Ann. Henri Poincaré 20 (2019), 1837–1875

1424-0637/19/061837-39
published online April 13, 2019
https://doi.org/10.1007/s00023-019-00794-3

Annales Henri Poincaré

Strongly Disordered Floquet Topological Systems

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Abstract. We study the strong disorder regime of Floquet topological systems in dimension two that describe independent electrons on a lattice subject to a periodic driving. In the spectrum of the Floquet propagator we assume the existence of an interval in which all states are localized—a mobility gap—extending previous studies which make the stronger spectral gap assumption. We devise a new approach to define the topological invariants by way of stretching the gap of a given system onto the whole circle. We show that such completely localized systems have natural indices that circumvent the relative construction and match with quantized magnetization and pumping observables from the physics literature. These indices obey a bulk-edge correspondence, which carries over to the stretched systems as well. Finally, these invariants are shown to coincide with those associated with the usual relative construction, which we also extend to the mobility gap regime.

1. Introduction

In the context of topological insulators, a Floquet system describes independent electrons on a lattice subject to a periodic driving beyond the adiabatic regime. The system is topological when one can define a stable index that captures some topological property of the sample, either in the bulk of an infinite one or at the edge of a semi-infinite one. For Floquet systems, the latter is sometimes associated with a transport observable and usually coincides with the bulk index (whose physical meaning is sometimes associated with magnetization, see below) through the bulk-edge correspondence [16, 29, 30]. Originally designed to induce topological properties on a trivial sample through the periodic driving [23], Floquet topological systems have recently become a topic of intense study when it was realized that this driving also allowed to engineer new topological phases of matter that have no static counterpart [29]; some proposals for experimental observation of these phases in cold atoms were recently suggested [22, 25].

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So far the main prerequisite to define topological indices in Floquet systems has been the presence of a gap in the spectrum of the unitary Floquet propagator, describing time evolution in the bulk after one period of driving. In this context the bulk-edge correspondence was first established in clean systems and then extended to weakly disordered samples, for various dimensions and symmetries [6,13,14,16,20,27–30]. By analogy with static systems, the effect of disorder is to progressively fill the spectral gap of the propagator by localized states [17,34], and all the previous results work only as long as the spectral gap remains open.

This paper deals with two-dimensional systems with no particular symmetry (class A of [3]). We address the problem of strong disorder, when the gap is completely filled by localized states (see Fig. 1). This is the so-called mobility gap regime that is characterized by Anderson localization and mathematically through the fractional moment condition [1]. The starting point of our work is a general and almost sure consequence of this condition, which we take as a deterministic assumption to define the mobility gap. This approach is analogous to the other few works on topological properties of strongly disordered systems in the static case, first studied for the integer quantum Hall effect [10] and more recently in chiral systems [15]. Moreover, the fractional moment condition has been already established for unitary random operators [5,17,19], as well as some numerical evidence of localization in Floquet topological models [34]. We note in passing that [24] also studied strongly disordered unitary topological systems in the bulk; however, they used a covariant probabilistic framework.

A zeroth step here is to verify that the so-called relative construction, developed in the spectral gap case in dimension two [16,29,30], can be extended to the mobility gap regime. This construction reduces the physical unitary

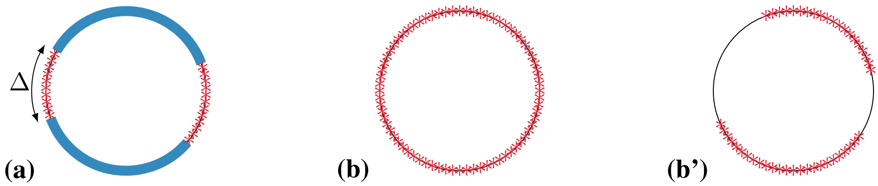

Figure 1. Examples of spectrum for the Floquet propagator $U(1)$. **a** Typical situation with a mobility gap Δ (possibly several), represented by red crosses. The remaining part of the spectrum, in blue, can be arbitrary. For completely localized operators, the mobility gap is the entire circle (**b**) or possibly with spectral gaps (special case of a mobility gap) ($\mathbf{b}'$). The relative construction applies to all cases, but can be avoided in **b**, $\mathbf{b}'$ through magnetization and pumping indices. The stretch function construction maps situation (**a**, **b**) by stretching Δ onto $\mathbb{S}^1\backslash\{1\}$ and mapping the remainder of the spectrum to 1 (color figure online)

evolution to a time periodic propagator in the bulk, which has a well-defined index. This requires a logarithm of the Floquet propagator that we prove to be well defined with a branch cut in the mobility gap. Thanks to the estimates coming from localization, the logarithm is weakly local—its matrix elements in the position basis have rapid off-diagonal decay, and possible diagonal blowup. With this we can adapt the proof in [16] of the bulk-edge correspondence from the spectral gap case, in which the Combes–Thomas estimate was used instead of localization.

The physical implementation of this relative construction is however not straightforward, and one can look for situations where it may be circumvented. For clean samples, a Floquet system is actually an insulator only when the Floquet propagator is exactly the identity operator $\mathbb{1}$, for which the relative construction is not required. In the spectral and mobility gap cases the system is not insulating anymore and the relative constructions somehow subtract the other transport contributions from the topological one [32]. In the strongly disordered case the analogue of $\mathbb{1}$ is to consider a Floquet propagator that is completely localized, namely that its entire spectrum is a mobility gap (see Fig. 1b, b′). It was shown in [34] that in contrast to the static case [10], such systems may still have edge modes and topological properties. Moreover, the indices can be computed without the relative construction and have a nice physical interpretation in terms of quantized orbital magnetization in the bulk [22] and quantized pumping at the edge [34]. The first result of this paper is a rigorous definition of these indices and a proof of their respective bulk-edge correspondence.

Our second result is to show that any mobility gap situation can actually be reduced to a fully localized case, for which the previous indices can be used, circumventing again the relative construction. This reduction is done through the smooth functional calculus with a particular function that stretches the mobility gap onto the entire circle (as in [30] who however used this construction only for the edge, in the spectral gap regime).

We finally show that the indices defined in this approach coincide with the ones of the relative construction. To that end we show the continuity of the bulk relative index along a specific path of deformation. We believe this continuity result in the mobility gap regime is important because it joins an extremely short list of results: the deterministic constancy of the quantum Hall conductivity w.r.t. the Fermi energy proven in [10]. Thus Theorem 2.19 opens interesting perspectives in the investigation of the topology of deterministic mobility gapped systems, of which very little is known.

We note in passing that quantum walks, namely finite sequences of unitary operators, can also be seen as discrete-time Floquet systems, for which topological indices have been already defined in clean and weakly disordered models [4,33]. In some cases the Floquet formalism can be applied to quantum walks [8,30], so that our result should in principle cover the strongly disordered version of these quantum walks.

The paper is organized as follows. After describing the setting and stating the results mentioned in Sect. 2, we detail their respective proofs in Sects. 3–5. Finally, "Appendices A.2–A.4" are results of independent interest for Floquet systems or more general unitary operators.

2. Setting and Main Results

Let a time-dependent periodic Hamiltonian $H : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ be given where $\mathcal{H} := \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ is the (bulk) Hilbert space and $N, d \in \mathbb{N}_{\geq 1}$ are the (fixed) number of degrees of freedom per lattice site and the space dimension. Here we use $\mathbb{S}^1 \cong [0,1]/\{0 \sim 1\}$. We assume the following conditions about H throughout:

Assumption 2.1 (*Continuity*). $t \mapsto H(t)$ is strongly continuous except for a finite number of jump discontinuities.

In what follows, let $(\delta_x)_{x\in\mathbb{Z}}$ be the canonical (position) basis of $\ell^2(\mathbb{Z}^d)$ and the map $H(t)_{xy} = \langle \delta_x, H(t)\delta_y \rangle : \mathbb{C}^N \to \mathbb{C}^N$ acts between the internal spaces of x and y; $\|\cdot\|$ is the trace norm of such maps and also the 1-norm on $\mathbb{R}^d$ when applied to some $x \in \mathbb{Z}^d$: $\|x\| \equiv \sum_{i=1}^d |x_i|$.

Assumption 2.2 (*Locality*). There are some constants $C < \infty, \mu > 0$ such that for any $t \in \mathbb{S}^1$ we have

$$\|H(t)_{xy}\| \leq C\mathrm{e}^{-\mu\|x-y\|} \qquad (x, y \in \mathbb{Z}^d). \tag{2.1}$$

We use the symbols C, μ for constants which may change from line to line, but which are otherwise independent of the other variables unless stated differently.

To deal with Floquet systems we consider the unitary propagator $U : [0,1] \to \mathcal{B}(\mathcal{H})$ generated by H, that is, the unique solution to $\mathrm{i}\dot{U} = HU$ with initial condition $U(0) = \mathbb{1}$. (It is a fact that even though H is periodic, U need not be and so its domain is a priori $[0,1]$.) A well-defined topological phase exists depending on spectral or dynamical properties that $U(1)$ may or may not satisfy. Such a phase was established (see, e.g., [16,29,30]) in the presence of a spectral gap:

Definition 2.3 (*Spectral gap*). $U(1)$ has a spectral gap iff its spectrum is not the entire circle:

$$\sigma(U(1)) \neq \mathbb{S}^1. \tag{2.2}$$

Since $\sigma(U(1))$ is a closed subset of $\mathbb{S}^1$, the existence of a point outside it implies the existence of a whole open interval outside of it, which is called *a spectral gap*. In contrast to the parameterization of the domain of H, here we rather use $\mathbb{S}^1 \cong \{z \in \mathbb{C} \,|\, |z| = 1\}$.

The main point of the present paper is that even when (2.2) does not hold, a weaker gap, called *a mobility gap*, may exist, which still allows for the definition of topological indices.

Definition 2.4 (*Mobility gap*). The interval $\Delta \subseteq \mathbb{S}^1$ is a mobility gap for $U(1)$ iff (1) there is some constant $\mu > 0$ such that for any $a \in \ell^1(\mathbb{Z}^d)$ there is some constant $C_a < \infty$ such that we have

$$\sup_{g\in B_1(\Delta)} \|g(U(1))_{x,y})\| \leq C_a|a(x)|^{-1}\mathrm{e}^{-\mu\|x-y\|} \qquad (x, y \in \mathbb{Z}^d) \tag{2.3}$$

where $B_1(\Delta)$ is the space of all Borel maps $g : \mathbb{S}^1 \to \mathbb{C}$ which are constant outside of Δ and obey $|g(z)| \leq 1$ for all $z \in \mathbb{S}^1$, and (2) all eigenvalues of $U(1)$ within Δ are of finite degeneracy.

Remark 2.5. Equation (2.3), by analogy with [10,15] for static systems, corresponds to an almost sure consequence of a probabilistic analysis in which (strong, dynamical) localization is established. Such proofs of localization for unitary models have so far been established in, e.g., [5,17,19], but for a supremum over continuous maps $C_1(\mathbb{S}^1)$ multiplied by χ_Δ instead of $B_1(\Delta)$ (without discussing finite degeneracy). There exists also a disordered Floquet topological toy model for which Definition 2.4 is valid. See "Appendix A.1" for a more detailed discussion.

Remark 2.6. The reason this condition is associated with dynamical localization becomes clear by considering the supremum over the family of functions $\mathbb{S}^1 \ni \lambda \mapsto \lambda^n\chi_\Delta(\lambda)$ indexed by $n \in \mathbb{N}$. Consequently $U(1)$ has pure point spectrum within Δ, due to a RAGE theorem analogue for unitaries [11, Theorem 3.2] in the deterministic setting, [17, Lemma 6.1] in the probabilistic. This is also detailed in "Appendix A.2."

Remark 2.7. Ignoring the supremum over g, decay condition (2.3) is weaker than locality condition (2.1) due to the presence of a which means that while the matrix elements decay in the off-diagonal direction, the rate is nonuniform in the diagonal direction. In the language of [26], we assume the operator $U(1)$ is *SUDL* within Δ. We call this decay property *weakly local*; see Definition 3.1. This is the best almost sure consequence that probability theory can provide (see Proposition A.1), as was formulated, e.g., in [10, Eq. (1.2)]. The reason we introduce the term weakly local rather than use SUDL is to emphasize the form of decay of the matrix elements of various operators rather than the dynamical properties of physical systems.

Remark 2.8. We need to require finite degeneracy for the eigenvalues in Δ because here we start with a deterministic assumption; within a probabilistic model such a zero-one law would come out as is usual for Anderson localization proofs (see [31]).

Remark 2.9. (2.2) implies Definition 2.4 via the Combes–Thomas estimate, so that a spectral gap is also a mobility gap and we may treat both by treating only the mobility gap.

The Edge Sample. In the edge picture the Hilbert space is $\mathcal{H}_\mathrm{E} = \ell^2(\mathbb{N} \times \mathbb{Z}^{d-1}) \otimes \mathbb{C}^N$ describing independent electrons on a half-space. The canonical embedding $\iota : \mathcal{H}_\mathrm{E} \hookrightarrow \mathcal{H}$ and truncation $\iota^* : \mathcal{H} \twoheadrightarrow \mathcal{H}_\mathrm{E}$ satisfy $\iota^*\iota = \mathbb{1}$ on $\mathcal{H}_\mathrm{E}$ and $\iota\iota^* = P_1$ where $P_1 := \Theta(X_1)$ is the projection in $\mathcal{H}$ onto states supported in the half-space, with X_i the ith component of the position operator X and Θ the step function. For A acting on $\mathcal{H}$ we denote the corresponding truncated operator on $\mathcal{H}_\mathrm{E}$ by $\widehat{A} := \iota^* A \iota$. In particular the edge Hamiltonian is

$$H_\mathrm{E}(t) := \widehat{H(t)} \tag{2.4}$$

corresponding to Dirichlet boundary condition, although other conditions could be implemented in principle. H_E inherits some properties of H; in particular, it satisfies (2.1), and generates a unitary propagator U_E on $\mathcal{H}_\mathrm{E}$ through $\mathrm{i}\dot{U}_\mathrm{E} = H_\mathrm{E} U_\mathrm{E}$ and $U_\mathrm{E}(0) = \mathbb{1}$. All these properties rely only on the fact that H is local by (2.1) and not on the existence of (any) gap of $U(1)$, so they remain true in the mobility gap regime.

In what follows the topological indices are defined through the use of *switch functions* [9] $\Lambda : \mathbb{Z} \to \mathbb{R}$ such that $\Lambda(n) = 1$ (resp. 0) for n large and positive (resp. negative). We denote by Λ the corresponding multiplication operator on $\ell^2(\mathbb{Z})$ and by $\Lambda_i = \Lambda(X_i)$ a switch function in direction i acting on $\ell^2(\mathbb{Z}^d)$ or $\ell^2(\mathbb{N} \times \mathbb{Z}^{d-1})$.

2.1. The Relative Construction

Here we finally specify to the case $d = 2$ and no symmetry. For this case, the bulk-edge correspondence is established in [16] first when $U(1) = \mathbb{1}$ (so $\mathbb{S}^1 \backslash \{1\}$ is a "special" spectral gap) and then when $U(1)$ has a general spectral gap. The latter case was reduced to the first one by constructing a *relative evolution*, generated by an effective Hamiltonian. It turns out that the same procedure can be followed in the mobility gap regime. The effective Hamiltonian is defined through a logarithm of the one-period propagator

$$H_\lambda := \mathrm{i} \log_\lambda(U(1)) \tag{2.5}$$

where $\lambda \in \Delta$ is chosen inside the mobility gap and used as a branch cut for the (principal) logarithm. We show in Corollary 3.8 that H_λ is *weakly local* as a weaker property of being local [i.e., satisfying (2.1)]. Note that in the spectral gap case the discontinuity of the logarithm, which is otherwise analytic, may be ignored since it occurs out of the spectrum of $U(1)$; since analytic functions of local operators are local [1, Appendix D], the logarithm is local too. For us, however, H_λ is merely weakly local via localization. This is enough to define the indices, as we shall see.

