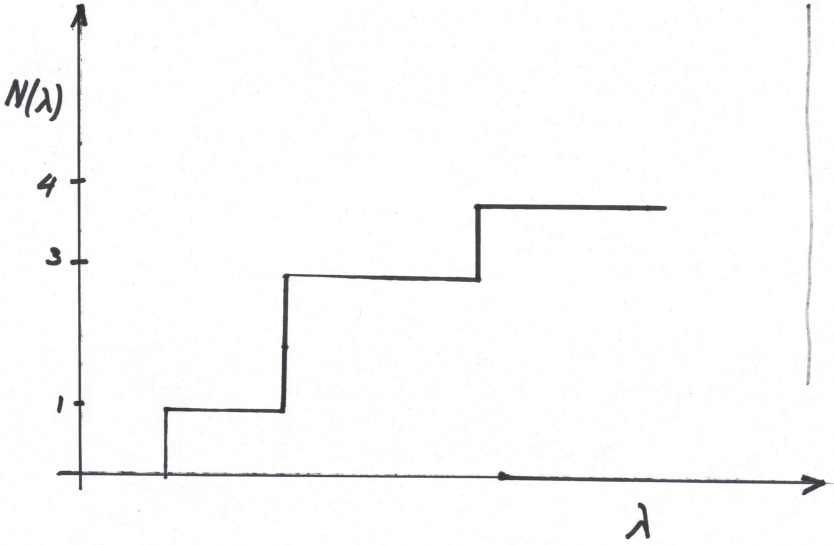
$$N(H_\Omega, \lambda) = \#\{\mu \in \mathrm{spec}(H_\Omega) : \mu \le \lambda\}$$
$$= \mathrm{Tr}(\chi_{(-\infty,\lambda]}(H_\Omega))$$

This is a step function, and the values $N(H_\Omega, E)$ is a **gap-label** whenever E lies in a spectral gap, i.e. for all $E \in \mathbb{R} \setminus \mathrm{spec}(H_\Omega)$.

Properties of the (spectral) counting function $N(H_\Omega, \lambda)$

1. $N(H_\Omega, \lambda)$ is non-decreasing;
2. $N(H_\Omega, \lambda) = 0$ for all $\lambda < \inf \mathrm{spec}(H_\Omega)$;
3. $N(H_\Omega, \lambda) \sim \lambda^{d/2}$ as $\lambda \to \infty$ (Weyl law);
4. $N(H_\Omega, \lambda)$ is constant on spectral gaps.

Integrated Density of States



Magnetic Schrodinger operators

What we would like to have is a counting function for the operator H on $\mathbb{R}^d$. This is trickier to define, as H contains continuous spectrum in general.

Define the **integrated density of states (IDS)**

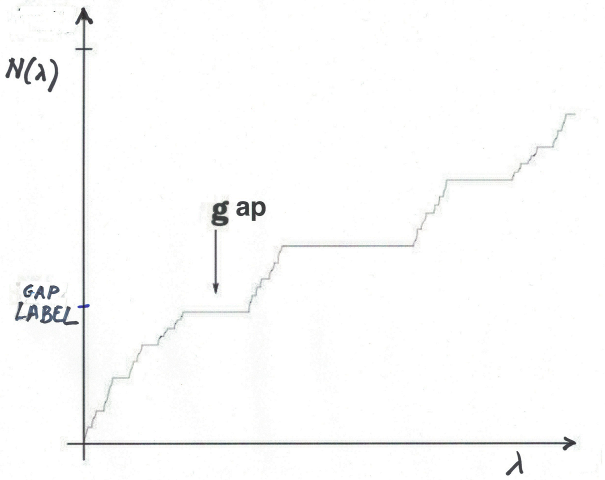
$$N(H, \lambda) = \lim_{\Omega \uparrow \mathbb{R}^d} \frac{1}{\mathrm{vol}(\Omega)} N(H_\Omega, \lambda)$$

(which exists because $\mathbb{R}^d$ is amenable, and such a sequence of open sets $\{\Omega\}$ is a **Folner sequence**.)

Properties of the IDS $N(H, \lambda)$

1. $N(H, \lambda)$ is non-decreasing;
2. $N(H, \lambda) = 0$ for all $\lambda < \inf \mathrm{spec}(H)$;
3. $N(H, \lambda) \sim \lambda^{d/2}$ as $\lambda \to \infty$ (Weyl law);
4. $N(H, \lambda)$ is constant on spectral gaps.

Integrated Density of States



NB Gap labels are no longer integers and more interesting!

Some C^*-algebras

Difficult to work with this definition of IDS - seek an alternatives.

Let $U(\gamma)$ denote the unitary operator on $L^2(\mathbb{R}^d)$ given by translation by $\gamma \in \mathbb{Z}^d$. Consider the set consisting of translates of the resolvent operator:

$$\Sigma_0(z) = \left\{ U(\gamma)(H - zI)^{-1} U(\gamma)^{-1} : \gamma \in \mathbb{Z}^d \right\}$$

& assume that it is precompact for some $z \in \mathbb{C}$ with $\Im(z) \neq 0$.

