RESEARCH STATEMENT

ROMÉO TROUBAT

My research interests lie at the intersection between geometry and representation theory. My goal is, in a broad sense, the study of pseudo-Riemannian geometry in non-Lorentzian signature. Part of my work consists in generalizing the concepts of causality and global hyperbolicity to pseudo-Riemannian spaces of general signature, while the other part consists in the study of Anosov subgroups of $\mathrm{SO}(p,q)$ and their associated geometric structures.

1. Context

Let (M,g) be a smooth conformal Lorentzian manifold. A smooth path $c : I \to M$ is said to be *causal* if for each $t \in I$, $g(\dot{c}(t)) \leqslant 0$. It is said to be *inextendible* if c does not converge at the boundaries of the interval I. When M admits a *time orientation*, one may also ask for causal paths to be *future oriented* or *past oriented*. The notion of future oriented causal path can be extended to a familly of non-smooth curves. We shall say that a subset $S \subset M$ is a *Cauchy surface* of M if for each inextendible future oriented causal curve c, there exists a unique $t \in I$ such that $c(t) \in S$. It is a well known fact in Lorentzian geometry due to Geroch ([Ger70]) that in a causal space M, the following properties are equivalent :

1. For each pair of points $x, y \in M$, the space of non-parametrized future oriented causal curves from x to y is either empty or compact for the uniform topology,
2. There exists a Cauchy surface $S \subset M$,
3. There exists a map $T : M \to \mathbb{R}$ such that for each $t \in \mathbb{R}$, $T^{-1}(t)$ is a Cauchy surface. Equivalently, for each inextendible future oriented causal curve c, $T \circ c : I \to \mathbb{R}$ is a bijection.

When M satisifes one of those properties, it is said to be *globally hyperbolic* and the map T is called a *Cauchy time function.* Conformal Lorentzian spaces of signature $(p,1)$ are geometric spaces which are infinitesimally modeled on $\mathrm{Ein}^{p,1}$ in the sense of Cartan geometries, $\mathrm{Ein}^{p,1}$ being the space of isotropic lines in $\mathbb{R}^{p+1,2}$. It is one of the three parabolic spaces associated with $\mathrm{SO}_0(p+1,2)$ $(p \geqslant 2)$, the two others being the space of photons and pointed photons in $\mathrm{Ein}^{p,1}$.

Let $p \geqslant q$ be two positive integers and let $\mathrm{SO}_0(p,q)$ be the neutral component of the indefinite orthogonal group $\mathrm{O}(p,q)$. It is split if and only if $p = q$ or $p = q+1$. Let $\Theta = \{i_1, ..., i_k\}$ be a subset of the roots of $\mathrm{SO}_0(p,q)$. The associated parabolic space $\mathrm{SO}_0(p,q)/P_\Theta$ is the space of partial flags of isotropic sub-spaces of $\mathbb{R}^{p+q}$, $V_{i_1} \subset ... \subset V_{i_k}$ with $\dim(V_{i_\ell}) = i_\ell$ for all ℓ. Of particular interests are the following spaces :

1. $\Theta = \{1\}$, $\mathrm{SO}_0(p,q)/P_1$ is the space of isotropic lines in $\mathbb{R}^{p,q}$, commonly refered to as the *Einstein space* $\mathrm{Ein}^{p-1,q-1}$. The spaces whose geometry are locally modeled on $\mathrm{Ein}^{p-1,q-1}$ in the Cartan sense are the conformal pseudo-Riemannian spaces of signature $(p-1,q-1)$.
2. $\Theta = \{1, ..., q-1\}$, $\mathrm{SO}_0(p,q)/P_{1,...,q-1}$ is the space of sub-maximal flags of isotropic sub-spaces of $\mathbb{R}^{p+q}$, $V_1 \subset ... \subset V_{q-1}$. We know from the works of Guichard and Wienhard ([GW18]) that this space admits a Θ-*positive structure*, i.e a family of connected components of the double Schubert cells on which a sub-semigroup of the radical unipotent U_Θ acts transitively.

(3) $\Theta = \{1, ..., q\}$ and $p = q + 1$, $\mathrm{SO}_0(q+1, q)/P_{1,...,q}$ is the space of complete flags of isotropic subspaces of $\mathbb{R}^{q+1+q}$. In this case, $\mathrm{SO}_0(q+1, q)$ is split and P_Θ is one of its Borel subgroups, meaning that the associated space also admits a Θ-positive structure.

The case $q = 2$, $\Theta = \{1\}$ brings us back to conformal Lorentzian geometry. This falls in the first two cases we considered, meaning that the space $\mathrm{Ein}^{p-1,1}$ of isotropic lines of $\mathbb{R}^{p,2}$ admits a Θ-positive structure. In this case, this structure is only the causal structure on $\mathrm{Ein}^{p-1,1}$ in the Lorentzian sense. My goal is two study the parabolic spaces associated with $\mathrm{SO}_0(p, q)$ and the spaces onto which they are modeled, with particular attention given to $\mathrm{Ein}^{p-1,q-1}$ with an attempt to generalize the notions of global hyperbolicity for non-Lorentzian signatures.