For two operators $A, B : [0, 1] \to \mathcal{B}(\mathcal{H})$, not necessarily periodic, we define the concatenation in time $A\#B : [0, 1] \to \mathcal{B}(\mathcal{H})$ by

$$(A\#B)(t) = \begin{cases} A(2t) & (0 \leq t \leq 1/2), \\ B(2(1-t)) & (1/2 \leq t \leq 1). \end{cases} \tag{2.6}$$

The two operators occur consecutively in time, the second backward. The relative bulk Hamiltonian is then defined by $H^{\mathrm{rel}} := 2(H\#H_\lambda)$. The effective Hamiltonian being time-independent, its unitary propagator is $U_\lambda(t) = \mathrm{e}^{-\mathrm{i}tH_\lambda}$

and satisfies $U_\lambda(1) = U(1)$ by construction. It follows that the relative evolution generated by H^{rel}, $U^{\mathrm{rel}} = U\#U_\lambda$, satisfies $U^{\mathrm{rel}}(1) = \mathbb{1}$. Similarly, the relative edge Hamiltonian is defined by $H^{\mathrm{rel}}_{\mathrm{E}} := 2(\widehat{H}\#\widehat{H_\lambda}) = \widehat{H^{\mathrm{rel}}}$, which generates $U^{\mathrm{rel}}_{\mathrm{E}}$. Note that $\widehat{H_\lambda}$ is the truncation of H_λ and *not* the logarithm of $U_{\mathrm{E}}(1)$ for which we do not assume a mobility gap to exist. In particular $U_{\mathrm{E}}(1) \neq \mathrm{e}^{-\mathrm{i}\widehat{H_\lambda}}$ so that $U^{\mathrm{rel}}_{\mathrm{E}}$ is not given by their concatenation.

Finally recall the noncommutative three-dimensional winding W defined for a unitary loop $\mathfrak{U} : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ is given by the formula

$$W(\mathfrak{U}) \equiv -\frac{1}{2}\int_0^1 \mathrm{d}t\, \varepsilon_{\alpha\beta}\, \mathrm{tr}\, \dot{\mathfrak{U}}\mathfrak{U}^*\mathfrak{U}_{,\alpha}\mathfrak{U}^*\mathfrak{U}_{,\beta}\mathfrak{U}^*. \tag{2.7}$$

Here and in what follows, we use the shorthand notation for the noncommutative spatial derivative in direction j of an operator A as $A_{,j} := \partial_j A \equiv -\mathrm{i}[\Lambda_j, A] \equiv -\mathrm{i}[\Lambda(X_j), A]$, since $X_j \mapsto \mathrm{i}\partial_{k_j}$ under the Fourier series. We also use the shorthand $\varepsilon_{\alpha\beta}$ for the totally antisymmetric tensor and repeating indices follow the summation convention for the two spatial directions $\alpha = 1, 2$.

(2.7) is equivalent to a pairing between the K_1 class defined by $\mathfrak{U}$ and a three-dimensional Chern character, as detailed [24]. The normalization is chosen such that W takes values in $\mathbb{Z}$.

Then the zeroth step in our analysis is:

Theorem 2.10. *Under Assumptions 2.1 and 2.2, additionally assuming Definition 2.4 holds for $U(1)$:* (1) *the bulk index $\mathcal{I} \equiv W(U^{\mathrm{rel}})$ is finite, integer valued and independent of the choices of switch functions and branch cut $\lambda \in \Delta$;* (2) *the edge index*

$$\mathcal{I}_{\mathrm{E}} \equiv \mathrm{i}\,\mathrm{tr}\, U^{\mathrm{rel}}_{\mathrm{E}}(1)^*\partial_2 U^{\mathrm{rel}}_{\mathrm{E}}(1) \tag{2.8}$$

is finite, integer valued and independent of the choice of switch function; (3) *the bulk-edge correspondence holds*

$$\mathcal{I} = \mathcal{I}_{\mathrm{E}}. \tag{2.9}$$

If H_λ were local, then so would be U_λ, U^{rel} and $U^{\mathrm{rel}}_{\mathrm{E}}$ and this theorem would be already covered by [16, Theorem 3.8]. Here instead we need to adapt the proof to weakly local operators. In particular we need to show that $\partial_j A$ also have a so-called confining property when A is weakly local, so that expressions involved in (2.7) and (2.8) are trace-class. Apart from that point, the other properties of $\mathcal{I}$ and $\mathcal{I}_{\mathrm{E}}$ as well as the bulk-edge correspondence follow the same route as in [16]. The proof of Theorem 2.10 is detailed in Sect. 3.4.

Remark 2.11. Even if the system has a gap, it is of interest to probe the system when placing the branch cut of the logarithm within the localized spectrum, in analogy with the explanation of the plateaus of the IQHE. Thus, if one day we could experimentally determine the position of the branch cut, our results would explain the corresponding plateaus which should be measured in $\mathcal{I}$.

In contrast to the integer quantum Hall effect (IQHE henceforth) where the mobility gap bulk-edge correspondence is quite different in the spectral and mobility gap regime (cf. [9] vs. [10]), the relative construction works similarly for both cases in Floquet topological insulators, once one generalizes from local to weakly local operators and shows the desired properties of the discontinuous logarithm, Corollary 3.8. The surprising ease of this extension to the mobility gap regime is almost suspicious, so that one starts wondering whether the relative construction is really the correct way to define the edge topology.

Indeed, one objection to the relative construction is the following: in defining $\mathcal{I}_{\mathrm{E}}$, the truncated generator of the bulk relative propagator, $H_{\mathrm{E}}^{\mathrm{rel}}$ (which depends on the logarithm of the *bulk* evolution), and not just the truncated bulk Hamiltonian, H_E, has been used, so that Theorem 2.10 actually connects between $\mathcal{I}$ and an edge index which contains significant information from the bulk. It is thus legitimate to ask for a more independent formulation where bulk and edge indices are strictly separated already at the level of the Hamiltonians, without intertwining their evolutions. This was our main motivation to look for an alternative approach for defining the indices, whose conclusion we show in the stretch function construction, see Sect. 2.3. We note nevertheless that the latter indices coincide with those of the relative one, as we show in Theorem 2.19, so that the relative construction remains equally valid.

2.2. Completely Localized Systems

As preparation for Sect. 2.3, and also of independent interest, we discuss one possible scenario in which the relative construction may be avoided when defining the topological indices.

Interestingly, formula (2.7) is finite also for unitary maps which are *not* periodic (whose domain is $[0,1]$ rather than $\mathbb{S}^1$), but is an integer (and hence possibly an index) only when the map is periodic [16, Proposition 3.3], *e.g.,* for the relative evolution U^{rel} which by construction has $U^{\mathrm{rel}}(1) = U^{\mathrm{rel}}(0) = \mathbb{1}$. In this section we propose an alternative definition of the bulk index when the physical evolution is not periodic, $U(1) \neq \mathbb{1}$, avoiding the relative construction.

Definition 2.12 (*Completely localized unitaries*). We call a unitary $\mathfrak{U} \in \mathcal{U}(\mathcal{H})$ *completely localized* iff its whole spectrum is one mobility gap in the sense of Definition 2.4, except possibly for one point of $\mathbb{S}^1$.

Namely the mobility gap arc Δ for $\mathfrak{U}$ as in Definition 2.4 is $\Delta = \mathbb{S}^1$ or $\Delta = \mathbb{S}^1 \backslash \{z_0\}$ for some $z_0 \in \mathbb{S}^1$. In particular $\mathfrak{U}$ has only pure point spectrum. We denote by $\mathcal{E} \subset \mathbb{S}^1$ its (countable) set of eigenvalues and by $P_z \equiv \chi_{\{z\}}(\mathfrak{U})$ the associated spectral projection onto an eigenvalue $z \in \mathcal{E}$. By Definition 2.4 all z's have $\dim \operatorname{im} P_z < \infty$ except possibly for P_{z_0}. The latter appears in the approach of Sect. 2.3, but never carries any topology (Chern number) by construction. In principle we could extend Definition 2.12 to a finite number of infinite degenerate eigenvalues, but we would then have to require that each of them has a trivial topology. This is the so-called anomalous phase [29,34]. Here we stick to a single z_0 to avoid extra-technical hypothesis and streamline the presentation.

Inspired by [22], we define the orbital magnetization corresponding to evolutions U whose endpoints $U(1)$ are completely localized.

Definition 2.13 (*Magnetization*). For the evolution $U : [0,1] \to \mathcal{B}(\mathcal{H})$ (which need not be periodic), define the magnetization operator

$$m(U) := -\int_0^1 \mathrm{d}t\, \mathbb{I}\mathrm{m}(U^* \Lambda_1 H \Lambda_2 U)\,, \tag{2.10}$$

where $\mathbb{I}\mathrm{m} A \equiv \frac{1}{2i}(A - A^*)$ is the imaginary part of A, and the total orbital magnetization (a number) as

$$M(U) := \sum_{z \in \mathcal{E}} \mathrm{tr}\, P_z m(U) P_z. \tag{2.11}$$

where P_z are the spectral projections onto the eigenvalues of $U(1)$, and $\mathcal{E}$ is the corresponding set, as above.

Note that the integrand in $m(U)$ can be rewritten

$$-\mathbb{I}\mathrm{m}(U^* \Lambda_1 H \Lambda_2 U) = \frac{1}{2}\big(U^* \Lambda_1 U \partial_t (U^* \Lambda_2 U) - 1 \leftrightarrow 2\big). \tag{2.12}$$

Pretending $\Lambda_i \sim X_i$, the position operator, the latter expression is the third component of $(1/2)\,\mathbf{X}(t) \times \dot{\mathbf{X}}(t)$, which (in natural units) corresponds to the orbital magnetization. The physical aspects of m and M, including a proposal for an experimental realization in cold atoms, were studied in detail in [22].

Theorem 2.14. *If $U(1)$ is completely localized in the sense of Definition 2.12, the magnetization $M(U)$ is finite, integer valued and independent of the choice of switch functions. Moreover $M(U) = \mathcal{I}$. If $U(1) = \mathbb{1}$, then $M(U) = W(U)$, and if H is time-independent, then $M(U) = 0$.*

Thus for completely localized systems $U(1)$ the computation of the index $M(U)$ does not require the relative construction, but the price to pay is that operator $m(U)$ is not trace-class anymore. However, it is summable in the eigenbasis of $U(1)$, with sum $M(U)$. We emphasize that a mobility gap also applies when $\sigma(U(1)) \neq \mathbb{S}^1$; i.e., for a spectrally gapped system obeying Definition 2.12 the relative construction can also be circumvented using magnetization.

The strategy of the proof is to use the relative construction by choosing an effective Hamiltonian H_λ for an arbitrary $\lambda \in \Delta$. As detailed in Sect. 4 we get

$$\mathcal{I} = M(U) - M(U_\lambda). \tag{2.13}$$

The effective Hamiltonian being time-independent, its corresponding magnetizations simplify to

$$M(U_\lambda) = -\sum_{z \in \mathcal{E}} \mathrm{tr}\,\big(P_z\, \mathbb{I}\mathrm{m}(\Lambda_1 H_\lambda \Lambda_2)\, P_z\big) \tag{2.14}$$

and the task is to show that this expression is well defined and vanishes. Note that a similar expression already appeared in the context of the IQHE as an extra term required to establish the bulk-edge correspondence of Hall conductivity in the mobility gap regime [10]. An interpretation in terms of magnetization was also proposed there for time-independent Hamiltonians. However, in that case the magnetization is not vanishing because the mobility gap is not the entire spectrum.

For completely localized systems, it is also possible to define an edge index without the relative construction, also related to the previous one through the bulk-edge correspondence.

Theorem 2.15. *Let $U : [0,1] \to \mathcal{U}(\mathcal{H})$ be a bulk evolution whose endpoint $U(1)$ is completely localized as in Definition 2.12, and $\mathfrak{U}_E : [0,1] \to \mathcal{U}(\mathcal{H}_{\mathrm{E}})$ be* any *edge evolution (e.g., U_E) such that $\sup_{t\in[0,1]} \|\partial_2(\mathfrak{U}_E(t) - \widehat{U}(t))\|_1 < \infty$.*

The time-averaged charge pumping

$$P_{\mathrm{E}}(\mathfrak{U}_E(1)) := \lim_{n\to\infty} \lim_{r\to\infty} \frac{1}{n} \mathrm{i}\,\mathrm{tr}\left(((\mathfrak{U}_E(1))^*)^n \partial_2 \mathfrak{U}_E(1)^n \Lambda_{1,r}^{\perp} \right) \tag{2.15}$$

where $\Lambda_{1,r}^{\perp} = \chi_{\leq r}(X_1)$ exists, is finite, integer valued and independent of the choice of switch function. Moreover the bulk-edge correspondence holds

$$P_{\mathrm{E}}(\mathfrak{U}_E(1)) = M(U). \tag{2.16}$$

The physical interpretation of P_{E} is a quantized pumping of charges, counted through $((\mathfrak{U}_E(1))^*)^n \Lambda_2 \mathfrak{U}_E(1)^n - \Lambda_2$ that is confined at the edge [16,34]: if the corresponding $U(1) = \mathbb{1}$, the pumping is quantized within a single cycle, whereas for completely localized $U(1)$, the quantization is true on average over time only and coincides with magnetization.

2.3. The Stretch Function Construction

The previous section extends the definition of bulk and edge indices beyond $U(1) = \mathbb{1}$ without using the relative construction. However, it only works for completely localized systems. Here we finally give a recipe for reducing the general situation ($\Delta \neq \mathbb{S}^1$) to the one described by Definition 2.12, which completes the story and results in a new approach to define the topology (in both spectrally and mobility gapped cases).

Definition 2.16 (*Stretch function*). Let Ω be an arc in $\mathbb{S}^1 \subseteq \mathbb{C}$. A stretch function $F_\Omega : \mathbb{S}^1 \to \mathbb{S}^1$ is a smooth function with $F_\Omega(z) = 1$ for $z \in \mathbb{S}^1 \backslash \Omega$, which winds once:

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{S}^1} F_\Omega(z)^{-1} \mathrm{d}F_\Omega(z) = 1. \tag{2.17}$$

The role of F is to stretch the arc Ω onto the entire circle except the point at 1, which is the image of $\mathbb{S}^1 \backslash \Omega$. In particular, if $\Omega = \mathbb{S}^1 \backslash \{1\}$, then the identity $F_\Omega(z) = z$ is an appropriate stretch function. We think of F_Ω as a function which selects the appropriate (mobility or spectral) gap, analogous to (a smooth deformation of) the function $\chi_{(-\infty, E_F)}$ with $E_F \in \mathbb{R}$ the Fermi energy, for the IQHE. For a given interval Ω we define

$$V(t) := F_\Omega(U(t)), \qquad V_{\mathrm{E}}(t) := F_\Omega(U_{\mathrm{E}}(t)) \quad (t \in [0,1]) \tag{2.18}$$

via the functional calculus. V and V_{E} are two unitary evolutions on $\mathcal{H}$ and $\mathcal{H}_{\mathrm{E}}$, respectively, that satisfy $V(0) = V_{\mathrm{E}}(0) = \mathbb{1}$.

Lemma 2.17. *If $\Omega = \Delta$ is a mobility gap of $U(1)$, then $V(1)$ is completely localized in the sense of Definition 2.12, with mobility gap $\mathbb{S}^1 \backslash \{1\}$.*

This is just a consequence of Definitions 2.4 and 2.16, in particular

$$B_1(\mathbb{S}^1) \circ F_{\Delta'} \subseteq B_1(\Delta) \tag{2.19}$$

for any $\Delta' \subsetneq \Delta$ which is a proper sub-arc. Thus the supremum in (2.3) over the LHS set is bounded by the supremum over the RHS set. Moreover $\{1\}$ is the image of $\mathbb{S}^1 \backslash \Delta$ by F_Δ, so it is not a finite degenerate eigenvalue of $V(1)$ in general. As noticed above, this particular point can be present in completely localized systems and is not problematic for the definition of magnetization and pumping indices.

Corollary 2.18. *If Δ is a mobility gap of $U(1)$ and F_Δ is a stretch function, $V(1)$ is completely localized so that*

$$\mathcal{I}' := M(V), \qquad \mathcal{I}'_{\mathrm{E}} := P_{\mathrm{E}}(V_{\mathrm{E}}(1)) \tag{2.20}$$

are well-defined indices according to Theorems 2.14 and 2.15. In particular the bulk-edge correspondence holds: $\mathcal{I}' = \mathcal{I}'_{\mathrm{E}}$.

Thus the composition of stretch function and magnetization or quantized pumping provides indices for any $U(1)$ with mobility gap Δ and circumvents the relative construction. Note that if Δ is a spectral gap, then $V(1) = \mathbb{1}$ so that $\mathcal{I}' = W(V)$ and $\mathcal{I}'_{\mathrm{E}}$ coincides with the edge index definition of [30] where a particular stretch function was used.

The proof of Corollary 2.18 is not straightforward as one has to check that the underlying assumptions of Theorems 2.14 and 2.15 are satisfied for V and V_{E}, namely that all the properties of U and U_{E} are correctly transferred through the stretch function construction. This is done in Sect. 5.1.

It is finally legitimate to ask whether the two constructions coincide since the relative indices, $\mathcal{I}$ and $\mathcal{I}_{\mathrm{E}}$, and the ones defined through stretch functions, $\mathcal{I}'$ and $\mathcal{I}'_{\mathrm{E}}$, are both defined in a general mobility gap situation.