This is the case for any $V \in L^\infty(\mathbb{R}^d, \mathbb{R}) \cap C^\infty(\mathbb{R}^d)$.
In the case when V is smooth and periodic, then this set is a point.

Some C^*-algebras

Let $\Sigma = \overline{\Sigma_0(z)}^s$ be the compact set that is the strong closure. It turns out to be independent of z with $\Im(z) \neq 0$, modulo homeomorphism. It is called the **disorder set** associated to H, or the **hull**. $\mathbb{Z}^d$ acts on Σ by homeomorphisms.

The most interesting case is when Σ is a **Cantor set**. and $\mathbb{Z}^d$ acts on Σ minimally (i.e. having dense orbit)

Let μ be a $\mathbb{Z}^d$-invariant probability Borel measure on Σ.

Twisted crossed product algebra

Let $\sigma : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow U(1)$ be a 2-cocycle on $\mathbb{Z}^d$, so it satisfies,

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1 + \gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2 + \gamma_3)\sigma(\gamma_2, \gamma_3), \quad \gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}^d$$

Then the twisted crossed product C^*-algebra $\mathcal{A} = C(\Sigma) \rtimes_\sigma \mathbb{Z}^d$ is constructed as follows. Let $a, b \in \mathcal{A}_0 = C_c(\Sigma \times \mathbb{Z}^d)$.

- Product:
  $ab(\omega, \gamma) = \sum_{\gamma' \in \mathbb{Z}^d} a(\omega, \gamma') b(\gamma'^{-1}\omega, \gamma - \gamma')\sigma(\gamma - \gamma', \gamma')$;
- The adjoint: $a^*(\omega, \gamma) = \overline{a(\gamma^{-1}\omega, -\gamma)}\sigma(-\gamma, \gamma)$;
- The regular representation for $a \in \mathcal{A}_0$ and $\psi \in L^2(\mathbb{Z}^d)$:
  $\pi_\omega(a)\psi(\gamma) = \sum_{\gamma' \in \mathbb{Z}^d} a(\gamma'^{-1}\omega, \gamma - \gamma')\psi(\gamma')\sigma(\gamma - \gamma', \gamma')$;
- The norm: $||a|| = \sup_{\omega \in \Sigma} ||\pi_\omega(a)||$;
- The twisted crossed product C^*-algebra: $\mathcal{A} = \overline{\mathcal{A}_0}^{||\cdot||}$

The trace functional

Let $\mu : C(\Sigma) \longrightarrow \mathbb{C}$ be an invariant measure on Σ.
Then it induces a trace τ_μ on the twisted crossed product C^*-algebra $\mathcal{A} = C(\Sigma) \rtimes_\sigma \mathbb{Z}^d$ as follows: define for $a \in \mathcal{A}$,

$$\tau_\mu(a) = \int_\Sigma a(\omega, 0) d\mu(\omega).$$

Then for $a, b \in \mathcal{A}$, one has

$$\tau_\mu(ab) = \tau_\mu(ba),$$

and if $a \geq 0$,

$$\tau_\mu(a) \geq 0.$$

The trace actually extends to the bigger von Neumann algebra, $L^\infty(\Sigma) \rtimes_\sigma \mathbb{Z}^d$, using the same formula.

Magnetic gap-labelling conjecture

Now the spectral projections of H are bounded measurable functions of H, therefore $\chi_{(-\infty,\lambda]}(H) \in L^\infty(\Sigma) \rtimes_\sigma \mathbb{Z}^d \otimes \mathcal{K}$, the von Neumann algebra. Then one has the useful

Theorem (Shubin)

IDS has the following expression, at a point of continuity:

$$N(H,\lambda) = \tau_\mu(\chi_{(-\infty,\lambda]}(H)), \qquad \lambda \in \mathbb{R}$$

When λ is in a **spectral gap** of H, i.e. $\lambda \in \mathbb{R} \setminus \mathrm{spec}(H)$, then the projection $\chi_{(-\infty,\lambda]}(H)$ is the smaller algebra $C(\Sigma) \rtimes_\sigma \mathbb{Z}^d \otimes \mathcal{K}$,

Lemma

$E \notin \mathrm{spec}(H) \Rightarrow \chi_{(-\infty,E]}(H)) \in C(\Sigma) \rtimes_\sigma \mathbb{Z}^d \otimes \mathcal{K}$.

Proof. Suppose that, $\mathrm{spec}(H) \subset [-A,\infty)$ and that the open interval (a,b) is a **spectral gap** of H, i.e. $(a,b) \cap \mathrm{spec}(H) = \emptyset$.

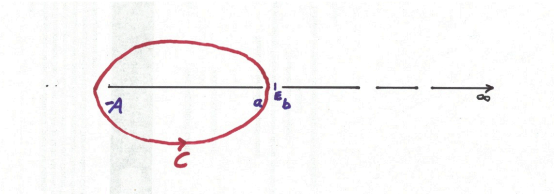
Magnetic gap-labelling conjecture

Suppose that $E \in (a, b)$ i.e. $E \notin \mathrm{spec}(H)$.

Then there is a holomorphic function ϕ on a neighbourhood of $\mathrm{spec}(H) \cap [-A, a]$ such that

$$\chi_{(-\infty,E]}(H) = \phi(H) = \oint_C \frac{d\lambda}{\lambda - H}$$

where C is a closed contour enclosing the interval $[-A, a]$ to the left of E, and is the **Riesz projection**.