Theorem 2.19. *If $U(1)$ has a mobility gap Δ and F_Δ is a stretch function, then*

$$\mathcal{I}' = \mathcal{I}. \tag{2.21}$$

In particular $\mathcal{I}'$ is independent of the choice of stretch function. Moreover by the respective bulk-edge correspondences one infers $\mathcal{I}'_{\mathrm{E}} = \mathcal{I}_{\mathrm{E}}$.

The poof is somewhat delicate. By Corollary 2.18 we know that $\mathcal{I}'$ coincides with the relative construction applied to V, namely $M(V) = W(V^{\mathrm{rel}})$. But in order to show that $W(V^{\mathrm{rel}}) = W(U^{\mathrm{rel}}) \equiv \mathcal{I}$ we use a smooth deformation of the stretch function from F_Δ to the identity. Then we have to show that W stays continuous under this deformation. The only other *deterministic* proof of continuity for indices in the mobility gap regime so far was in [10] for the deformation corresponding to tuning the Fermi energy E_F within

Figure 2. Operator content in the relative construction (**a**) and stretch function approach (**b**)

the mobility gap. Thus the proof of Theorem 2.19 provides another continuity proof for the index W along a different path and paves the way for further development of locally constant indices at strong disorder.

Remark 2.20. It is worth pointing out that it is Theorem 2.19 which implies that $\mathcal{I}$ is independent of the choice of branch cut $\lambda \in \Delta$ (part of item (a) of Theorem 2.10), since $\mathcal{I}'$ is manifestly independent of λ.

Summary. To conclude, the objection at the end of Sect. 2.1 was that the index $\mathcal{I}_{\mathrm{E}}$ from the relative construction is calculated using $U_{\mathrm{E}}^{\mathrm{rel}} = U_{\mathrm{E}} \# \mathrm{e}^{-\mathrm{i}\cdot \widehat{H}_\lambda}$ that contains a truncation of the effective *bulk* Hamiltonian (Fig. 2a). Instead the stretch function approach removes this intertwining since the index $\mathcal{I}'_{\mathrm{E}}$ is calculated using $V_{\mathrm{E}} = F_\Delta(U_{\mathrm{E}})$ (Fig. 2b). The only knowledge here from the bulk is the position of the gap $\Delta \subset \mathbb{S}^1$. The latter approach is then more natural for the bulk-edge correspondence, although the two are equivalent by Theorem 2.19.

3. Bulk-Edge Correspondence for the Relative Evolution

The bulk-edge correspondence was established in [16] in the case where $U(1)$ has a spectral gap. In that case all the operators involved are local in the sense of Assumption 2.2. In particular U, H_{E} and U_{E} are local, uniformly in $t \in [0,1]$ (see [16, Proposition 4.7]). These properties are *independent* of the existence of (any) gap of $U(1)$ since they probe the dynamics only in a compact time interval, and hence, at a finite distance from the spectrum, so they remain true also in our setting.

Furthermore when the branch cut of the logarithm is taken inside a spectral gap, H_λ (and thus U_λ) are also local. This is not the case anymore in the mobility gap regime. However the logarithm still has some off-diagonal decay properties that suffice to generalize the proof of the bulk-edge correspondence in the relative construction, as we shall now show.

3.1. The Weakly Local Star Algebra

Definition 3.1 (*Weakly local operators*). The operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *weakly local* iff there is some $\nu \geq 0$ such that for any $\mu > 0$ sufficiently large there is some constant $C_\mu < \infty$ with

$$\|A_{xy}\| \leq C_\mu (1 + \|x-y\|)^{-\mu} (1 + \|x\|)^\nu \quad (x, y \in \mathbb{Z}^d). \tag{3.1}$$

In our application, the sufficiently large value of μ will usually be 2 (Corollary A.8) and fixed throughout for all operators. However, to discuss the algebraic properties we allow this value to be arbitrary.

Remark 3.2. In the above definition, when $\nu = 0$, we call the operator strictly polynomially local, as opposed to the condition in Assumption 2.2 which is strictly exponentially local or just local.

In summary, the "weak" qualifier refers to the possible diagonal blowup, which is a consequence of deterministic conditions of localization, whereas the "polynomial" qualifier refers to having applied the smooth functional calculus on any (localized or not) local operator, see Corollary A.8.

This form of any-power-polynomial decay that comes from the smooth functional calculus is associated with probing the system at infinite times in the weakest possible way (cf. the Borel bounded functional calculus which probes this directly, and only localization could guarantee that it then takes local operators to (exponentially) weakly local operators).

Remark 3.3. Clearly Definition 2.4 entails that $g(U(1))$ is weakly local uniformly as g varies in $B_1(\Delta)$, by picking, e.g., $a(x) := (1 + \|x\|)^{-(d+1)}$ in d dimensions.

Remark 3.4. One could choose various other ways to encode the off-diagonal decay of an operator. Compare with [10, Section 3.3], which illustrates how to encode (exponential) decay either with bounds on matrix elements or by estimates on the operator norm of a space-weighted version of the operator. Here we refrain from reformulating (3.1) in different ways for polynomial decay in order to reach our goal as quickly as possible.

Lemma 3.5. *The transpose of a weakly local operator is again weakly local.*

Proof. Assume A obeys Definition 3.1. Then picking $\mu > \nu$,

$$\begin{aligned}\|A_{xy}\| &\leq C_\mu (1 + \|x-y\|)^{-\mu} (1 + \|x\|)^\nu \\ &\leq C_\mu (1 + \|x-y\|)^{-(\mu-\nu)} (1 + \|y\|)^\nu (1 + \|y\|)^{-\nu} (1 + \|x-y\|)^{-\nu} (1 + \|x\|)^\nu.\end{aligned}$$

But now, $\frac{1+\|x\|}{(1+\|y\|)(1+\|x-y\|)} \leq \frac{1+\|x\|}{1+\|y\|+\|x-y\|}$ and using the reverse triangle inequality, $\|x-y\| \geq \|x\| - \|y\|$ so that this fraction is smaller than or equal to one. So is its νth power.

We find that $\|A_{xy}\| \leq C_{\mu+\nu} (1 + \|y-x\|)^\mu (1 + \|y\|)^\nu$ for all μ sufficiently large; in other words, A^T is weakly local (though with different constants). □

Lemma 3.6. *The weakly local operators form a star algebra.*

Proof. Due to Lemma 3.5 the linearity of taking matrix elements and the triangle inequality of the matrix norm, we only verify the product property. Let A, B be two given weakly local operators with constants C_μ^A, C_μ^B, respectively. Then for any $\mu > 0$ sufficiently large (for both A and B) and $\nu := \max(\{\nu_A, \nu_B\})$ we have,

$$\begin{aligned}\|(AB)_{xy}\| &\leq \sum_z \|A_{xz}\|\|B_{zy}\| \\ &\leq \sum_z C_\mu^A (1+\|x-z\|)^{-\mu}(1+\|x\|)^\nu C_\mu^B (1+\|z-y\|)^{-\mu}(1+\|z\|)^\nu \\ &\leq C_\mu^A C_\mu^B (1+\|x\|)^\nu \sum_z (1+\|x-z\|)^{-\mu}(1+\|z-y\|)^{-\mu}(1+\|z\|)^\nu. \end{aligned} \tag{3.2}$$

Now note that $(1+\|x-z\|)(1+\|z-y\|) \geq 1+\|x-z\|+\|z-y\| \geq 1+\|x-y\|$ so that

$$\begin{aligned}\|(AB)_{xy}\| \leq C_\mu^A C_\mu^B &(1+\|x\|)^\nu (1+\|x-y\|)^{-\mu/2} \\ &\cdot \sum_z (1+\|x-z\|)^{-\mu/2}(1+\|z-y\|)^{-\mu/2}(1+\|z\|)^\nu. \end{aligned}$$

Assume further that $\mu > 2\nu$ has been chosen. Then $(1+\|x-z\|)^{-\mu/2}(1+\|z\|)^\nu \leq (\frac{1+\|z\|}{1+\|x-z\|})^\nu \leq (1+\|x\|)^\nu$ by $(1+\|x\|)(1+\|x-z\|) \geq 1+\|x\|+\|x-z\| \geq 1+\|z\|$. We conclude that

$$\begin{aligned}\|(AB)_{xy}\| &\leq C_\mu^A C_\mu^B (1+\|x\|)^{2\nu}(1+\|x-y\|)^{-\mu/2} \sum_z (1+\|z-y\|)^{-\mu/2} \\ &\leq C_\mu^A C_\mu^B \left(\sum_{z\in\mathbb{Z}^d} (1+\|z\|)^{-\mu/2}\right)(1+\|x-y\|)^{-\mu/2}(1+\|x\|)^{2\nu}. \end{aligned}$$

If now we also pick μ large enough so that the sum in the first parenthesis is finite [e.g., $\mu > 2(d+1)$], then we find our result. □

3.2. The Logarithm is Weakly Local

For the rest of this section we assume that there is some nonempty interval $\Delta \subseteq \mathbb{S}^1$ which is a mobility gap for $U(1)$ in the sense of Definition 2.4. We further assume that $\lambda \in \Delta$, where λ is the position of the branch cut used in the definition of H_λ from (2.5).

Lemma 3.7. *$f(U(1))$ is also weakly local for all bounded $f : \mathbb{S}^1 \to \mathbb{C}$ which are smooth outside of Δ and piecewise smooth with a finite number of jump discontinuities within Δ.*

Proof. Assume that f has *one* jump discontinuity at some $\lambda_0 \in \Delta$ and is otherwise smooth. Since Δ is an interval, pick some other $\lambda_1 \in \Delta\backslash\{\lambda_0\}$. For each $\Omega \in \{(\lambda_0,\lambda_1), \mathbb{S}^1\backslash(\lambda_0,\lambda_1)\} =: S$, the restriction $f|_\Omega : \Omega \to \mathbb{C}$ is smooth and so has a smooth extension $f_\Omega^s : \mathbb{S}^1 \to \mathbb{C}$ (that is $f|_\Omega = f_\Omega^s|_\Omega$). Hence $\chi_\Omega f = \chi_\Omega f_\Omega^s$ and $f = \sum_{\Omega\in S} f\chi_\Omega = \sum_{\Omega\in S} f_\Omega^s \chi_\Omega$. We note that $U(1)$ is (strictly exponentially) always local, regardless of localization, via the Schrödinger equation and Assumption 2.2 (see [16, Proposition 4.7]). Moreover, any smooth function of a local unitary operator is also weakly local by Corollary A.8. On the other hand, $\chi_\Omega \in B_1(\Delta)$, so the corresponding operator is weakly local by the assumption entailed in Definition 2.4. Thus Lemma 3.6 allows us to conclude about the whole of f. □

Corollary 3.8. *Both H_λ and U_λ are weakly local.*

Proof. Since $\lambda \in \Delta$, we get that $\log_\lambda$ is analytic except for a jump discontinuity within Δ as f of Lemma 3.7. Now $U_\lambda(t) = (\mathrm{e}^{-\mathrm{i}t\,\cdot} \circ \mathrm{i}\log_\lambda)(U(1))$, $\mathrm{e}^{-\mathrm{i}t\,\cdot}$ is analytic, so that for fixed t, the composition $\mathrm{e}^{-\mathrm{i}t\,\cdot} \circ \mathrm{i}\log_\lambda$ is again analytic apart from one jump discontinuity within Δ, which is covered by Lemma 3.7. □

3.3. The Weakly Local and Confined Two-Sided Ideal

Definition 3.9 (*Weakly Local and Confined Operators*) The operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *weakly local and confined in direction* i for $i = 1, \ldots, d$ iff there is some $\nu > 0$ such that for any $\mu > 0$ sufficiently large there is some constant $C_\mu < \infty$ with

$$\|A_{xy}\| \leq C_\mu (1 + \|x - y\|)^{-\mu} (1 + |x_i|)^{-\mu} (1 + \|x\|)^\nu \qquad (x, y \in \mathbb{Z}^d) \tag{3.3}$$

We see that adding the "confined" condition guarantees that the operator has also diagonal decay at least in one direction.

Lemma 3.10. *If A is weakly local and confined in direction i, then so is A^T.*

Proof. Since $\|x\| \geq |x_i|$, we have $(1 + \|x\|)^{-\mu} \leq (1 + |x_i|)^{-\mu}$, and hence,

$$\begin{aligned} \|A_{xy}\| &\leq C_\mu (1 + \|x - y\|)^{-\mu/2} (1 + |x_i - y_i|)^{-\mu/2} (1 + |x_i|)^{-\mu} (1 + \|x\|)^\nu \\ &\leq C_\mu (1 + \|x - y\|)^{-\mu/2} ((1 + |x_i - y_i|)(1 + |x_i|))^{-\mu/2} (1 + \|x\|)^\nu \end{aligned}$$

Now, $(1 + |x_i - y_i|)(1 + |x_i|) \geq 1 + |x_i - y_i| + |x_i| \geq 1 + |y_i|$, so that

$$\|A_{xy}\| \leq C_\mu (1 + \|x - y\|)^{-\mu/2} (1 + |y_i|)^{-\mu/2} (1 + \|x\|)^\nu$$

Now we can follow the same procedure as in Lemma 3.5 to replace the $(1+\|x\|)^\nu$ factor with a $(1 + \|y\|)^\nu$ (by worsening the constants). □

Lemma 3.11. *If A is weakly local and confined in direction i, then for all μ sufficiently large and ν as in Definition 3.9 we have $\|(1+\|X\|)^{-\nu}(1+|X_i|)^\mu A\| < \infty$.*

Proof. We use Holmgren's bound (with $\max_{x\leftrightarrow y} A_{xy} := \max(\{A_{xy}, A_{yx}\})$) and the assumed bound in Definition 3.9 to get

$$\begin{aligned} \|(1 + \|X\|)^{-\nu}(1 + |X_i|)^\mu A\| &\leq \max_{x\leftrightarrow y} \sup_y \sum_x \|((1 + \|X\|)^{-\nu}(1 + |X_i|)^\mu A)_{xy}\| \\ &\leq \max_{x\leftrightarrow y} \sup_y \sum_x (1 + \|x\|)^{-\nu}(1 + |x_i|)^\mu \|A_{xy}\| \\ &\leq \max_{x\leftrightarrow y} \sup_y \sum_x (1 + \|x\|)^{-\nu}(1 + |x_i|)^\mu \\ &\quad \cdot C_\mu (1 + \|x - y\|)^{-\mu}(1 + |x_i|)^{-\mu}(1 + \|x\|)^\nu \end{aligned}$$

$$= C_\mu \sum_{x\in\mathbb{Z}^d} (1+\|x\|)^{-\mu} < \infty\,,$$

assuming μ is chosen sufficiently large so that this last sum is finite. □

Lemma 3.12. *The space of weakly local and confined in direction i operators forms a star-closed two-sided ideal within the star algebra of weakly local operators.*

Proof. The additive subgroup property follows by the linearity of taking matrix elements as well as the triangle inequality of the matrix norm associated with $\mathbb{C}^N$. The star closure follows due to Lemma 3.10.

Let now A be weakly local and confined in direction i and B be merely weakly local. Then pick $\mu > 0$ sufficiently large for both A and B and let $\nu := \max(\{\nu_A, \nu_B\})$, to get

$$\begin{aligned}
\|(AB)_{xy}\| &\leq \sum_z \|A_{xz}\|\|B_{zy}\| \\
&\leq \sum_z C_\mu^A (1+\|x-z\|)^{-\mu}(1+|x_i|)^{-\mu}(1+\|x\|)^{\nu_A} C_\mu^B (1+\|z-y\|)^{-\mu}(1+\|z\|)^{\nu_B} \\
&= C_\mu^A C_\mu^B (1+|x_i|)^{-\mu} \sum_z (1+\|x-z\|)^{-\mu}(1+\|x\|)^{\nu}(1+\|z-y\|)^{-\mu}(1+\|z\|)^{\nu}.
\end{aligned}$$

Everything after $(1+|x_i|)^{-\mu}$ is identical to (3.2) (after which we showed that the remainder expression is estimated as weakly local), so that we find AB is also weakly local and confined in direction i.