Since $C(\Sigma) \rtimes_\sigma \mathbb{Z}^d \otimes \mathcal{K}$ is closed under the continuous functional calculus, it follows that $\chi_{(-\infty,E]}(H) \in C(\Sigma) \rtimes_\sigma \mathbb{Z}^d \otimes \mathcal{K}$.

Magnetic gap-labelling conjecture

It follows that the spectral gap-labels of H are contained in the countable subgroup of $\mathbb{R}$, $\tau_\mu(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^d))$.

Then the **magnetic gap-labelling conjecture** is about finding an expression for

$$\tau_\mu(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^d)) = ????$$

K-theory of C^*-algebras

Two projections P, Q in the C^*-algebra $\mathcal{A} \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space, are said to be (Murray-von Neumann) equivalent $P \sim Q$ whenever there is an element $a \in \mathcal{A}$ such that $P = a^* a \quad$ and $Q = aa^*$.

Recall that the K-theory $K_0(\mathcal{A})$ is defined as stable equivalence classes of pairs of projections (P, Q) in $\mathcal{A} \otimes \mathcal{K}$,
where (P, Q) and (P', Q') are stably equivalent whenever $P \oplus Q' \oplus R \sim P' \oplus Q \oplus R$, for some projection R in $\mathcal{A} \otimes \mathcal{K}$.
$K_0(\mathcal{A})$ is a countable group.

Clearly a trace on the algebra $\mathcal{A}$, $\tau_\mu : \mathcal{A} \to \mathbb{C}$, induces a morphism $\tau_\mu : K_0(\mathcal{A}) \longrightarrow \mathbb{R}$.

Motivation: Magic formula

Let $\Lambda[dx] = \Lambda[dx_1, \ldots, dx_d]$ denote the exterior algebra with generators $dx_1, \ldots, dx_d$. It has basis the monomials $dx_I = dx_{i_1}, \ldots, dx_{i_p},\ I = \{i_1, \ldots, i_p\},\ i_1 < \cdots < i_p$, $1 \leq p \leq d$.

Given a skew-symmetric matrix Θ, we can associate a quadratic element $\frac{1}{2}dx^t\Theta dx$ in $\Lambda[dx]$.

Recall that the Pfaffian of the skew-symmetric matrix Θ, $\mathrm{Pf}(\Theta)$ can be defined as

$$\frac{1}{m!}\left(\frac{1}{2}dx^t\Theta dx\right)^m = \mathrm{Pf}(\Theta)dx_1 \wedge dx_2 \wedge \ldots \wedge dx^d$$

where $d = 2m$.

Motivation: Magic formula

By section 1 in [Mathai-Quillen86],

$$e^{\frac{1}{2}dx^t\Theta dx} = \sum_I \mathrm{Pf}(\Theta_I)dx_I$$

where I runs over subsets of $\{1, \ldots, d\}$ with an even number of elements, and Θ_I denotes the submatrix of $\Theta = (\Theta_{ij})$ with $i, j \in I$, which is clearly also skew-symmetric.

This was a key formula in the paper above, to construct the Chern-Weil representative of the Thom class of an oriented vector bundle.

Motivation: Magic formula

sketch of proof.

To verify this identity, fix such a multiindex I and consider the onto algebra homomorphism

$$\Lambda[dx] \longrightarrow \Lambda[dx_j : j \in I]$$

which kills components containing dx_k for $k \notin I$.
In degree $|I|$ this map kills all monomials except dx_I , and it maps the Gaussian expression $e^{\frac{1}{2}dx^t\Theta dx}$ onto the corresponding Gaussian expression constructed from the submatrix Θ_I . Thus the coefficient of dx_I in the Gaussian expression $e^{\frac{1}{2}dx^t\Theta dx}$ is just

$$\mathrm{Pf}(\Theta_I)$$

as claimed. □

Magnetic gap-labelling conjecture

The subgroup of the real line $\mathbb{R}$ which is generated by μ-measures of clopen subsets of Σ is denoted $\mathbb{Z}[\mu]$.

NB $\mathbb{Z}[\mu]$ is a **countable set** as all the clopen subsets of a Cantor set form a countable set.

$\mathbb{Z}[\mu]$ is also the image under (the integral associated with) the invariant probability measure μ of $C(\Sigma, \mathbb{Z})$, the group of continuous integer valued functions on Σ. That is,

$$\mathbb{Z}[\mu] = \left\{ \int_\Sigma f(z) d\mu(z) \middle| f \in C(\Sigma, \mathbb{Z}) \right\} = \mu(C(\Sigma, \mathbb{Z}))$$

Magnetic gap-labelling conjecture

Recall that if a group Γ acts on a module M, then the **coinvariants** of M is defined as the quotient

$$M/\{m - gm | m \in M, g \in \Gamma\},$$

and the **invariants** of M is defined as

$$\{m \in M | m = gm \text{ for all } g \in \Gamma\}.$$

Let I be an ordered subset of $\{1, \ldots, p\}$ with an even number of elements, and let $C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}$ denote the coinvariants under the subgroup $\mathbb{Z}^{I^c}$ of $\mathbb{Z}^d$, where I^c denotes the complementary index to I. Let $\left(C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}\right)^{\mathbb{Z}^I}$ denote the subset of $C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}$ invariant under the subgroup $\mathbb{Z}^I$ of $\mathbb{Z}^d$. Define the countable group

$$\mathbb{Z}_I[\mu] = \mu\left(\left(C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}\right)^{\mathbb{Z}^I}\right).$$

Magnetic gap-labelling conjecture

Let Σ be a Cantor set with a minimal action of $\mathbb{Z}^d$ that preserves a Borel probability measure μ. Let σ be the multiplier on $\mathbb{Z}^d$ associated to a skew-symmetric $(d \times d)$ matrix Θ.

1. If d is even, then the **magnetic frequency group** is defined as follows:
$$\mathbb{Z}[\mu] + \sum_{0<|I|<d} \mathrm{Pf}(\Theta_I)\mathbb{Z}_I[\mu] + \mathrm{Pf}(\Theta)\mathbb{Z}.$$
2. If d is odd, then the **magnetic frequency group** is defined as follows:
$$\mathbb{Z}[\mu] + \sum_{0<|I|\leq d} \mathrm{Pf}(\Theta_I)\mathbb{Z}_I[\mu].$$

Here, $|I|$ is even, and Θ_I denotes the skew-symmetric submatrix of $\Theta = (\Theta_{ij})$ with $i, j \in I$, $\mathrm{Pf}(\Theta_I)$ denotes the Pfaffian of Θ_I.

Magnetic gap-labelling conjecture

The **magnetic gap-labelling group** is defined as the range of the trace on K-theory,

$$\tau_\mu \left(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^d) \right).$$

The **magnetic gap-labelling conjecture [BM]** asserts that:

magnetic gap-labelling group $\trianglelefteq$ magnetic frequency group

Magnetic gap-labelling conjecture

When $\Theta = 0$, this is the case when there is **no** magnetic field. It was first formulated in the early 1980s by Jean Bellissard, and the statement reduces to

$$\tau_\mu\left(K_0(C(\Sigma) \rtimes \mathbb{Z}^d)\right) = \mathbb{Z}[\mu].$$

It was proved by Bellissard and collaborators when $d = 1, 2, 3$ in the 1990s.

In early 2000s, there were 3 groups who published proofs of the conjecture in all dimensions.

J. Bellissard, R. Benedetti, J-M. Gambaudo;

M.-T. Benameur ,H. Oyono-Oyono;

J. Kaminker, I. Putnam.

Magnetic gap-labelling conjecture: existence of gaps

Raikov et al., consider the 2D magnetic Schrödinger operator,

$$H = -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - \theta x\right)^2 + V(x).$$

Here $B = \theta\, dx \wedge dy$, $\theta \neq 0$ is a constant magnetic field, and V is a real valued, non-constant smooth periodic electric potential that is independent of the y variable. The self-adjoint operator H on $L^2(\mathbb{R}^2)$ is proved to generically have *infinitely* many open spectral gaps.

This is in stark contrast to the Bethe-Sommerfeld conjecture (proved recently by L. Parnovski), which says that there are only a **finite** number of gaps in the spectrum of any Schrödinger operator with smooth **periodic** potential V on Euclidean space, in the case when the magnetic field vanishes, i.e. $\theta = 0$, whenever the dimension is greater than or equal to 2.

Magnetic gap-labelling conjecture: existence of gaps

In fact Raikov et al, also study the Hamiltonian $H_{\pm} = H \pm W$, where $W \in L^{\infty}(\mathbb{R}^2) \cap C^{\infty}(\mathbb{R}^2)$ is non-negative and decays at infinity and $\theta \neq 0$, so that $H_{\pm}$ is the sort of Hamiltonians that we consider in our paper. They find that there are *infinitely* many discrete eigenvalues of $H_{\pm}$ in any open gap in the spectrum of spec(H), and the convergence of these eigenvalues to the corresponding endpoint of the spectral gap is asymptotically Gaussian.

This shows that the spectral gaps of magnetic Schrödinger operators (of the type considered in this paper) can be rather interesting even in higher dimensions.

Evidence for the conjecture: the 2D case

We now compute the magnetic gap-labelling group in a physically relevant case when $p = 2$.

- Let $\mathbb{Z}^2 \curvearrowright \Sigma$ be a minimal action with invariant probability measure μ on Σ.

- Let σ be a multiplier on $\mathbb{Z}^2$. Then the group cohomology class of $[\sigma] \in H^2(\mathbb{Z}^2; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$ can be identified with a real number θ, $0 \leq \theta < 1$. More precisely, we take $\sigma = e^{2\pi i\theta\omega}$ where ω is the standard symplectic form on $\mathbb{Z}^2$.

The magnetic gap-labelling conjecture in 2D reduces to

Theorem (2D case)

$$\tau_\mu(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2) = \mathbb{Z}[\mu] + \mathbb{Z}\theta$$

Evidence for the conjecture: the 2D case

Proof.