Since $BA = (A^*B^*)^*$, A^* is weakly local and confined in direction i, B^* is weakly local, so that by the previous paragraph, A^*B^* belongs to this ideal as well, and hence by the star closure, BA as well. □

Lemma 3.13. *If A, B are weakly local and confined in direction i, j, respectively, then AB is weakly local and confined in directions i and j simultaneously.*

Proof. Due to Lemma 3.10 we may interchange which of the indices of the matrix element we want to represent the confinement. Thus we are allowed to write, for $\mu > 0$ sufficiently large for both A and B and $\nu := \max(\{\nu_A, \nu_B\})$

$$\begin{aligned}
&\|(AB)_{xy}\| \\
&\quad\leq \sum_z \|A_{xz}\|\|B_{zy}\| \\
&\quad\leq \sum_z C_\mu^A (1+\|x-z\|)^{-\mu}(1+|x_i|)^{-\mu}(1+\|x\|)^{\nu} C_\mu^B (1+\|z-y\|)^{-\mu} \\
&\qquad\cdot (1+|y_j|)^{-\mu}(1+\|z\|)^{\nu} \\
&\quad\leq C_\mu^A C_\mu^B (1+|x_i|)^{-\mu}(1+|y_j|)^{-\mu} \sum_z (1+\|x-z\|)^{-\mu}(1+\|x\|)^{\nu} \\
&\qquad\cdot (1+\|z-y\|)^{-\mu}(1+\|z\|)^{\nu}.
\end{aligned}$$

Now, by Lemma 3.6 we know that the expression from $\sum_z$ and after is estimated by something which is weakly local. Then we may again use Lemma 3.10 to replace the $(1+|y_j|)^{-\mu}$ factor with $(1+|x_j|)^{-\mu}$. $\square$

As in [10], multiplying d weakly local and confined operators (each in a distinct direction of all possible directions in $\mathbb{Z}^d$) gives trace-class operators. Here our notion of confined is however weaker because we have merely polynomial decay, which changes very little. We denote the trace norm by $\|\cdot\|_1$.

Since this paper uses $d = 2$, that is the scope of the lemma, whose generalization to arbitrary d is straightforward.

Lemma 3.14. *If A, B are both weakly local, A also confined in direction 1 and B in direction 2, then $\|AB\|_1 < \infty$.*

Proof. Assume $\mu > 0$ is sufficiently large for both A and B and let $\nu := \max(\{\nu_A, \nu_B\})$. Using the freedom that Lemma 3.10 affords, we may estimate

$$\begin{aligned}
&\|AB\|_1 \\
&\quad \le \sum_{xyz} \|A_{xz}\|\|B_{zy}\| \\
&\quad \le \sum_{xyz} C^A_\mu (1+\|x-z\|)^{-\mu}(1+|z_1|)^{-\mu}(1+\|z\|)^{\nu} C^B_\mu (1+\|z-y\|)^{-\mu} \\
&\qquad \cdot (1+|z_2|)^{-\mu}(1+\|z\|)^{\nu}.
\end{aligned}$$

Now, $(1+|z_1|)(1+|z_2|) \ge 1+|z_1|+|z_2| \equiv 1+\|z\|$ so that we find, by summing first over z and then using translation invariance for the x and y sums,

$$\|AB\|_1 \le \left(\sum_x (1+\|x\|)^{-\mu}\right)\left(\sum_y (1+\|y\|)^{-\mu}\right)\left(\sum_z (1+\|z\|)^{-(\mu-2\nu)}\right).$$

If we pick $\mu > 0$ sufficiently large so that all three sums are finite (e.g., $\mu \ge 2\nu + d + 1$), then AB is indeed trace-class. $\square$

Lemma 3.15. *From [15, Proof of Lemma 2] we use: for any switch function $\Lambda : \mathbb{R} \to \mathbb{R}$ we have the estimate: for any $\mu > 0$ we have some $C_{\Lambda\mu} < \infty$ such that $|\Lambda(n) - \Lambda(m)| \le C_{\Lambda\mu}(1+|n-m|)^{+\mu}(1+\frac{1}{2}|n|)^{-\mu}$ for all $n, m \in \mathbb{Z}$.*

Corollary 3.16. *If A is weakly local, then $\partial_i A$ is weakly local and confined in direction i.*

Proof. We have by the previous estimate on Λ, for any $\mu' > 0$ and $\mu > 0$ sufficiently large for A,

$$\begin{aligned}
\|(\partial_i A)_{xy}\| &= |\Lambda(x_i) - \Lambda(y_i)|\|A_{xy}\| \\
&\le C_{\Lambda\mu'}(1+|x_i-y_i|)^{+\mu'}\left(1+\frac{1}{2}|x_i|\right)^{-\mu'} C_{A\mu}(1+\|x-y\|)^{-\mu}(1+\|x\|)^{\nu} \\
&\le C_{\Lambda\mu'} C_{A\mu}(1+\|x-y\|)^{-(\mu-\mu')}\left(1+\frac{1}{2}|x_i|\right)^{-\mu'}(1+\|x\|)^{\nu}.
\end{aligned}$$

By adjusting the constants we can remove the factor $\frac{1}{2}$, and then always taking the worst rate of decay we find the form of (3.3). $\square$

Corollary 3.17. *As in [15, Lemma 2] for one dimension, we have in two dimensions: if A is weakly local, then $\|(\partial_1 A)\partial_2 A\|_1 < \infty$. Moreover if A is also confined in direction i, then $\|\partial_j A\|_1 < \infty$ for $j \neq i$.*

Remark 3.18. If $(A_n)_{n\in\mathbb{N}}$ is a family of weakly local operators with a uniform estimate [i.e., the constants C, μ and ν in (3.1) do not depend on n] then $(\|(\partial_1 A_n)\partial_2 A_n\|_1)_{n\in\mathbb{N}}$ is a bounded sequence.

Proof. One goes through the entire procedure that leads to Corollary 3.17 and verifies that since there is a uniform bound on the xy matrix elements $\|(A_n)_{xy}\|$ (which does not depend on n), all estimates are uniform in n, including the final one. □

3.4. The Bulk-Edge Correspondence in the Relative Construction

The central ingredient of the bulk-edge correspondence is the relation between truncated bulk and edge propagator. For H local, the difference $D = U_{\mathrm{E}} - \widehat{U}$ is local and confined in direction 1, uniformly in $t \in [0,1]$. This result is also independent of the existence of (any) gap of $U(1)$ (see [16, Proposition 4.10]) and is generalized from local to weakly local operators.

Lemma 3.19. *Let $\mathfrak{H} : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ be some weakly local Floquet Hamiltonian, with its associated $\mathfrak{H}_{\mathrm{E}}$, $\mathfrak{U}$ and $\mathfrak{U}_E$ as in Sect. 2. Then $\mathfrak{D} := \mathfrak{U}_E - \widehat{\mathfrak{U}}$ is weakly local and confined in direction 1, uniformly in $t \in [0,1]$.*

Proof. To deal with $\mathfrak{D}$ we recall that $\widehat{\mathfrak{U}} \equiv \iota^* \mathfrak{U} \iota$. Since $\mathfrak{U}_E$ is weakly local, by Lemma 3.6 it suffices to deal with $\mathbb{1} - \iota^* \mathfrak{U}(t) \iota \mathfrak{U}_E(t)^* = \int_{s=0}^{t} \partial_s (\iota^* \mathfrak{U}(s) \iota \mathfrak{U}_E(s)^*) \mathrm{d}s$. Since all estimates about the weak locality of the involved operators are uniform in time (the time interval being compact), the weak locality and confined property of the integrand implies that of the integral. With the shorthand notation ∂ for derivative w.r.t. time and using the defining property $\mathrm{i}\partial U = HU$ and the adjoint of this equation, and finally the fact that a Hamiltonian and the semi-group which it generates commute, we have

$$\begin{aligned}
\partial(\iota^* \mathfrak{U} \iota (\mathfrak{U}_E)^*) &= \iota^* (\partial \mathfrak{U}) \iota (\mathfrak{U}_E)^* + \iota^* \mathfrak{U} \iota \partial (\mathfrak{U}_E)^* \\
&= \iota^* (-\mathrm{i} \mathfrak{H} \mathfrak{U}) \iota (\mathfrak{U}_E)^* + \iota^* \mathfrak{U} \iota (+\mathrm{i} (\mathfrak{U}_E)^* \iota^* \mathfrak{H} \iota) \\
&= -\mathrm{i} \iota^* \mathfrak{U} \mathfrak{H} \iota (\mathfrak{U}_E)^* + \mathrm{i} \iota^* \mathfrak{U} \iota \iota^* \mathfrak{H} \iota (\mathfrak{U}_E)^* \\
&= \mathrm{i} \iota^* \mathfrak{U} (\iota \iota^* - \mathbb{1}) \mathfrak{H} \iota (\mathfrak{U}_E)^*.
\end{aligned}$$

We note that $\iota\iota^* - \mathbb{1} = -(|\iota^*|^2)^\perp = \Theta(X_1)^\perp$ where Θ is the step function—a valid choice of switch function. In fact all that matters now is that we found a factor $\Lambda_1^\perp \mathfrak{H} \iota$, and then, using $\Lambda_1^\perp \iota = 0$, this factor equals $[\Lambda_1^\perp, \mathfrak{H}]\iota = -\mathrm{i}(\partial_1 \mathfrak{H})\iota$. But $\mathfrak{H}$ is weakly local, and we now invoke Corollary 3.16 to assert $\partial_1 \mathfrak{H}$ is confined. □

In particular this lemma applies to $\mathfrak{H} = H_\lambda$ and $\mathfrak{H} = H^{\mathrm{rel}}$.

Proof of Theorem 2.10. The first step is to show that the indices are well defined and have the claimed properties. This has been the focus of most of the results in this section, and we may now finally put them to use.

By Lemma 3.19 applied to H^{rel} we deduce that $U_{\mathrm{E}}^{\mathrm{rel}}(1) = \mathbb{1} + D^{\mathrm{rel}}(1)$, and consequently $[\Lambda_2, U_{\mathrm{E}}^{\mathrm{rel}}(1)] = \mathrm{i}\partial_2 D^{\mathrm{rel}}(1)$ which is trace-class by Corollary 3.17, so that $\mathcal{I}_{\mathrm{E}}$ is well defined. The invariance under the choice of switch function is a simple computation, and the fact it is integer valued is achieved by choosing $\Lambda_2 = \Theta(X_2)$ and identifying $\mathcal{I}_{\mathrm{E}}$ with an index of a pair of projections. Then by Lemma 3.12, Corollary 3.16 and Lemma 3.14, $[\Lambda_i, U^{\mathrm{rel}}](U^{\mathrm{rel}})^*[\Lambda_j, U^{\mathrm{rel}}] = -\partial_i(U^{\mathrm{rel}})(U^{\mathrm{rel}})^*\partial_j U^{\mathrm{rel}}$ is trace-class for all $t \in [0,1]$ and $i \neq j$ so that $\mathcal{I} = W(U^{\mathrm{rel}})$ is finite. Similarly, the switch function independence and the integer value follow, with proofs similar to those of [16, Proposition 3.3].

The second step is the proof of the bulk-edge correspondence. This is an algebraic computation that involves trace-class operators. We comment on how to generalize to our case [16, Theorem 3.4] (the bulk-edge duality for the case of *local* unitary evolutions where the bulk is periodic). Passing from local to weakly local operators does not modify the trace-class of the expression, because the switch functions cure the nonuniform off-diagonal decay characteristic of weakly local operators, as we have already seen in Lemma 3.19. We first rewrite $\mathcal{I}_{\mathrm{E}} = \mathrm{i}\lim_{r\to\infty} \mathrm{tr}(U_{\mathrm{E}}^{\mathrm{rel}}(1)^*\partial_2 U_{\mathrm{E}}^{\mathrm{rel}}(1)\Lambda_{1,r}^{\perp})$ where $\Lambda_{1,r}^{\perp} = \Theta(r - X_1)$ is a cutoff in direction 1 on $\mathcal{H}_{\mathrm{E}}$ for $r \in \mathbb{N}$. At finite r the previous expression becomes trace-class for every $t \in [0,1]$ so we rewrite it as the integral of its derivative. After some algebra we end up with

$$\mathrm{i}\,\mathrm{tr}(U_{\mathrm{E}}^{\mathrm{rel}}(1)^*\partial_2 U_{\mathrm{E}}^{\mathrm{rel}}(1)\Lambda_{1,r}^{\perp}) = W_r(U^{\mathrm{rel}}) + o(r) \tag{3.4}$$

where W_r is given by (2.7) for $\Lambda_1 = \mathbb{1} - \Lambda_{1,r}^{\perp}$. Since this quantity is independent of the choice of switch function, we conclude $W_r(U^{\mathrm{rel}}) = W(U^{\mathrm{rel}})$ so that $\mathcal{I}_{\mathrm{E}} = \mathcal{I}$ in the $r \to \infty$ limit. Equality (3.4) only uses $H_{\mathrm{E}} = \widehat{H}$, Lemma 3.19 and some updated version of it, [16, Lemma 5.5], namely that

$$[\Lambda_2, U_{\mathrm{E}}^{\mathrm{rel}}(t)]U_{\mathrm{E}}^{\mathrm{rel}}(t)^* - \iota^*\,[\Lambda_2, U^{\mathrm{rel}}(t)]U^{\mathrm{rel}}(t)^*\,\iota \tag{3.5}$$

is trace-class for $t \in [0,1]$. All the rest follows by algebraic manipulations and Lemma A.9.

□

We note that, surprisingly, identity (3.4) remains true when applied to U and U_{E} instead of the relative evolutions, even if $U(1) \neq \mathbb{1}$. One still has $W_r(U) = W(U)$, but this quantity is not quantized anymore. Yet the left-hand side of (3.4) converges to it in the $r \to \infty$ limit but does not coincide with any edge index because the $\partial_2 U_{\mathrm{E}}(1)$ is not anymore trace-class. Although not relevant here, this identity will be used below.

4. The Completely Localized Case

This section is dedicated to the proof of Theorems 2.14 and 2.15. We start by studying the bulk part M. Let us assume that U is completely localized in the sense of Definition 2.12, and $\lambda \in \mathbb{S}^1\backslash\{1\} \subseteq \Delta$. According to Theorem 2.10, the bulk index $\mathcal{I} = W(U^{\mathrm{rel}})$ is well defined. By (2.6) and (2.7), W is manifestly additive under concatenation, so we deduce

$$\mathcal{I} = W(U) - W(U_\lambda)$$

where $U_\lambda \equiv \mathrm{e}^{-\mathrm{i}\cdot H_\lambda}$. Here we have extended W to nonperiodic families with the same formula (2.7). Indeed, since both U and U_λ are weakly local, $W(U)$ and $W(U_\lambda)$ are finite, although they are not separately integers. Before we proceed with U let us rewrite the expression for W, for any weakly local $\mathfrak{U} : [0,1] \to \mathcal{U}(\mathcal{H})$ and its generator $\mathfrak{H} := \mathrm{i}\dot{\mathfrak{U}}\mathfrak{U}^*$

$$W(\mathfrak{U}) = \frac{1}{2\mathrm{i}} \int_0^1 \mathrm{d}t \, \mathrm{tr} \, \varepsilon_{\alpha\beta} \mathfrak{H} (\partial_\alpha \mathfrak{U}) \partial_\beta \mathfrak{U}^*$$

where we have used $\partial_\alpha \mathfrak{U}^* = -\mathfrak{U}^*(\partial_\alpha \mathfrak{U})\mathfrak{U}^*$ and recall $\varepsilon_{\alpha\beta}$ the anti-symmetric tensor with $\varepsilon_{12} = 1$. (Summation over $\alpha, \beta \in 1, 2$ is understood when indices appear twice.) Since $\mathrm{tr}\, \partial_\alpha(\mathfrak{H}\mathfrak{U}\partial_\beta\mathfrak{U}^*) = 0$ and $\varepsilon_{\alpha\beta}\mathfrak{H}\mathfrak{U}\partial_\alpha\partial_\beta\mathfrak{U}^* = 0$ (as the non-commutative derivatives commute) we deduce

$$W(\mathfrak{U}) = \frac{\mathrm{i}}{2} \int_0^1 \mathrm{d}t \, \mathrm{tr} \, \varepsilon_{\alpha\beta} (\partial_\alpha \mathfrak{H}) \mathfrak{U} \partial_\beta \mathfrak{U}^*.$$

Defining $\delta_\alpha^{\mathfrak{U}} := \mathfrak{U}^* \partial_\alpha \mathfrak{U}$ (the logarithmic derivative of $\mathfrak{U}$ in direction α) and noticing that

$$\dot{\delta}_\alpha^{\mathfrak{U}} = \frac{1}{\mathrm{i}} \mathfrak{U}^* (\partial_\alpha \mathfrak{H}) \mathfrak{U} \tag{4.1}$$

we deduce

$$W(\mathfrak{U}) = \frac{1}{2} \int_0^1 \mathrm{d}t \, \mathrm{tr} \, \varepsilon_{\alpha\beta} \dot{\delta}_\alpha^{\mathfrak{U}} \delta_\beta^{\mathfrak{U}}. \tag{4.2}$$

Consequently for U^{rel} we find,

$$\mathcal{I} = \frac{1}{2} \, \mathrm{tr} \int_0^1 \mathrm{d}t \, \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha \delta_\beta - \dot{\delta}_\alpha^\lambda \delta_\beta^\lambda \big)$$

where we use the shorthand notation δ_α (resp. δ_α^λ) for δ_α^U (resp. $\delta_\alpha^{U_\lambda}$). Note that there is no problem to exchange trace and integral here since both $\varepsilon_{\alpha\beta}\dot{\delta}_\alpha\delta_\beta$ and $\varepsilon_{\alpha\beta}\dot{\delta}_\alpha^\lambda\delta_\beta^\lambda$ are trace-class for all $t \in [0,1]$. Finally

$$\begin{aligned}\int_0^1 \mathrm{d}t \, \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha \delta_\beta - \dot{\delta}_\alpha^\lambda \delta_\beta^\lambda \big) = {} & -\mathrm{i} \int_0^1 \mathrm{d}t \, \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha U^* \Lambda_\beta U - \dot{\delta}_\alpha^\lambda U_\lambda^* \Lambda_\beta U_\lambda \big) \\ & + \mathrm{i} \int_0^1 \mathrm{d}t \, \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha - \dot{\delta}_\alpha^\lambda \big) \Lambda_\beta.\end{aligned}$$

The last term vanishes because it is a total derivative, and by the fact that $\delta_\alpha(0) = \delta_\alpha^\lambda(0) = 0$ and $\delta_\alpha(1) = \delta_\alpha^\lambda(1)$ since $U(1) = U_\lambda(1)$ (note however that $\delta_\alpha(1) \neq 0$ in general). Hence, by (4.1) and (2.10),

$$\mathcal{I} = \mathrm{tr}(m(U) - m(U_\lambda)).$$

This relation is general and does not require $U(1)$ to be completely localized. However it is equivalent to (2.13) only in the latter case. Indeed $m(U)$ and

$m(U_\lambda)$ are not separately trace-class, only their difference is. When $U(1)$ is completely localized, the trace of this difference can be computed through its eigenbasis:

$$\mathcal{I} = \sum_{z\in\mathcal{E}} \operatorname{tr} P_z \big(m(U) - m(U_\lambda)\big) P_z$$

with $P_z \equiv \chi_{\{z\}}(U(1))$ the projection onto the eigenvalue z. What remains to be shown is that the sum can actually be split into two parts, leading to (2.13).