By results of Packer-Raburn and the Connes-Thom isomorphism, it follows that

$$\mu_\theta : K^0(X) \longrightarrow K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z})$$

is an isomorphism, where $X = \Sigma \times_{\mathbb{Z}^2} \mathbb{R}^2$ is a fibre bundle over the torus $\mathbb{R}^2/\mathbb{Z}^2$ with typical fibre the Cantor set Σ; X is also called a **solenoidal torus**. By the foliated twisted L^2-index theorem [BM15], for any vector bundle ξ over X, one has

$$\tau_\mu(\mu_\theta(\xi)) = \int_X d\mu(\vartheta)\, e^{\theta dx_1 \wedge dx_2} \wedge \mathrm{Ch}(\xi)$$

Now X is a connected space since the $\mathbb{Z}^2$-action is minimal. □

Evidence for the conjecture: the 2D case

Proof.

Therefore

$$\tau_\mu(\mu_\theta(\xi)) = \theta\mu(\Sigma)\int_{\mathbb{T}^2} dx_1 \wedge dx_2 \,\mathrm{rank}(\xi) + \int_X \mathrm{c}_1(\xi)$$
$$= \theta\,\mathrm{rank}(\xi) + \int_X \mathrm{c}_1(\xi)$$

Varying over all (virtual) vector bundles ξ over $\mathbb{T}^2$, and using the fact that the **zero**-magnetic field gap-labelling in 2D holds, i.e.

$$\left\{\int_X \mathrm{c}_1(\xi) : \xi \in K^0(X)\right\} = \mathbb{Z}[\mu]$$

we conclude that the result follows. □

A particular 2D example

Suppose that $0 < \alpha_1 < \alpha_2 < 1$ are two rationally independent irrational numbers. Then $T_j x = x + \alpha_j \pmod 1$, $j = 1, 2$ defines a minimal $\mathbb{Z}^2$-action on the circle $\mathbb{R}/\mathbb{Z}$.

Define the Cantor set Σ to be the circle disconnected along the dense orbit of $\mathbb{Z}^2$ through the origin. Then by fiat, $\mathbb{Z}^2$ also acts minimally on Σ and this example has a unique invariant probability measure μ. In this case, one can show that,

$$\mathbb{Z}[\mu] = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2.$$

The magnetic gap-labelling theorem in this 2D example is:

$$\tau^\mu(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2)) = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\theta.$$

Evidence for the conjecture: Jordan block diagonal case

We also deduce the magnetic gap-labelling conjecture when $\Theta = \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix}$ is in **Jordan block diagonal form**, for any n.

This essentially follows from the 2D case, and the Kunneth theorem in K-theory.

Evidence for the conjecture: the periodic case

Let $\Sigma = \{pt\}$ i.e. V is periodic. Then $C(\Sigma) \rtimes_\sigma \mathbb{Z}^d$ is the noncommutative torus A_Θ.

Theorem

1. *If d is even, then*

$$\tau(K_0(A_\Theta)) = \mathbb{Z} + \sum_{0<|I|<d} \mathrm{Pf}(\Theta_I)\mathbb{Z} + \mathrm{Pf}(\Theta)\mathbb{Z},$$

2. *If d is odd, then*

$$\tau(K_0(A_\Theta)) = \mathbb{Z} + \sum_{0<|I|<d} \mathrm{Pf}(\Theta_I)\mathbb{Z},$$

where I runs over subsets of $\{1, \dots, d\}$ with an even number of elements, and Θ_I denotes the submatrix of $\Theta = (\Theta_{ij})$ with $i, j \in I$.

Evidence for the conjecture: the periodic case

Proof.

Since the Baum-Connes conjecture with coefficients is true for $\mathbb{Z}^d$ (assume that d is even), it follows that

$$\mu_\Theta : K^0(\mathbb{T}^d) \xrightarrow{\sim} K_0(A_\Theta)$$

is an isomorphism. Then by the twisted L^2-index theorem [Mathai99] and equation [MathaiQuillen86],

$$\begin{aligned}\tau(\mu_\Theta(\xi)) &= \int_{\mathbb{T}^d} e^{\frac{1}{2} dx^t \Theta dx} \wedge \mathrm{Ch}(\xi) \\ &= \sum_I \mathrm{Pf}(\Theta_I) \int_{\mathbb{T}^d} dx_I \wedge \mathrm{Ch}(\xi)\end{aligned}$$

Since the Chern character is an **integral isomorphism** on the torus $\mathbb{T}^d$, the result follows by varying ξ over all K-theory classes.

Remarks

The proposition above is a result of [Elliott82], however we both give a new proof of it, as well as a significantly neater expression for the range of the trace that is better suited to our paper.

Key difficulty: integrality of the Chern character

The Chern character is an integral isomorphism on the torus and is well understood for manifolds in general.

However for the fibre bundle $\Sigma \to X \to \mathbb{T}^d$, where Σ is a Cantor set, X is only a solenoidal torus and the Chern character is not well understood in this case.

By the (foliated) index theorem, if one can prove **integrality** of all of the components of the Chern character, $\int_X dx_I \wedge \mathrm{Ch}(\xi)$ for all $\xi \in K^0(X)$, then it turns out that the MGL conjecture can be proved. This is hard work!

(For the precise definition of the Chern character in this context, see Moore-Schochet, *Global Analysis on Foliated Spaces*)

Sketch of the proof of the the conjecture in 3D

The 3D case is technically much more involved and is the main theorem in **[BM].**

Theorem

Let $\mathbb{Z}^3 \curvearrowright \Sigma$ be a minimal action with invariant probability measure μ on Σ.

$$\tau_\mu(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^3)) \subset$$

$$\mathbb{Z}[\mu] \,+\, \Theta_{12}\,\mathbb{Z}_{12}[\mu] \,+\, \Theta_{13}\,\mathbb{Z}_{13}[\mu] \,+\, \Theta_{23}\,\mathbb{Z}_{23}[\mu].$$

This proves the magnetic gap-labelling conjecture in 3D.

Sketch of the proof of the the conjecture in 3D

Step 1. The **twisted Connes-Thom isomorphism**,

$$\text{Index} : K^1(X) \to K_0(C(\Sigma) \rtimes_\Theta \mathbb{Z}^3),$$

is an isomorphism, where $\Sigma \hookrightarrow X \to \mathbb{T}^3$ is a fibre bundle over the torus $\mathbb{T}^3$ with typical fibre Σ. More precisely, $X = \mathbb{R}^3 \times_{\mathbb{Z}^3} \Sigma$.

Step 2. By the **measured twisted foliated index theorem** in [BM15], we see that

$$\tau_\mu(\text{Index}(\partial_\Theta \otimes U) = \Theta_{13} \int_X d\mu(\vartheta)\, dx_1 \wedge dx_3\, ch_1^{odd}(U) + \Theta_{12} \int_X d\mu(\vartheta)\, dx_1 \wedge dx_2\, ch_1^{odd}(U)$$

$$+\Theta_{23} \int_X d\mu(\vartheta)\, dx_2 \wedge dx_3\, ch_1^{odd}(U) + \int_X d\mu(\vartheta)\, ch_3^{odd}(U)$$

where $U : X \to U(\infty)$ is continuous and represents a class $[U] \in K^1(X)$, where $ch_{2j+1}^{odd}(U) \in H^{2j+1}(X)$ are the components of the odd Chern character, $ch^{odd} : K^1(X) \to H^{odd}(X)$.

Sketch of the proof of the the conjecture in 3D

Therefore the range of the trace τ_μ, $\mathrm{range}(\tau_\mu)$ is equal to the set,

$$\Big\{\Theta_{13}\int_X d\mu(\vartheta)\, dx_1 \wedge dx_3\, ch_1^{odd}(U) + \Theta_{12}\int_X d\mu(\vartheta)\, dx_1 \wedge dx_2\, ch_1^{odd}(U)$$

$$+\Theta_{23}\int_X d\mu(\vartheta)\, dx_2 \wedge dx_3\, ch_1^{odd}(U) + \int_X d\mu(\vartheta)\, ch_3^{odd}(U)\Big|[U] \in K^1(X)\Big\}$$

When $\Theta = 0$, the usual gap-labelling conjecture in this context asserts that,

$$\mathrm{range}(\tau_\mu) = \left\{\int_X d\mu(\vartheta)\, ch_3^{odd}(U)\Big|[U] \in K^1(X)\right\} = \mathbb{Z}[\mu],$$

and this has a complete proof by Bellissard et al. [1998]

Sketch of the proof of the the conjecture in 3D

It suffices to compute,

$$\Big\{\Theta_{13}\int_X d\mu(\vartheta)\, dx_1 \wedge dx_3\, ch_1^{odd}(U) + \Theta_{12}\int_X d\mu(\vartheta)\, dx_1 \wedge dx_2\, ch_1^{odd}(U)$$

$$+\Theta_{23}\int_X d\mu(\vartheta)\, dx_2 \wedge dx_3\, ch_1^{odd}(U)\Big|[U] \in K^1(X)\Big\}$$

Step 3. To do this, we use the following Lemma in homological algebra, where we assume that $M = C(\Sigma, \mathbb{Z})$, which is a free $\mathbb{Z}[\Gamma]$-module where $\Gamma = \mathbb{Z}^3$ acts minimally on Σ.

Sketch of the proof of the the conjecture in 3D

The following is standard and can be found in any book on group cohomology, $\Gamma = \mathbb{Z}^3$

Lemma

If M is a $\mathbb{Z}[\Gamma]$-module, then the following hold:

1. *The cohomology groups $H^n(\Gamma; M)$ and homology groups $H_n(\Gamma; M)$, are trivial except for $0 \leq n \leq 3$.*
2. $H_0(\Gamma; M) = M/\{m - gm | m \in M, g \in \Gamma\}$*, the coinvariants of M.*
3. $H^0(\Gamma; M) = \{m \in M | m = gm \text{ for all } g \in \Gamma\}$*, the invariants of M.*
4. *There is a natural isomorphism $PD : H^n(\Gamma; M) \cong H_{3-n}(\Gamma; M)$ (**Poincaré duality**) for $0 \leq n \leq 3$.*

In particular, $H^0(\Gamma; M) = M^\Gamma$ and $H^3(\Gamma, M) = M_\Gamma$.
So it remains to compute $H^1(\Gamma, M)$.