Proposition 4.1. *If $U(1)$ is completely localized in the sense of Definition 2.12, then the effective evolution magnetization*

$$M(U_\lambda) = -\sum_{z\in\mathcal{E}} \operatorname{tr} P_z \int_0^1 \mathrm{d}t \, \mathbb{Im}(U_\lambda^* \Lambda_1 H_\lambda \Lambda_2 U_\lambda) P_z$$

is absolutely convergent and vanishes.

Thus we are left with $\mathcal{I} = M(U)$, so that $M(U)$ is well defined and shares all the properties of $\mathcal{I}$. This proves the main statement of Theorem 2.14. In the particular case where $U(1) = \mathbb{1}$, we have $H_\lambda = 0$ and $\delta_\alpha(1) = 0$, so that $M(U)$ is trace-class by the previous computation, and $\mathcal{I} = W(U) = \operatorname{tr}(m(U)) = M(U)$. Finally the case where H is time-independent is a direct consequence of the latter proposition.

Proof of Proposition 4.1. Since $U_\lambda \equiv \mathrm{e}^{-\mathrm{i}\cdot H_\lambda}$ and $U_\lambda(1) = U(1)$ is completely localized, then so are H_λ and $U_\lambda(t)$ for $t \in [0,1]$ with the same eigenbasis as $U(1)$. Thus for a fixed $z \in \mathbb{S}^1$ one has by replacing $U_\lambda(t) P_z = z^t$ and $P_z U_\lambda(t)^* = z^{-t}$ for all $t \in [0,1]$:

$$P_z \int_0^1 \mathrm{d}t \, \mathbb{Im}(U_\lambda^* \Lambda_1 H_\lambda \Lambda_2 U_\lambda) P_z = P_z \mathbb{Im}(\Lambda_1 H_\lambda \Lambda_2) P_z. \tag{4.3}$$

By construction H_λ is bounded with a real spectrum that unwinds the circular one of $U(1)$ with respect to the branch cut λ. For each eigenvalue of $U(1)$, $z \in \mathbb{S}^1$, $r := \mathrm{i}\log_\lambda(z) \in \mathbb{R}$ is an eigenvalue of H_λ with same eigenprojection P_z. For $x \in \mathbb{R}$ we define the Fermi projection up to x by $P_{<x} := \chi_{(-\infty,x)}(H_\lambda)$, so that $P_{<x} = 0$ for $x < \inf \sigma(H_\lambda)$ and $P_{<x} = \mathbb{1}$ for $x \geq \sup \sigma(H_\lambda)$. We use the following representation of H_λ

$$H_\lambda = C - \int_{\sigma(H_\lambda)} P_{<x} \mathrm{d}x \tag{4.4}$$

where $C = \sup(\sigma(H_\lambda)) \in \mathbb{R}$. This representation comes from the following functional equality

$$\int_\Omega \chi_{(-\infty,x)}(y) \mathrm{d}x = \int_\Omega \chi_{(y,\infty)}(x) \mathrm{d}x = \sup(\Omega) - y \tag{4.5}$$

for any interval Ω and $y \in \Omega$. Inserting (4.4) into (4.3) we get

$$P_z \mathbb{Im}(\Lambda_1 H_\lambda \Lambda_2) P_z = \frac{\mathrm{i}}{2} \int_{\sigma(H_\lambda)} \varepsilon_{\alpha\beta} P_z \Lambda_\alpha P_{<x} \Lambda_\beta P_z \mathrm{d}x.$$

Consider $z \in \mathcal{E}$ and $x \in \sigma(H)$ fixed, and define $P^{\perp}_{<x} = \mathbb{1} - P_{<x}$. Then either $\mathrm{i}\log_{\lambda}(z) > x$, in which case $P_z P_{<x} = 0$ and $P_z P^{\perp}_{<x} = P_z$, or $\mathrm{i}\log_{\lambda}(z) \leq x$, in which case $P_z P_{<x} = P_z$ and $P_z P^{\perp}_{<x} = 0$. Therefore

$$\frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta} P_z \Lambda_\alpha P_{<x} \Lambda_\beta P_z = \frac{\mathrm{i}}{2} P_z \varepsilon_{\alpha\beta} P^{\perp}_{<x} \Lambda_\alpha P_{<x} \Lambda_\beta P^{\perp}_{<x} P_z - \frac{\mathrm{i}}{2} P_z \varepsilon_{\alpha\beta} P_{<x} \Lambda_\alpha P^{\perp}_{<x} \Lambda_\beta P_{<x} P_z \tag{4.6}$$

$$=: \frac{1}{2} P_z T(x) P_z. \tag{4.7}$$

Moreover, $T(x)$ is trace-class for every $x \in \sigma(H_\lambda)$. Indeed, after some algebra

$$T(x) = -\mathrm{i} P^{\perp}_{<x}\big[\partial_1 P^{\perp}_{<x}, \partial_2 P^{\perp}_{<x}\big] + \mathrm{i} P_{<x}\big[\partial_1 P_{<x}, \partial_2 P_{<x}\big] \tag{4.8}$$

and each term is separately trace-class by Corollary 3.17. Indeed,

$$P_{<x} = \chi_{(\lambda, \mathrm{e}^{-\mathrm{i}x})}(U(1))$$

is weakly local according to Definition 2.4 since $U(1)$ is completely localized. [Even if the single point of infinite degeneracy z_0 has $z_0 \in (\lambda, \mathrm{e}^{-\mathrm{i}x})$, then $P_{<x} = \mathbb{1} - P^{\perp}_{<x}$ is weakly local from the fact that $P^{\perp}_{<x}$ is.] Thus for every $z \in \mathcal{E}$, $P_z T(x) P_z$ is trace-class (even if z is infinitely degenerate) and moreover

$$\frac{1}{2}\sum_{z\in\mathcal{E}} \operatorname{tr} P_z T(x) P_z = \frac{1}{2}\operatorname{tr}(T(x)) = \frac{1}{4\pi}(\operatorname{Chern}(P_{<x}) - \operatorname{Chern}(P^{\perp}_{<x}) = \frac{1}{2\pi}\operatorname{Chern}(P_{<x}).$$

with the sum on the left-hand side that is absolutely convergent due to the trace-class property. The quantity on the right-hand side is the Chern number [10] defined by

$$\operatorname{Chern}(P) \equiv 2\pi\mathrm{i}\operatorname{tr} P[\partial_1 P, \partial_2 P] = -\operatorname{Chern}(P^{\perp})$$

that is well defined and integer valued for any weakly local projection P. Since H_λ is bounded we have

$$\frac{1}{2}\int_{\sigma(H_\lambda)} \sum_{z\in\mathcal{E}} |\operatorname{tr} P_z T(x) P_z| \mathrm{d}x \leq \frac{1}{2}\int_{\sigma(H_\lambda)} \|T(x)\|_1 \mathrm{d}x \leq \frac{1}{2}|\sigma(H_\lambda)| \sup_{x\in\sigma(H_\lambda)} \|T(x)\|_1 < \infty \tag{4.9}$$

due to the fact that $\sigma(H_\lambda)$ is compact and $x \mapsto \|T(x)\|_1$ is bounded. Indeed, this last fact is a nontrivial consequence of the fact that the estimate of complete localization in mobility gap, (2.3), includes a supremum over all Borel bounded functions $|f| \leq 1$ (which are constant outside of the mobility gap, but since here the entire circle except one point is a mobility gap, this constraint is vacuously true). One could then include in this supremum the spectral projections $P_{<x}$ with supremum over x since the functions $(\chi_{(\lambda, \mathrm{e}^{\mathrm{i}x})})_{x\in\sigma(H_\lambda)}$ are definitely part of this set of functions. Thus $P_{<x}$ has weakly local estimates which

are uniform in x, which in turn implies via Remark 3.18 that $x \mapsto \|T(x)\|_1$ is bounded. [One may be worried about the point of infinite degeneracy in the spectrum of $U(1)$ which is allowed in Definition 2.12, throughout which estimate (2.3) might fail to hold uniformly, but this is not a problem since we could always just remove this point in the integral before (4.9).]

Hence by Fubini's theorem we may exchange $\sum_z$ and $\int_x$. Putting everything together, we deduce that $M(U_\lambda)$ is defined by an absolutely convergent sum. Moreover

$$M(U_\lambda) = -\int_{\sigma(H_\lambda)} \mathrm{Chern}(P_{<x})\mathrm{d}x.$$

It was shown in [10, Prop. 2] that $\mathrm{Chern}(\chi_\Omega(H)) = 0$ for any interval Ω inside the mobility gap of H that contains only finite-multiplicity eigenvalues. Here the entire spectrum of H_λ is a mobility gap, but it might contain one infinite degenerated eigenvalue corresponding to z_0. However if $z_0 \in (\lambda, \mathrm{e}^{-\mathrm{i}x})$, then $\mathrm{Chern}(P_{<x}) = -\mathrm{Chern}(P_{<x}^\perp) = 0$ as the latter projection only contains finite degenerated eigenvalues. Thus $M(U_\lambda) = 0$. □

Proof of Theorem 2.15. Let n be a fixed integer. From (3.4) in the proof of Theorem 2.10 and the fact that $U_\mathrm{E}(1)^n = U_\mathrm{E}(n)$, we have the following identity:

$$\lim_{r\to\infty} \mathrm{i}\,\mathrm{tr}\left((U_\mathrm{E}(1)^*)^n \partial_2 U_\mathrm{E}(1)^n \Lambda_{1,r}^\perp\right) = -\frac{1}{2}\int_0^n \mathrm{d}t\,\mathrm{tr}\,\dot{U}U^*[(\partial_1 U)U^*, (\partial_2 U)U^*].$$

In particular, on the left-hand side, the expression is trace-class for every finite r because of the cutoff $\Lambda_{1,r}^\perp$ and confinement in direction 2 through Λ_2. The right-hand side is expression (2.7) of W but on a time interval $[0, n]$ instead of $[0, 1]$. In particular it is independent of switch function Λ_1, which is why the limit $r \to \infty$ is finite. If $U(n) = \mathbb{1}$, W would be quantized and define the bulk index, and the limit on the right would be equal to edge index. Nevertheless the previous equation is true for any pair of bulk and edge operators U and U_E, as long as they are weakly local and related by Lemma 3.19, although it is not integer valued. From now we assume $U(1)$ completely localized. Rewriting W as in (4.2)

$$\begin{aligned}
&-\frac{1}{2}\int_0^n \mathrm{d}t\ \mathrm{tr}\,\dot{U}U^*[(\partial_1 U)U^*, (\partial_2 U)U^*]\\
&= \frac{1}{2}\int_0^n \mathrm{d}t\ \mathrm{tr}\,\varepsilon_{\alpha\beta}\dot{\delta}_\alpha\delta_\beta\\
&= \frac{1}{2}\sum_{z\in\mathcal{E}} \mathrm{tr}(P_z \int_0^n \mathrm{d}t\,\varepsilon_{\alpha\beta}\dot{\delta}_\alpha U^*\Lambda_\beta U P_z) - \frac{1}{2}\sum_{z\in\mathcal{E}} \mathrm{tr}(P_z \int_0^n \mathrm{d}t\,\varepsilon_{\alpha\beta}\dot{\delta}_\alpha \Lambda_\beta P_z)
\end{aligned} \tag{4.10}$$

where $\delta_\alpha = U^*\Lambda_\alpha U - \Lambda_\alpha$. Since $\dot{\delta}_\alpha\delta_\beta$ is trace-class, we permute trace and time integral and then compute this trace in the eigenbasis of $U(1)$. What remains to be shown is that the two terms in the last formula obtained by splitting δ_β are separately finite and then study their $n \to \infty$ limit. Note that the eigenbasis of $U(1)$ and $U(n)$ is the same since $U(n) = U(1)^n$, although the eigenvalues are different. The first term in (4.10) is close to magnetization

$$\frac{1}{2}\sum_{z\in\mathcal{E}} \operatorname{tr}\left(P_z \int_0^n \mathrm{d}t\, \varepsilon_{\alpha\beta}\dot{\delta}_\alpha U^* \Lambda_\beta U P_z\right) = -\sum_{z\in\mathcal{E}} \operatorname{tr}\left(P_z \int_0^n \mathrm{d}t\, \mathbb{I}\mathrm{m}(U^*\Lambda_1 H \Lambda_2 U) P_z\right)$$
$$=: M_n(U).$$

Then we use the facts that $U(t) = U(t-k)U(1)^k$ for $k \leq t < k+1$ and $k \in \{0,\ldots,n-1\}$ and $U(1)^k P_z = z^k P_z$ for $z \in \mathcal{E} \subset \mathbb{S}^1$. Similarly $U^*(t) = (U(1)^*)^k U^*(t-k)$ and $(U(1)^*)^k P_z = z^{-k} P_z$. Moreover $H(t+k) = H(t)$. Applying these relations to the previous time integral that we cut into n parts, we get up to a change of variable

$$M_n(U) = nM(U)$$

so that $\mathcal{M}_n(U)$ is finite and shares all the properties of $M(U)$ from Theorem 2.14. Moreover $n^{-1}M_n(U) \to M(U)$ trivially when $n \to \infty$.

The second term of (4.10) is a total derivative and can be simplified to

$$\frac{1}{2}\sum_{z\in\mathcal{E}} \operatorname{tr}\left(P_z \int_0^n \mathrm{d}t\, \varepsilon_{\alpha\beta}\dot{\delta}_\alpha \Lambda_\beta P_z\right) = \frac{1}{2}\sum_{z\in\mathcal{E}} \operatorname{tr}(P_z \varepsilon_{\alpha\beta} U(n)^* \Lambda_\alpha U(n) \Lambda_\beta P_z) \tag{4.11}$$

since $\delta_\beta(0) = 0$ and $\varepsilon_{\alpha\beta}\Lambda_\alpha\Lambda_\beta = 0$. Note that $U(n) = U(1)^n = \mathrm{e}^{-\mathrm{i}nH_\lambda}$ for H_λ defined in (2.5) and any $\lambda \in \mathbb{S}^1 = \Delta$. Then we use the following functional equality for a continuously differentiable $f : [a,b] \to \mathbb{C}$:

$$f(y) = f(b) - \int_a^b f'(x)\chi_{(a,x)}(y)\mathrm{d}x$$

for $y \in [a,b]$, which is a generalization of (4.5), see also [10]. Consequently

$$U(n) = \mathrm{e}^{-\mathrm{i}nH_\lambda} = \mathrm{e}^{-\mathrm{i}nb}\mathbb{1} + \mathrm{i}n\int_{\sigma(H_\lambda)} \mathrm{e}^{-\mathrm{i}nx} P_{<x}\mathrm{d}x \tag{4.12}$$

where $P_{<x} = \chi_{(-\infty,x)}(H_\lambda)$ and $b = \sup(\sigma(H_\lambda))$. When inserting this expression for $U(n)$ in (4.11), the first term vanishes by antisymmetry. In order to show that the second one is finite, we claim that

$$\frac{\mathrm{i}n}{2}\int_{\sigma(H_\lambda)} \sum_{z\in\mathcal{E}} \mathrm{e}^{\mathrm{i}nx} \operatorname{tr}(P_z\varepsilon_{\alpha\beta}U(n)^*\Lambda_\alpha P_{<x}\Lambda_\beta P_z)\mathrm{d}x$$

is absolutely convergent for any fixed n. Indeed since $U(n)^*$ commutes with P_z, one has

$$P_z\mathrm{i}\varepsilon_{\alpha\beta}U(n)^*\Lambda_\alpha P_{<x}\Lambda_\beta P_z = P_z U(n)^* T(x) P_z$$

where $T(x)$ is defined in (4.6). Moreover $T(x)$ is trace-class as pointed out in (4.8) so the previous sum over z is absolutely convergent for every $x \in \sigma(H_\lambda)$. The integral is then also absolutely convergent for the same reasons as in (4.9).