Sketch of the proof of the the conjecture in 3D

To compute $H^1(\Gamma, M)$, we use the following homological algebra lemma iteratively (long, so we skip the details here):

Lemma

$$0 \to H^1(\mathbb{T}_d; H^{n-1}(\mathbb{T}^{d-1}_{12\ldots(d-1)}; M)) \to H^n(\mathbb{T}^d; M) \to$$
$$\to H^0(\mathbb{T}_2; H^n(\mathbb{T}^{d-1}_{12\ldots(d-1)}; M)) \to 0.$$

for $n = 1, \ldots, d$.

This is proved by basic topology by cutting the dth circle factor in the classifying space $\mathbb{T}^d$ for $\mathbb{Z}^d$ into two semicircles gives rise to Mayer-Vietoris exact sequences giving rise to the short exact sequences indicated.

We will be concerned with $d \leq 3$.

Sketch of the proof of the the conjecture in 3D

Therefore, the magnetic gap-labelling group coincides with $\mathbb{Z}[\mu]$ plus the range of the map

$$H^1(\mathbb{Z}^3, C(\Sigma, \mathbb{Z})) \xrightarrow{\cup\Theta} H^3(\mathbb{Z}^3, C(\Sigma, \mathbb{R})) \xrightarrow{PD} C(\Sigma, \mathbb{R})_{\mathbb{Z}^3} \xrightarrow{\mu} \mathbb{R}.$$

This completes the sketch of proof for the 3D case.

A general theorem in all dimensions

The integrality hypothesis: *(IH) The range of the Chern character* $Ch : K^p(X) \longrightarrow H^{[p]}(X, \mathbb{Q}) \simeq \oplus_{k \geq 0} H^{p+2k}(\mathbb{Z}^p, C(\Sigma, \mathbb{Q}))$ *is contained in* $H^{p+2k}(\mathbb{Z}^p, C(\Sigma, \mathbb{Z}))$.

We will show that IH is satisfied for principal solenoidal tori in all dimensions.

Theorem

Suppose that the integralty hypothesis (IH) *stated above is satisfied, then the Magnetic Gap-Labelling Conjecture is true.*

A general theorem in all dimensions

The proof uses as in the 3D case,

1. the twisted Connes-Thom isomorphism;
2. the twisted measured foliated index theorem in [BM15]
3. various homological algebra arguments in group cohomology with coefficients in modules

Chern character for principal solenoidal tori

For each $j \in \mathbb{N}$, let $X_j = \mathbb{T}^n$. Define the finite regular covering map $f_{j+1} : X_{j+1} \to X_j$ to be a finite covering map such that degree of f_{j+1} is greater than 1 for all j. For example, let $f_{j+1}(z_1, ..., z_n) = (z_1^{p_1}, ..., z_n^{p_n})$,where each $p_j \in \mathbb{N} \setminus \{1\}$.
Set (X_∞, f_∞) to be the inverse limit, $\varprojlim(X_j, f_j)$. Then X_∞ is a solenoid torus, and $X_\infty \subset \prod_{j\in\mathbb{N}} X_j$, where the right hand side is a compact space when given the Tychonoff topology, therefore X_∞ is also compact. Let $G_j = \mathbb{Z}^n/\Gamma_j$ be the finite covering space group of the finite cover $p_j : X_j \to X_1 = \mathbb{T}^n$. Then the inverse limit $G_\infty = \varprojlim G_j = \varprojlim \mathbb{Z}^n/\Gamma_j$ is the profinite completion of $\mathbb{Z}^n$ that is homeomorphic to the Cantor set, cf. Lemma 5.1 in [McCord]. Moreover $G_\infty \to X_\infty \to \mathbb{T}^n$ is a principal fibre bundle, cf. [McCord]. We call such an X a **principal solenoidal torus**.

Chern character for principal solenoidal tori

Now K-theory is continuous under taking inverse limits in the category of compact Hausdorff spaces, which follows from Proposition 6.2.9 in [Wegge-Olsen], see also [NCPhillips],

$$K^\bullet(X_\infty) \cong \varinjlim K^\bullet(X_j).$$

Now the Chern character

$$Ch : K^\bullet(X_j) \to H^\bullet(X_j, \mathbb{Z})$$

maps to integral cohomology, as shown earlier, since X_j is a torus. Therefore

$$\varinjlim Ch : K^\bullet(X_\infty) \cong \varinjlim K^\bullet(X_j) \to \varinjlim H^\bullet(X_j, \mathbb{Z}). \cong H^\bullet(X_\infty, \mathbb{Z})$$

by the continuity for Čech cohomology under taking inverse limits in the category of compact Hausdorff spaces, [Spanier].

Chern character for principal solenoidal tori

Now $H^\bullet(X_j, \mathbb{Z})$ are torsion-free Abelian groups, therefore the direct limit $\varinjlim H^\bullet(X_j, \mathbb{Z})$ is again a torsionfree Abelian group. So we have proved the following,

Theorem (Integrality of the Chern character)

Let X be a principal solenoidal torus as above. Then the Chern character,

$$\varinjlim Ch : K^\bullet(X_\infty) \to H^\bullet(X_\infty, \mathbb{Q})$$

is integral, that is, the range is contained in $H^\bullet(X_\infty, \mathbb{Z})$.