Consequently, (4.11) can be rewritten as

$$\frac{1}{2}\sum_{z\in\mathcal{E}} \operatorname{tr}(P_z \varepsilon_{\alpha\beta} U(n)^* \Lambda_\alpha U(n) \Lambda_\beta P_z) \\ = \frac{n}{2}\sum_{z\in\mathcal{E}} z^n \int_{\sigma(H_\lambda)} \mathrm{e}^{\mathrm{i}nx} \operatorname{tr}(P_z T(x) P_z) \mathrm{d}x$$

with absolute convergence. We finally claim that

$$\lim_{n\to\infty} \sum_{z\in\mathcal{E}} z^n \int_{\sigma(H_\lambda)} \mathrm{e}^{\mathrm{i}nx} \operatorname{tr}(P_z T(x) P_z) \mathrm{d}x = 0. \tag{4.13}$$

First for $z \in \mathcal{E}$ denote $g_z(x) := \operatorname{tr}(P_z T(x) P_z)$ that is ℓ^1 on $\sigma(H_\lambda)$ by (4.9). Then

$$\int_{\sigma(H_\lambda)} \mathrm{e}^{\mathrm{i}nx} \operatorname{tr}(P_z T(x) P_z) \mathrm{d}x = 2\pi(\mathcal{F}^{-1}(g_z))(n) \underset{n\to\infty}{\longrightarrow} 0$$

where $\mathcal{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ is the Fourier series. As such it indeed vanishes in the limit $n \to \infty$ by the Riemann–Lebesgue lemma. Finally $z^n 2\pi(\mathcal{F}^{-1}(g_z))(n)$ is summable in z and vanishes when $n \to \infty$ for fixed z. Moreover

$$|z^n 2\pi(\mathcal{F}^{-1}(g_z))(n)| \leq \int_{\sigma(H_\lambda)} |\operatorname{tr}(P_z T(x) P_z)| \mathrm{d}x\,,$$

the RHS is summable in z since $T(x)$ is trace-class, so that we may use it as a dominating function (in z) on the sequence $(z^n 2\pi(\mathcal{F}^{-1}(g_z))(n))_n$ when applying the dominated-convergence lemma to exchange $\lim_n$ and $\sum_z$. This leads to $\sum_z 0 = 0$ which gives (4.13), concluding the proof. □

5. The Stretch Function Construction

5.1. Proof of Corollary 2.18

Corollary 2.18 is a consequence of Theorems 2.14 and 2.15 that both rely on Theorem 2.10 applied to $V \equiv F_\Delta(U)$ and $V_\mathrm{E} \equiv F_\Delta(U_\mathrm{E})$ for a given stretch function F_Δ. By Definition 2.4, $V(1)$ is completely localized since $F_\Delta \in B_1(\Delta)$, but in order to replace U and U_E by V and V_E in the previous theorems we need to show that they satisfy all the required properties concerning locality and confinement. We note that Lemmas 5.1 and 5.2 are true regardless of the existence of (any) gap of $U(1)$ and moreover all the operators involved are (strictly polynomially) local since H is (strictly exponentially) local by Assumption 2.2, see Corollary A.8.

The existence of the gap only becomes relevant when we use localization to assert the (weak) locality of the logarithm, which is when we apply Corollary 3.8 to $V(1)$. When we do that, we actually get expressions like $\log_\lambda \circ F_\Delta$ applied to $U(1)$, which, as in Lemma 3.7, gets decomposed to sums of functions such as $g \circ F_\Delta$ with g smooth, which is a smooth function, or $\chi_{[\lambda,\lambda']} \circ F_\Delta$ which is in $B_1(\Delta)$ and so Definition 2.4 applies. The conclusion is that the logarithm of $V(1)$ is also weakly local so that the relative construction could just as well be applied to V.

Lemma 5.1. *V and V_{E} are (strictly polynomially) local if U and U_{E} are (strictly exponentially) local. Moreover the maps $t \mapsto V(t)$ and $t \mapsto V_{\mathrm{E}}(t)$ are strongly differentiable, and their respective generators $H_V = \mathrm{i}\dot{V}V^*$ and $H_{V_{\mathrm{E}}} = \mathrm{i}\dot{V}_{\mathrm{E}}V_{\mathrm{E}}^*$ are weakly local.*

Proof. The first fact is a direct consequence of Corollary A.8, F_Δ being smooth. For the derivatives we compute for $t, s \in [0,1]$, using Lemma A.7 and the resolvent identity,

$$V(s) - V(t) = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}z\mathrm{d}\bar{z}(\partial_{\bar{z}}\tilde{F}_\Delta(z))R_{U(s)}(z)(U(s) - U(t))R_{U(t)}(z)$$

where $\tilde{F}_\Delta$ is a quasi-analytic extension of F_Δ and $R_{U(s)}(z) = (U(s) - z)^{-1}$ that is norm-continuous in s. Hence

$$\partial_t V(t) = \mathrm{s} - \lim_{s\to t} \frac{V(s) - V(t)}{s - t} = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}z\mathrm{d}\bar{z}(\partial_{\bar{z}}\tilde{F}_\Delta(z))R_{U(t)}(z)(\partial_t U)(t)R_{U(t)}(z).$$

Since $\|R_{U(t)}(z)\| \leq C||z| - 1|^{-1}$ and $|\partial_{\bar{z}}\tilde{F}| \leq C||z| - 1|^N$ for some $N \geq 2$, the integral is convergent. Moreover H and U are local thus so are $\partial_t U = -\mathrm{i}HU$ and $R_{U(t)}$, the latter by the Combes–Thomas estimate. Since $\tilde{F}_\Delta$ is compactly supported we deduce that $\partial_t V$ is (polynomially) local, and so is H_V by Lemma 3.6. We proceed similarly for V_{E}. □

Lemma 5.2. *The differences $V_{\mathrm{E}} - \iota^* V \iota$ and $H_{V_{\mathrm{E}}} - \iota^* H_V \iota$ are weakly local and confined in direction 1, uniformly in $t \in [0,1]$.*

Proof. This looks like a consequence of Lemma 3.19 (see also [16, Prop. 4.10]). However since $\iota^* U \iota$ is not a unitary, it is not obvious how to directly implement functional calculus on it. Instead we should first reformulate this result in the bulk picture. In what follows D denotes an operator that is local and confined in direction 1. We claim that

$$U = \iota U_{\mathrm{E}} \iota^* + jU_- j^* + D \tag{5.1}$$

where $j : \mathcal{H}_- \hookrightarrow \mathcal{H}$ and $j^* : \mathcal{H} \twoheadrightarrow \mathcal{H}_-$ with $\mathcal{H}_- = \ell^2((\mathbb{Z}\backslash\mathbb{N}) \times \mathbb{Z}) \otimes \mathbb{C}^N$ is the left half-space. Note that $jj^* = \mathbb{1} - P_1$, $j^* j = \mathbb{1}$, and $j^* i = \iota^* j = 0$. Finally U_- is generated by $H_- := j^* H j$, so that both are local like H_{E} and U_{E} are. The proof of (5.1) is completely analogue to Lemma 3.19 and relies on the fact that $[P_1, H]$ is local and confined in direction 1.

Then we consider the unitary $U_{\mathrm{d}} := \iota U_{\mathrm{E}} \iota^* + jU_- j^*$ that satisfies $R_{U_{\mathrm{d}}}(z) = \iota R_{U_{\mathrm{E}}}(z)\iota^* + jR_{U_-}(z)j^*$ where $R_U(z) = (U - z)^{-1}$. By (5.1) and the resolvent identity we deduce

$$R_U(z) - R_{U_{\mathrm{d}}}(z) = -R_U(z)DR_{U_{\mathrm{d}}}(z) \tag{5.2}$$

We compute $F_\Delta(U)$ and $F_\Delta(U_{\mathrm{d}})$ through quasi-analytic functional calculus, see Lemma A.7, leading to

$$F_\Delta(U) - F_\Delta(U_{\mathrm{d}}) = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}z\mathrm{d}\bar{z}(\partial_{\bar{z}}\tilde{F}_\Delta)R_U(z)DR_{U_{\mathrm{d}}}(z).$$

On the right-hand side the integral is convergent because of the decaying behavior of $\partial_{\bar{z}}\tilde{F}_\Delta$ around $\mathbb{S}^1$, similarly to the previous proof. Moreover both resolvents are local by Combes–Thomas estimate so that the integral is weakly local and confined in direction 1 by Corollary A.8. On the left-hand side we have $F_\Delta(U_\mathrm{d}) = \iota F(U_\mathrm{E})\iota^* + jF(U_-)j^*$, so that the difference $\iota^*(F_\Delta(U) - F_\Delta(U_\mathrm{d}))\iota = \iota^* V\iota - V_\mathrm{E}$ has the expected property.

It is then easy to show that $\partial_t V_\mathrm{E} - \iota^*\partial_t V\iota$ is also weakly local and confined in direction 1, by using quasi-analytic functional calculus of Lemma 5.1 and the fact that both $\partial_t U_\mathrm{E} - \iota^*\partial_t U\iota$ and $R_{U_\mathrm{E}}(z) - \iota^* R_U(z)\iota$ are local and 1-confined, respectively, coming from Lemma 3.19 and (5.2). We deduce that $H_{V_\mathrm{E}} - \iota^* H_V\iota$ has the expected property. □

5.2. Stretch Function Invariance

Proof of Theorem 2.19. We assume that $F : \mathbb{C} \to \mathbb{C}$ is a stretch function and have $V \equiv F \circ U$. As mentioned we assume F is smooth. Above we have shown that $\mathcal{I}' = W(V^{\mathrm{rel}})$, so that our task now is to show that $W(V^{\mathrm{rel}}) = W(U^{\mathrm{rel}})$. Let $[0, 1] \ni s \mapsto \mathfrak{F}_s(\cdot)$ be a homotopy that interpolates smoothly between the identity map $\mathbb{C} \ni z \mapsto z$ at $s = 0$ and F at $s = 1$. Since F itself is a "stretching" of the mobility gap $\Delta \subseteq \mathbb{S}^1$ onto the entire circle, we pick this interpolation such that it stretches *about* the branch cut $\lambda \in \Delta$. This point is crucial and will be used later on, in that it means no eigenvalue of $\mathfrak{F}_s(U(1))$ crosses λ as s changes. The gist of the argument is as follows. All maps involved are continuous (even smooth) except one, $\log_\lambda$. While this map indeed has a jump discontinuity, the particular form of deformation which we choose does not ever cross this point of discontinuity—in other words, λ is a fixed point of the deformation (see Fig. 3).

The smoothness assumption means that, in particular, for fixed z, $s \mapsto \mathfrak{F}_s(z)$ is differentiable, for all s, $z \mapsto \mathfrak{F}_s(z)$ is smooth [so $\mathfrak{F}_s(U(t)]$ is local for all t and it makes sense to take the derivative of $t \mapsto \mathfrak{F}_s(U(t)))$ and for fixed z, $s \mapsto \mathfrak{F}'_s(z)$ is differentiable. In addition, because $s \mapsto \mathfrak{F}_s$ interpolates between $\mathbb{1}$ and F, the mobility gap never closes (it only gets stretched from $\Delta \to \mathbb{S}^1 \setminus \{1\}$) for all s, λ is within the mobility gap of $\mathfrak{F}_s(U(1))$ so that $\log_\lambda(\mathfrak{F}_s(U(1)))$ is weakly local for all s.

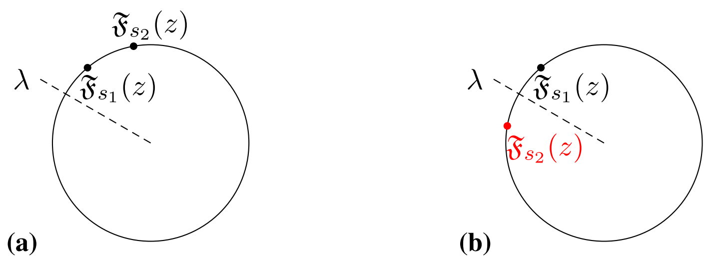

FIGURE 3. In the proof, situation **a** happens as a rule and situation **b** never occurs by choice of $\mathfrak{F}$

Hence $W(\mathfrak{F}_s(U)^{\mathrm{rel}})$ is well defined and integer valued for all s, so it suffices to prove that $|W(\mathfrak{F}_{s_1}(U)^{\mathrm{rel}}) - W(\mathfrak{F}_{s_2}(U)^{\mathrm{rel}})| < 1$ for any $s_1, s_2 \in [0,1]$ with $|s_1 - s_2|$ sufficiently small. Recall that $W(\mathcal{V}^{\mathrm{rel}}) = W(\mathcal{V}) - W(\mathcal{V}_\lambda)$ so that by the triangle inequality we can work separately with $|W(\mathfrak{F}_{s_1}(U)) - W(\mathfrak{F}_{s_2}(U))|$ and $|W(\mathfrak{F}_{s_1}(U)_\lambda) - W(\mathfrak{F}_{s_2}(U)_\lambda)|$, though each part is not separately an integer. To probe the smallness of these expressions we use Lemma 5.3.

We will use Lemma A.9. This is boosted, using the weakly local properties, to Lemmas A.11 and A.12. Note that in order to use these lemmas, one must have uniform exponents μ and ν which is certainly not part of the context in Sect. 3.1. However, this is actually not a problem since the form of weak locality that is produced by Corollary A.8 gives us the ability to choose the minimal exponents μ once and for all. The exponent ν is actually not even necessary here since the deformation is always applied on $U(1)$ which is honestly local and not just weakly local, but even if that were not the case, one can just choose a universal ν which makes $\sum_x (1 + \|x\|)^{-\nu}$ finite.

Since $s \mapsto \mathfrak{F}_s(z)$ is differentiable,

$$\text{s-lim}_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathfrak{F}_{s+\varepsilon}(U(t)) - \mathfrak{F}_s(U(t))) = \partial_s \mathfrak{F}_s(U(t)),$$

so that $\frac{1}{\varepsilon} T(\mathfrak{F}_{s+\varepsilon}(U(t)) - \mathfrak{F}_s(U(t))) \to T\partial_s \mathfrak{F}_s(U(t))$ in trace-class norm for any trace-class T. Similarly we handle also $\partial_t \mathfrak{F}_s(U(t))) = \mathfrak{F}'_s(U(t)))\dot{U}(t)$ which is also differentiable as a function of s. Since $\mathfrak{F}_s(U(t)))$ is weakly local for any value of s, we also have similar convergence for the spatial derivatives: $\frac{1}{\varepsilon} T_\alpha \partial_\beta(\mathfrak{F}_{s+\varepsilon}(U(t)) - \mathfrak{F}_s(U(t))) \to T_\alpha \partial_\beta \partial_s \mathfrak{F}_s(U(t))$ in trace-class norm for any T_α which is weakly local and confined in the α direction. We conclude that $|W(\mathfrak{F}_{s_1}(U)) - W(\mathfrak{F}_{s_2}(U))|$ can be made arbitrarily small as $s_2 \to s_1$.

When dealing with $|W(\mathfrak{F}_{s_1}(U))_\lambda) - W(\mathfrak{F}_{s_2}(U)_\lambda)|$, it might appear that we are stuck, since $(\mathfrak{F}_{s_1}(U))_\lambda(t) \equiv \exp(t \log_\lambda(\mathfrak{F}_{s_1}(U(1))))$ and $\log_\lambda$ is *not* continuous. Furthermore, algebraic laws like $\log(\frac{z_1}{z_2}) = \log(z_1) - \log(z_2)$ only hold mod $2\pi\mathrm{i}$ in general, which could introduce jump discontinuities. Since $\mathfrak{F}_{s_1}(U(1))$ and $\mathfrak{F}_{s_2}(U(1))$ are functions of the same operator $U(1)$, they commute and hence have the same diagonalization. Indeed, let P be the projection-valued spectral measure of $U(1)$. Then

$$\begin{aligned} &\log_\lambda(\mathfrak{F}_{s_1}(U(1))) - \log_\lambda(\mathfrak{F}_{s_2}(U(1))) \\ &\quad = \int_{z \in \mathbb{S}^1} \log_\lambda(\mathfrak{F}_{s_1}(z)) \mathrm{d}P(z) - \int_{z \in \mathbb{S}^1} \log_\lambda(\mathfrak{F}_{s_2}(z)) \mathrm{d}P(z) \\ &\quad = \int_{z \in \mathbb{S}^1} (\log_\lambda(\mathfrak{F}_{s_1}(z)) - \log_\lambda(\mathfrak{F}_{s_2}(z))) \mathrm{d}P(z). \end{aligned}$$

Now, since λ is a fixed point of the deformation in s and since the deformation is continuous in s, $\mathfrak{F}_{s_1}(z)$ and $\mathfrak{F}_{s_2}(z)$ (for sufficiently small $|s_1 - s_2|$) are sufficiently close on the circle and on the same "side" of the cut so that the algebraic rule of the logarithm holds without the mod $2\pi\mathrm{i}$. Hence

$$\log_\lambda(\mathfrak{F}_{s_1}(U(1))) - \log_\lambda(\mathfrak{F}_{s_2}(U(1))) = \int_{z\in\mathbb{S}^1} \log_\lambda(\mathfrak{F}_{s_1}(z)(\mathfrak{F}_{s_2}(z))^{-1})\mathrm{d}P(z)$$
$$= \log_\lambda(\mathfrak{F}_{s_1}(U(1))(\mathfrak{F}_{s_2}(U(1)))^{-1}).$$

This gives

$$(\mathfrak{F}_{s_1}(U))_\lambda(t) - (\mathfrak{F}_{s_2}(U))_\lambda(t) \equiv \exp(t\log_\lambda(\mathfrak{F}_{s_1}(U(1)))) - \exp(t\log_\lambda(\mathfrak{F}_{s_2}(U(1))))$$
$$= (\mathfrak{F}_{s_1}(U))_\lambda(t)(\mathbb{1} - \mathrm{e}^{t\log_\lambda(\mathbb{1}+(\mathfrak{F}_{s_2}(U(1))(\mathfrak{F}_{s_1}(U(1)))^{-1}-\mathbb{1})}).$$

We thus find that

$$\text{s-lim}_{\varepsilon\to 0}\frac{1}{\varepsilon}((\mathfrak{F}_{s+\varepsilon}(U))_\lambda(t) - (\mathfrak{F}_s(U))_\lambda(t)) = t(\mathfrak{F}_s(U))_\lambda(t)(\partial_s\mathfrak{F}_s)(U(1))(\mathfrak{F}_s(U(1)))^{-1},$$

which is weakly local, as λ always falls within the mobility gap of $\mathfrak{F}_s(U(1)))$. For the time derivative we get similar formulas and following the same argument as above, we find that $|W(\mathfrak{F}_{s_1}(U))_\lambda) - W(\mathfrak{F}_{s_2}(U)_\lambda)|$ can also be made arbitrarily small. ☐

Lemma 5.3. *For any two unitary maps $A, B : [0,1] \to \mathcal{U}(\mathcal{H})$ which are differentiable, whose derivatives are bounded too, and which are weakly local, we have*

$$|W(A) - W(B)| \leq \sup_{[0,1]} \|T_1(A-B)\|_1 + \|T_2(\dot{A}-\dot{B})\|_1$$
$$+ \sup_{\alpha,\beta}(\|T_{3\alpha}(A-B)_{,\beta}\|_1 + \|(A-B)^*_{,\alpha}T_{4\beta}\|_1$$
$$+ \|(A-B)^*_{,\alpha}(A-B)_{,\beta}\|_1) \tag{5.3}$$

where T_1, T_2 are some (time-dependent) trace-class operators depending on A, B their derivatives w.r.t. time and their spatial derivatives; the supremum over α, β is over the two possibilities where $\alpha \neq \beta$. Then $T_{3\alpha}, T_{4\beta}$ is a weakly local operator confined in the α, β direction.

Proof. We start from (2.7) which says

$$W(A) = -\frac{1}{2}\int_0^1 \mathrm{d}t\ \mathrm{tr}\,\varepsilon_{\alpha\beta}\dot{A}A^*A_{,\alpha}A^*A_{,\beta}A^*$$

to get

$$|W(A) - W(B)| \leq \frac{1}{2}\sup_{[0,1]}\|\varepsilon_{\alpha\beta}(\dot{A}A^*A_{,\alpha}A^*A_{,\beta}A^* - \dot{B}B^*B_{,\alpha}B^*B_{,\beta}B^*)\|_1$$
$$\leq \frac{1}{2}\sup_{\alpha,\beta,[0,1]}\|\dot{A}A^*A_{,\alpha}A^*A_{,\beta}A^* - \dot{B}B^*B_{,\alpha}B^*B_{,\beta}B^*\|_1$$
$$\leq \frac{1}{2}\sup_{\alpha,\beta,[0,1]}(\|\dot{A}A^*(A_{,\alpha}A^*A_{,\beta}A^* - B_{,\alpha}B^*B_{,\beta}B^*)\|_1$$
$$+ \|(\dot{A}A^* - \dot{B}B^*)B_{,\alpha}B^*B_{,\beta}B^*\|_1)$$
$$\leq \frac{1}{2}\sup_{\alpha,\beta,[0,1]}(\|\dot{A}\|\|A_{,\alpha}A^*A_{,\beta}A^* - B_{,\alpha}B^*B_{,\beta}B^*\|_1$$
$$+ \|\dot{A}\|\|(A-B)B_{,\alpha}B^*B_{,\beta}B^*\|_1 + \|(\dot{A}-\dot{B})B^*B_{,\alpha}B^*B_{,\beta}\|_1).$$

The supremum is over all times in $[0,1]$ and all α,β equal to $1,2$ (without $\alpha=\beta$).

We concentrate on the term $\|A_{,\alpha}A^*A_{,\beta}A^* - B_{,\alpha}B^*B_{,\beta}B^*\|_1$ since the two other terms are in their final desired form. Because A,B are unitary we have $A_{,\alpha}A^* = -AA^*_{,\alpha}$ so that

$$\begin{aligned} A_{,\alpha}A^*A_{,\beta}A^* - B_{,\alpha}B^*B_{,\beta}B^* &= - AA^*_{,\alpha}A_{,\beta}A^* + BB^*_{,\alpha}B_{,\beta}B^* \\ &= (B-A)A^*_{,\alpha}A_{,\beta}A^* \\ &\quad - B(A^*_{,\alpha}A_{,\beta} - B^*_{,\alpha}B_{,\beta})A^* \\ &\quad + BB^*_{,\alpha}B_{,\beta}(B-A)^*. \end{aligned}$$

Only the middle line is not in the form we want, so that we write,

$$A^*_{,\alpha}A_{,\beta} - B^*_{,\alpha}B_{,\beta} = A^*_{,\alpha}(A-B)_{,\beta} + (A-B)^*_{,\alpha}A_{\beta} - (A-B)^*_{,\alpha}(A-B)_{,\beta}.$$

□

Acknowledgements

The authors thank Gian Michele Graf for many useful discussions. J. S. thanks Netanel H. Lindner for his hospitality at the Technion in Israel and for elaborations on the completely localized edge index. C. T. thanks Alain Joye for his invitation at institut Fourier and fruitful discussions on localization for unitary operators.

A. Appendix

A.1. Dynamical Localization Implies Almost Surely the Mobility Gap Condition

Here we discuss the validity of Definition 2.4 with respect to the existing literature on localization of random unitary models. The probabilistic localization study could start from a random time-dependent Hamiltonian $H:\Omega\times\mathbb{S}^1\to\mathcal{B}(\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^N)$ (with appropriate regularity, locality and measurability assumptions to be made explicit once the model has been chosen), where Ω is a probability space. Solving the Schrödinger equation and evaluating it after some time period, we end up with a random unitary operator $U(1)$ on $\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^N$. Alternatively, one could study an a priori given random unitary $U(1)$. Then we say dynamical localization on $\Delta\subseteq\mathbb{S}^1$ is established if (A.1) holds with $I:=B_1(\Delta)$ and $A_g:=g(U(1))$ for all $g\in I$. The following standard proposition (see [26, Theorem 7.6] for the self-adjoint case with simple-spectrum, but the argument is the same; [17, Prop. 3.2] covers the unitary case) gives its almost sure consequence:

Proposition A.1. *Let $A_i : \Omega \to \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$ be a family of random operators indexed by $i \in I$ for some set I. If*

$$\mathbb{E}\left[\sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\|\right] \leq C\mathrm{e}^{-\mu\|x-y\|} \quad (x, y \in \mathbb{Z}^d) \tag{A.1}$$

for some constants $C < \infty, \mu > 0$, then almost surely, for some (random) constant $C' > 0$, and any (deterministic) $\mu' \in (0, \mu), a \in \ell^1(\mathbb{Z}^d)$ we have

$$\sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \leq C' \frac{1}{|a(x)|} \mathrm{e}^{-\mu'\|x-y\|}$$

Proof. Let $a \in \ell^1$, $\mu' < \mu$. By Fatou's lemma we can bound

$$\begin{aligned} &\mathbb{E} \sum_{x,y\in\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)| \\ &\quad \leq \liminf_{\Lambda\to\mathbb{Z}^d} \sum_{x,y\in\Lambda} \mathbb{E} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)|. \end{aligned}$$

We now use our hypothesis (A.1) to get that this last expression is estimated by

$$\liminf_{\Lambda\to\mathbb{Z}^d} \sum_{x,y\in\Lambda} C\mathrm{e}^{-(\mu-\mu')\|x-y\|} |a(x)| < \infty.$$

Hence $\mathbb{E} \sum_{x,y\in\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)|$ is finite. But an integrable nonnegative function must be finite almost everywhere, that is, there is some (random) constant $C' < \infty$ such that almost surely,

$$\sum_{x,y\in\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)| < C'. \qquad \square$$

Thus the validity of (2.3) in our mobility gap definition relies on the validity of condition (A.1) with $I = B_1(\Delta)$. However, so far, starting from an a priori given random $U(1)$, condition (A.1) was proven for *continuous* functions multiplied by a projection onto Δ [5,17,19], i.e., for

$$I = \left\{ f : \mathbb{S}^1 \to \mathbb{C} | f = g\chi_\Delta \exists g \in C(\mathbb{S}^1; \mathbb{C}) : \|g\|_\infty \leq 1 \right\}$$

with Δ fixed and $A_f := f(U(1))$ for $f \in I$. (Coincidentally this is sufficient for Lemma 3.7.) This is in contrast to self-adjoint random case, where indeed the range of the supremum may be expanded to *Borel* functions (see [1, Eq-n (E.6)]). This difference is ultimately traced to the representation of the functional calculus (cf. [1, Eq-n (E.7)] with [17, Eq-n (5.28)]). Extending the unitary dynamical localization results from continuous to Borel bounded functions seems within a reach, but would require some work. Furthermore, these existing results do not include statements or proofs regarding the finite degeneracy of eigenvalues within the mobility gap which have been standard lore for self-adjoint random operators [31]. This should be the result of a zero-one argument once finite-rank perturbation theory for a chosen random model has been set up. Thus we conjecture Definition 2.4 to almost surely hold for the models in [5,17,19] but postpone what remains to be proved to future work.

Instead we provide an alternative example, taken from [34, Section III], that is simple from the point of view of localization theory, but at least exhibits a disordered model obeying Definition 2.4 with nontrivial topology. We note in passing that it is not clear to us if the previous unitary models [5,17,19] are associated with a unitary evolution whose index is nontrivial. Here the model may be described as a time-dependent Hamiltonian on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$ which is piece-wise constant

$$[0,1) \ni t \mapsto H(t) = H_i, \qquad \frac{i-1}{5} \leq t < \frac{i}{5},$$

such that its four first parts $H_1, \ldots, H_4$ are deterministic and tuned so that the propagator ends at $U(\frac{4}{5}) = \mathbb{1}$, although $U(t) \neq \mathbb{1}$ for $0 < t < 4/5$. In particular $[0, \frac{4}{5}] \ni t \mapsto U(t)$ has some nontrivial Floquet index (see the "clean" model in [29]). The fifth part is a disordered Hamiltonian of the form

$$H_5 = \lambda\omega(X) \otimes \mathbb{1}_2 + \delta\mathbb{1} \otimes \sigma_3,$$

where X is the position operator, $\lambda > 0$ is the disorder strength, $\delta > 0$ is the energy separation of two nonequivalent sites (bipartite lattice), σ_3 is the third Pauli matrix, and $\{\omega(x)\}_{x\in\mathbb{Z}^2}$ is a sequence of i.i.d.r.v. with some regular probability distribution. Since $U(\frac{4}{5}) = \mathbb{1}$, it is possible to calculate explicitly the full propagator and find that it is equal to

$$\begin{aligned} U(1) &= \exp\left(-\mathrm{i}\frac{1}{5}H_5\right) \\ &= \exp\left(-\mathrm{i}\frac{1}{5}(\lambda\omega(X)+\delta)\right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \exp\left(-\mathrm{i}\frac{1}{5}(\lambda\omega(X)-\delta)\right) \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

For our question of dynamical localization, the presence of the $\mathbb{C}^2$ factor is actually irrelevant, and the main point about this last expression is that it is diagonal in the position basis, i.e., we can explicitly write down any of Borel bounded function of it and it will be diagonal as well, and hence trivially exponentially decaying. This shows that (A.1) holds, actually even before taking the expectation, and for *all* realizations (not just almost surely). By adjusting δ and λ we can even close the gap with localized spectrum and yet the Floquet topology persists. Moreover in this model the finite degeneracy of the spectrum holds as long as the probability distribution for the onsite random potential is continuous. The spectrum is actually simple here.

The validity of Definition 2.4 away from the tuning point where $U(\frac{4}{5}) = \mathbb{1}$ or for more general topological Floquet models, such as the driven half-BHZ model with disorder ([12]), is a natural question that we conjecture to be true from the physics literature [34] but postpone its rigorous proof to separate work.

A.2. Floquet's RAGE

In this section we prove that our deterministic dynamical localization assumption implies pure point spectrum (so that it is not necessary to also have the latter as an assumption). This entails importing the analysis of the RAGE theorem to the unitary Floquet case. Most of this was already done in [11, Theorem 3.2] in the deterministic setting and in [17, Lemma 6.1] in the probabilistic setting. However we wanted to provide here a more concrete presentation, focused on the deterministic aspects, compared to the abstract one in [11]. We also set up the notation for Theorem A.6, with a proof as short as possible.

Within this section, let a unitary $U \in \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$ be given such that it is *localized*. For our purposes in this section it is enough to make the following

Definition A.2. U is deterministically dynamically localized *in the interval* $I \subseteq S^1$ iff there is some $\mu > 0$ such that for any $a \in \ell^1(\mathbb{Z}^d)$ there is some $C_a < \infty$ such that the following holds

$$\sup_{n\in\mathbb{N}} \sum_{x,\, y\in\mathbb{Z}^d} \|\langle \delta_x,\, U^n \chi_I(U)\delta_y\rangle\| \, e^{\mu\|x-y\|}|a(x)| = C_a \tag{A.2}$$

Lemma A.3 (Discrete Wiener). *Let μ be a complex measure on S^1. For $m \in \mathbb{N}$, we define its mth complex moment as $\mu_m := \int_{z\in S^1} z^m \mathrm{d}\mu(z)$. Then $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^n |\mu_m|^2 = \sum_{z\in S^1} |\mu(\{z\})|^2$ that is, the RHS gives the pure point part of $\left|\mu(S^1)\right|^2$.*

Proof. We have

$$\begin{aligned}
\frac{1}{n}\sum_{m=1}^{n} |\mu_m|^2 &\equiv \frac{1}{n}\sum_{m=1}^{n} \int_{z\in S^1} z^m \mathrm{d}\mu(z) \int_{w\in S^1} \overline{w}^m \mathrm{d}\overline{\mu}(w) \\
&= \frac{1}{n}\sum_{m=1}^{n} \int_{w\in S^1}\int_{z\in S^1} (z\overline{w})^m \mathrm{d}\mu(z)\mathrm{d}\overline{\mu}(w) \\
&= \int_{w\in S^1}\int_{z\in S^1} \frac{1}{n}\sum_{m=1}^{n} (z\overline{w})^m \mathrm{d}\mu(z)\mathrm{d}\overline{\mu}(w)
\end{aligned}$$

Note that the sequence of functions $\left\{S^1 \ni z \mapsto \frac{1}{n}\sum_{m=1}^n z^m\right\}_{n\in\mathbb{N}}$ is uniformly bounded by 1 and converges pointwise to $\delta(\cdot - 1)$. We may thus use the dominated-convergence theorem to find

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} |\mu_m|^2 &= \int_{w\in S^1}\int_{z\in S^1} \delta(z\overline{w}-1)\mathrm{d}\mu(z)\mathrm{d}\overline{\mu}(w) \\
&= \int_{z\in S^1} \mathrm{d}\mu(z)\overline{\mu(\{z\})} = \sum_{z\in S^1} |\mu(\{z\})|^2 .
\end{aligned}$$

□

Lemma A.4. *Let U be unitary and K compact. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \|KU^m\psi\|^2 = 0$$

for all $\psi \in \mathcal{H}^c$, the continuous part of the Hilbert space for U.