So the MGL is true for principal solenoidal tori

Overview of the measured index theorem

One of the main steps towards proving the magnetic GL conjecture is the measured twisted index theorem, which is a twisted analog of ***Connes measured index theorem***.

The suspension $X = \mathbb{R}^p \times_{\mathbb{Z}^d} \Sigma$ is a compact foliated space with transversal the Cantor set Σ, and with invariant transverse measure induced from μ. The monodromy groupoid is

$$\mathcal{G} = (\mathbb{R}^p \times \mathbb{R}^p \times \Sigma)/\mathbb{Z}^p.$$

Overview of the measured index theorem

Consider functions f in $L^2(\mathbb{R}^p \times \Sigma; dxd\mu)$ and the operators defined on it as follows,

1. $S_\gamma f(x,\vartheta) = e^{i\varphi_\gamma(x)} f(x,\vartheta)$;
2. $U_\gamma f(x,\vartheta) = f(x.\gamma, \vartheta.\gamma)$.

Then for all $\gamma \in \mathbb{Z}^p$, the bounded operators $T_\gamma = U_\gamma \circ S_\gamma$ satisfy the relation

$$T_{\gamma_1} T_{\gamma_1} = \sigma(\gamma_1, \gamma_2)\, T_{\gamma_1\gamma_2}$$

where $\sigma(\gamma_1,\gamma_2) = \phi_{\gamma_1}(\gamma_2)$ is a multiplier on $\mathbb{Z}^p$.

Overview of the measured index theorem

Let $\partial\!\!\!/$ denote the Dirac operator on $\mathbb{R}^p$ and $\nabla = d + 2i\pi\eta$ the connection on the trivial line bundle on $\mathbb{R}^p$, ∇^E the lift to $\mathbb{R}^p \times \Sigma$ of a connection on a vector bundle $E \to X$. Consider the twisted Dirac operator along the leaves of the lifted foliation,

$$D = \partial\!\!\!/ \otimes \nabla \otimes \nabla^E : L^2(\mathbb{R}^p \times \Sigma, \mathcal{S}^+ \otimes E) \longrightarrow L^2(\mathbb{R}^p \times \Sigma, \mathcal{S}^- \otimes E).$$

Then one computes that $T_\gamma \circ D = D \circ T_\gamma$ for $\gamma \in \mathbb{Z}^p$.
The **heat kernel** of D, denoted $k(t, x, y, \vartheta)$, since it is smooth for $t > 0$, it has a well defined μ-trace

$$\tau_\mu(k_t) = \int_X k(t, x, x, \vartheta) d\mu(\vartheta) dx$$

Overview of the measured index theorem

For $t > 0$, define the **Wasserman or index** idempotent

$$e_t(D^+) \in M_2(\mathcal{A})$$

as follows:

$$e_t(D^+) = \begin{pmatrix} e^{-tD^-D^+} & e^{-\frac{t}{2}D^-D^+}\dfrac{(1-e^{-tD^-D^+})}{D^-D^+}D^+ \\ e^{-\frac{t}{2}D^+D^-}D^+ & 1-e^{-tD^+D^-} \end{pmatrix},$$

Here $\mathcal{A} \cong C^*(X, \mathcal{F}, \sigma) \otimes \mathcal{K} \cong C(\Sigma) \rtimes_\sigma \mathbb{Z}^p \otimes \mathcal{K}$.
Then the $\mathcal{A}$-*twisted foliated analytic index* is defined as

$$\mathrm{Index}_{\mathcal{A}}(D^+) = [e_t(D)] - [E_0] \in K_0(\mathcal{A}), \qquad (1)$$

where $t > 0$ and E_0 is the idempotent

$$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathcal{A}).$$

Overview of the measured index theorem

Note:

$$\mathrm{Index}_{\mathcal{A}} : K^0(X) \longrightarrow K_0(\mathcal{A}).$$

is an isomorphism (**Packer-Raeburn+Connes' Thom isom.**).
A **McKean-Singer** type argument shows that

$$\tau_\mu(\mathrm{tr}_s(k(t,\cdots))) = \tau_\mu(e^{-tD^-D^+}) - \tau_\mu(e^{-tD^+D^-}) = \tau_\mu^s(\mathrm{Index}_{\mathcal{A}}(D^+))$$

is independent of $t > 0$ and represents the **twisted measured foliated index.** By **Getzler's local index** and the **M-Q formula**,

$$\begin{aligned}\lim_{t\downarrow 0} \tau_\mu(\mathrm{tr}_s(k(t,\cdots))) &= \frac{1}{(2\pi)^p}\int_X \exp\left(\frac{1}{2}dx^t\Theta dx\right) \wedge Ch(F_E)d\mu(\vartheta),\\ &= \frac{1}{(2\pi)^p}\sum_I \mathrm{Pf}(\Theta_I)\int_X dx_I \wedge Ch(F_E)_{I^c}d\mu(\vartheta).\end{aligned}$$

Here I runs over subsets of $\{1,\ldots,p\}$ with an even number of elements, and Θ_I is the submatrix of $\Theta = (\Theta_{ij})$ with $i, j \in I$.