Proof. This is [2, Lemma 2.7] in our setting of discrete rather than continuous time. We thus omit the proof. □

Theorem A.5 (Unitary RAGE). *Let U be unitary and $\{A_L\}_L$ be a sequence of compact operators strongly converging to $\mathbb{1}$. Then*

$$\mathcal{H}^c = \left\{ \psi \in \mathcal{H} | \lim_{L\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \|A_L U^n \psi\|^2 = 0 \right\},$$

and

$$\mathcal{H}^p = \left\{ \psi \in \mathcal{H} | \lim_{L\to\infty} \sup_{n\in\mathbb{N}} \|(\mathbb{1} - A_L) U^n \psi\| = 0 \right\}.$$

Proof This is [2, Theorem 2.6] in our setting of discrete rather than continuous time, but the same proof goes through with very slight modifications. □

The following theorem and the remark after it are the reason for this section.

Theorem A.6 (Deterministic dynamical localization implies spectral localization). *If U is deterministically dynamically localized in the interval I, then it has pure point spectrum within that interval, that is,*

$$\sigma(U) \cap I = \sigma_{pp}(U) \cap I$$

Proof. Since $\{\delta_x\}_{x\in\mathbb{Z}^d}$ is an ONB for $\mathcal{H}$, and we want to show that $\chi_I(U)\mathcal{H} \subseteq \mathcal{H}^p$, let $y \in \mathbb{Z}^d$ be given. We claim that $\chi_I(U)\delta_y \in \mathcal{H}^p$. Let A_L be the projection onto a box of total volume $(2L+1)^d$ centered about the origin of $\mathbb{Z}^d$. Using Theorem A.5 it suffices to show

$$\lim_{L\to\infty} \sup_{n\in\mathbb{N}} \left\| A_L^\perp U^n \chi_I(U) \delta_y \right\| = 0.$$

By (A.2) we have *for any* $n \in \mathbb{N}$,

$$\sum_{x,\, y\in\mathbb{Z}^d} \|\langle \delta_x,\, U^n \chi_I(U)\delta_y \rangle\| \, e^{\mu\|x-y\|} |a(x)| \le C_a.$$

This in turn implies that

$$\begin{aligned} \|\langle \delta_x,\, U^n \chi_I(U)\delta_y \rangle\| \, e^{\mu\|x-y\|} |a(x)| &\le \sum_{x',y'} \|\langle \delta_{x'},\, U^n \chi_I(U)\delta_{y'} \rangle\| \, e^{\mu\left\|x'-y'\right\|} |a(x')| \\ &\le C_a \end{aligned}$$

since all terms are positive. Hence, $\|\langle \delta_x,\, U^n \chi_I(U)\delta_y \rangle\| \le C_a |a(x)|^{-1} \mathrm{e}^{-\mu\|x-y\|}$ uniformly in n.

Now we have

$$\begin{aligned} \left\| A_L^\perp U^n \chi_I(U)\delta_y \right\|^2 &= \sum_{x\in\mathbb{Z}^d : \|x\|>L} \|\langle \delta_x,\, U^n \chi_I(U)\delta_y \rangle\|^2 \\ &\text{(Using } \|\langle \delta_x,\, U^n \chi_I(U)\delta_y \rangle\| \le 1) \\ &\le \sum_{x\in\mathbb{Z}^d : \|x\|>L} \|\langle \delta_x,\, U^n \chi_I(U)\delta_y \rangle\| \le \sum_{x\in\mathbb{Z}^d : \|x\|>L} C_\varepsilon |a(x)|^{-1} \mathrm{e}^{-\mu\|x-y\|}. \end{aligned}$$

Hence since the square root is monotone increasing and continuous, and using $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, we find

$$\left\| A_L^{\perp} U^n \chi_I(U) \delta_y \right\| \le \sqrt{C_a} \sum_{x \in \mathbb{Z}^d : \|x\| > L} |a(x)|^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\mu \|x-y\|}$$

for any $n \in \mathbb{N}$ so that taking the supremum on both sides (redundant on the RHS) and then the limit $L \to \infty$ we get zero indeed. This follows because (for appropriate a) $|a(x)|^{-\frac{1}{2}} e^{-\frac{1}{2}\mu\|x-y\|}$ is summable in x, and hence taking the limit $L \to \infty$ gives zero. □

In our application of the stretch function, since $F_\Delta \in B_1(\Delta)$, $V(1)$ is actually dynamically localized as in Definition 2.4 on $\mathbb{S}^1 \backslash \{1\}$, and thus one may invoke Theorem A.6 to conclude $\sigma(V(1)) = \sigma_{pp}(V(1))$.

A.3. Helffer–Sjöstrand Formula for Unitary Operators

Helffer–Sjöstrand formula extends holomorphic functional calculus to smooth functions. It was developed for Hermitian operators but can be easily adapted to unitaries, with the simplification that the latter are always bounded. A formula was already proposed in [21] for functions on $\mathbb{S}^1 \backslash \{1\}$ and based on Cayley transformation. Here we provide another proof for any smooth function on $\mathbb{S}^1$ using a conformal mapping.

Lemma A.7. *Let $f : \mathbb{S}^1 \to \mathbb{C}$ be a smooth function. There exists a quasi-analytic extension $\tilde{f} : \mathbb{C} \to \mathbb{C}$, i.e., $\tilde{f}|_{\mathbb{S}^1} = f$ and $\partial_{\bar{z}} \tilde{f}|_{\mathbb{S}^1} = 0$, such that for any unitary operator U*

$$f(U) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} (\partial_{\bar{z}} \tilde{f}(z))(z-U)^{-1} \mathrm{d}z \mathrm{d}\bar{z} \tag{A.3}$$

Moreover $\tilde{f}$ is compactly supported around $\mathbb{S}^1$ and satisfies $|\partial_{\bar{z}} \tilde{f}| \le C||z|-1|^N$ for any $N \ge 2$.

Proof. Any function $f : \mathbb{S}^1 \to \mathbb{C}$ can be equivalently described by a periodic function $g : \mathbb{R} \to \mathbb{C}$, through the conformal mapping $w \mapsto z = \mathrm{e}^{\mathrm{i}w}$ by $g(w) = f(\mathrm{e}^{\mathrm{i}w})$, satisfying $g(w + 2\pi) = g(w)$ by construction. This bijective mapping extends to the annulus $\mathcal{A}_r$ where $\mathrm{e}^{-r} < |z| < \mathrm{e}^r$ corresponding to the strip $-r < \mathbb{Im}(w) < r$. In both cases the smoothness of f and g is the same. Let $\chi : \mathbb{R} \to \mathbb{C}$ be a smooth function supported in $(-r, r)$ and with $\chi(x) = 1$ near 0. On the real line, we know from Ref. [18] that for $N \ge 2$

$$\tilde{g}(\theta, \tau) = \sum_{k=0}^{N-1} g^{(k)}(\theta) \frac{(\mathrm{i}\tau)^k}{k!} \chi(\tau) \tag{A.4}$$

is a quasi-analytic extension of g on the strip, namely $\tilde{g}(\theta, 0) = g(\theta)$ and $\partial_{\bar{w}} g|_{\tau=0} = 0$, for $w = \theta + \mathrm{i}\tau$ and $\partial_{\bar{w}} = 1/2(\partial_\theta + \mathrm{i}\partial_\tau)$. Moreover, $|\partial_{\bar{w}} g| \le C|\tau|^N$. We claim that $\tilde{f}(z = \mathrm{e}^{\mathrm{i}(\theta + \mathrm{i}\tau)}) := \tilde{g}(\theta, \tau)$ is a quasi-analytic extension of f on the annulus. Indeed $\tilde{f}$ coincides with f on $\mathbb{S}^1$ and

$$\partial_{\bar{w}} \tilde{g} = \partial_{\bar{w}} (\overline{\mathrm{e}^{\mathrm{i}w}}) \partial_{\bar{z}} \tilde{f} = -\mathrm{i}\bar{z} \partial_{\bar{z}} \tilde{f} \tag{A.5}$$

so that $\partial_{\bar z}\tilde f|_{\tau=0} = 0$. Moreover on the annulus one has $\mathrm{e}^{-r} < |-\mathrm{i}\bar z| < \mathrm{e}^{r}$ and $|\ln x| \le \mathrm{e}^{r}|x-1|$ for $x \in (\mathrm{e}^{-r}, \mathrm{e}^{r})$ applied to $x = |z| = \mathrm{e}^{-\tau}$ we infer $|\tau| \le C||z|-1|$ so that

$$|\partial_{\bar z}\tilde f| \le C||z|-1|^N \tag{A.6}$$

with a different constant C. With the fact that $\left\|(z-U)^{-1}\right\| \le ||z|-1|^{-1}$ for a unitary U we deduce that the integral in (A.3) is absolutely convergent in norm. Then we claim that for $z_0 \in \mathbb{S}^1$

$$f(z_0) = \frac{1}{2\pi}\int_{\mathbb{C}} (\partial_{\bar z}\tilde f(z))(z-z_0)^{-1}\mathrm{d}z\mathrm{d}\bar z \tag{A.7}$$

The integral is reduced to the annulus $\mathcal{A}_r$ since $\tilde f$ is supported inside it and has to be understood as an improper integral on $\mathcal{A}_r \backslash \mathcal{A}_\varepsilon$ when $\epsilon \to 0$. The equality follows by [21, Cor. 2.3], and (A.3) follows by the functional calculus. □

Corollary A.8. *The smooth functional calculus of a strictly exponentially local unitary is strictly polynomially local.*

Proof. This is a direct consequence of Helffer–Sjöstrand formula (A.3), the fact that $\tilde f$ is smooth and compactly supported, and Combes–Thomas estimate [7] ([17] in the context of unitaries): if U is local then it exists $0 < C < \infty$ such that

$$|R_U(z)|_{x,y} \le \frac{C}{||z|-1|}\mathrm{e}^{-\mu(z)\|x-y\|} \tag{A.8}$$

for $\mu > 0$ small enough. For example one can take $\mu(z) = c||z|-1|$ as in [9]. According to Lemma A.7 the quasi-analytic extension of f satisfies $|\partial_{\bar z}\tilde f(z)| \le C||z|-1|^N$ for $N \ge 2$ so that

$$|f(U)|_{x,y} \le \frac{1}{2\pi}\int \mathrm{d}z\mathrm{d}\bar z|\partial_{\bar z}\tilde f(z)||R_U(z)|_{x,y} \le C_N(1+c\,\|x-y\|)^{-N} \tag{A.9}$$

□

A.4. Convergence Properties of Weakly Local Operators

The following claim may be found, e.g., in [9, Eq. (56)]:

Lemma A.9. *Let $\mathcal{H}$ be any separable Hilbert space, $A_n \to A$ strongly in $\mathcal{B}(\mathcal{H})$, and $B_n \to B$ in trace-class norm. Then $A_nB_n \to AB$ in trace-class norm.*

Lemma A.10. *If $A_n \to A$ strongly within the star algebra of weakly local operators (so A is also assumed to be weakly local), then $\partial_j A_n \to \partial_j A$ strongly within the ideal of weakly local and confined in direction j operators.*

Proof. We already know that $\partial_j A_n$ (for all n) and $\partial_j A$ are weakly local and confined in direction j by the results of Sect. 3.3. Now let $\psi \in \mathcal{H}$ be given. We have

$$
\begin{aligned}
\|\partial_j A_n\psi - \partial_j A\psi\| &\le \|\Lambda_j(A_n - A)\psi - (A_n - A)\Lambda_j\psi\| \\
&\le \|(A_n - A)\psi\| + \|(A_n - A)\Lambda_j\psi\| \\
&\to 0.
\end{aligned}
$$

□

Lemma A.11. *If $A_n \to A$ strongly within the ideal of weakly local and confined in direction 1 operators, all having a uniform both ν and sufficiently large μ as in Definition 3.9, and T is weakly local and confined in direction 2, then $TA_n \to TA$ in trace-class norm.*

Proof. We have $TA_n = T(1+|X_1|)^{-\mu}(1+\|X\|)^{\nu}(1+\|X\|)^{-\nu}(1+|X_1|)^{\mu}A_n$. WLOG, we also pick μ such that $T(1+|X_1|)^{-\mu}(1+\|X\|)^{\nu}$ is trace-class, and note that $(1+\|X\|)^{-\nu}(1+|X_1|)^{\mu}A_n \to (1+\|X\|)^{-\nu}(1+|X_1|)^{\mu}A$ strongly. We verify these two statements:

$$
\begin{aligned}
\|T(1+|X_1|)^{-\mu}(1+\|X\|)^{\nu}\|_1 &\le \sum_{xy}\|T_{xy}\|(1+|y_1|)^{-\mu}(1+\|y\|)^{\nu} \\
&\le \sum_{xy} C^T_\mu(1+\|x-y\|)^{-\mu}(1+|y_2|)^{-\mu} \\
&\quad\cdot(1+\|y\|)^{\nu}(1+|y_1|)^{-\mu}(1+\|y\|)^{\nu} \\
&< \infty.
\end{aligned}
$$

For the second statement, let $C_n := A_n - A$. Then

$$
\begin{aligned}
&\|(1+\|X\|)^{-\nu}(1+|X_1|)^{+\mu}C_n\psi\|^2 \\
&\quad\equiv \langle(1+\|X\|)^{-\nu}(1+|X_1|)^{+\mu}C_n\psi, (1+\|X\|)^{-\nu}(1+|X_1|)^{+\mu}C_n\psi\rangle \\
&\quad= \langle(1+\|X\|)^{-2\nu}(1+|X_1|)^{+2\mu}C_n\psi, C_n\psi\rangle \\
&\quad\le \|(1+\|X\|)^{-2\nu}(1+|X_1|)^{+2\mu}C_n\psi\|\|C_n\psi\| \\
&\quad\le \|(1+\|X\|)^{-2\nu}(1+|X_1|)^{+2\mu}C_n\|\|C_n\psi\|.
\end{aligned}
$$

The first norm is finite (for each n) by Lemma 3.11, and the second goes to zero because $C_n \to 0$ strongly.

Then we use the result that if S is trace-class and $B_n \to B$ strongly then $SB_n \to SB$ in trace-class norm with $S := T(1+|X_1|)^{-\mu}(1+\|X\|)^{\nu}$ and $B_n := (1+\|X\|)^{-\nu}(1+|X_1|)^{\mu}A_n$. □

Lemma A.12. *If $A_n \to A, B_n \to B$ strongly within the ideals of weakly local and confined in directions 1 and 2, respectively, all having a uniform both ν and sufficiently large μ as in Definition 3.9, then $A_nB_n \to AB$ in trace-class norm.*

Proof. We again write the factorization

$$
\begin{aligned}
A_nB_n &= A_n(1+|X_1|)^{\mu}(1+\|X\|)^{-\nu} \\
&\quad\cdot(1+|X_1|)^{-\mu}(1+\|X\|)^{2\nu}(1+|X_2|)^{-\mu} \\
&\quad\cdot(1+\|X\|)^{-\nu}(1+|X_2|)^{\mu}B_n \\
&= A_n(1+|X_1|)^{\mu}(1+\|x\|)^{-\nu}\cdot(1+|X_1|)^{-\mu/2}(1+\|X\|)^{\nu}(1+|X_2|)^{-\mu/2} \\
&\quad\cdot(1+|X_1|)^{-\mu/2}(1+\|X\|)^{\nu}(1+|X_2|)^{-\mu/2}\cdot(1+\|X\|)^{-\nu}(1+|X_2|)^{\mu}B_n.
\end{aligned}
$$

Now if μ is chosen sufficiently large, then the last expression is the product of four factors. The first one converges strongly as shown in the lemma. The second and third are trace-class, and the fourth also converges strongly. Thus we conclude the statement based on the properties of products of limits and the previous lemma. □

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Communicated by Alain Joye.
Received: September 17, 2018.
Accepted: January 28, 2019.