2. Global hyperbolicity in pseudo-Riemannian spaces

In a work not yet published, I extend the notion of global hyperbolicity to the conformal pseudo-Riemannian spaces of signature (p, q) for $q \geqslant 2$. Let N_0 be a q-dimensional smooth manifold and let $f : N_0 \to M$ be an immersion. It is said to be *causal* if for each $x \in N_0$, the restriction of $g_{f(x)}$ to $Im(df_x)$ is non-positive, potentially degenerate. The bundle $Gr_q^{\leqslant 0}(TM)^+ \to M$ of oriented non-positive q tangent vector spaces has every fiber made of two connected components; when the bundle is trivial, M is said to be *time orientable* and the choice of a connected component gives a *time orientation* of M. A causal immersion $f : N_0 \to M$ where N_0 is oriented is said to be future oriented if for each $x \in N_0$, $Im(df_x)$ is in the right connected component of $Gr_q^{\leqslant 0}(T_xM)^+$. As in the Lorentzian case, we then extend this notion to a familly of non-necessarely smooth maps f.

Finally, we will say that f is *inextendible* if for each inextendible curve c in N_0, $f \circ c$ is inextendible in M. We may then define the notion of Cauchy surface.

Definition 2.1. A subset $S \subset M$ is a *Cauchy surface* is for each inextendible future oriented causal map $f : N_0 \to M$, there exists a unique $x \in N_0$ such that $f(x) \in S$.

Definition 2.2. Let N be a q-dimensional manifold. A map $T : M \to N$ is said to be a *Cauchy time function* if for each $y \in N$, $T^{-1}(y)$ is a Cauchy surface. Equivalently, for each inextendible future oriented causal map $f : N_0 \to M$, the map $T \circ f : N_0 \to N$ is bijective. When such a map exists, M is said to be *globally hyperbolic.*

We provide examples of globally hyperbolic spaces, as well a spaces admitting a Cauchy surface which are not globally hyperbolic, showing that the equivalence no longer holds in higher signature.

Proposition 2.3.
- *$M = \mathbb{R}^{p,q}$, the projection $\pi : \mathbb{R}^{p,q} \to \mathbb{R}^q$ is a Cauchy time function,*
- *$M = P \times N$ where P is a complete Riemannian manifold of dimension p, N is a simply connected anti-Riemannian manifold of dimension q, the projection $\pi : M \to N$ is a Cauchy time function,*
- *Let $\pi : (M, g_M) \to (N, g_N)$ be a complete Riemannian submersion with N simply connected and for each $x \in M$, $v \in T_xM$, let $g(v) = g_M(v) - 2g_M(p|_{Ker(\pi)^\perp}(v))$ where $p|_{Ker(\pi)^\perp}$ is the orthogonal projection. Then (M, g) is a pseudo-Riemannian manifold of signature (p, q) and $\pi : M \to N$ is a Cauchy time function. In particular, every fiber bundle over a simply connected base can be endowed with a conformal pseudo-Riemannian metric for which the projection is a Cauchy time function.*
- *Let $q \geqslant 3$ and $M = \mathbb{R}^{p,q} \setminus \{0\}$. Then every non-trivial translation of $\mathbb{R}^{p,0}$ is a Cauchy surface of M but M is not globally hyperbolic.*

I prove a result regarding the topological structure of globally hyperbolic spaces, which is to be compared to a result of Geroch ([Ger70]) in the Lorentzian setting. Let $T : M \to N$ be a smooth Cauchy time function on M admitting at least one compact level set and let $\gamma : I \to N$ be a smooth path in N. Let $\gamma^*M = \bigsqcup_{t \in I} T^{-1}(\gamma(t))$ endowed with the Lorentzian metric coming from the decomposition $T_x\gamma^*M = Ker(dT) \oplus^\perp \mathrm{Vect}(dT^{-1}(\dot\gamma(t)))$ and the projection $\gamma^*T : \gamma^*M \to I$.

Theorem 2.4. *Under those assumptions, the map γ^*T is a Cauchy time function in the Lorentzian sense on the space γ^*M. Since this holds for each γ, in particular, $T : M \to N$ is a fiber bundle on N.*

I also introduce the notion of causal diamonds in higher signature. Let $g : N_1 \to M$ be a causal immersion with N_1 a closed smooth manifold of dimension $q-1$. Let $J(g)$ be the set of maps $f : N_0 \to M$ where N_0 is a smooth compact q-dimensional manifold with boundary $\partial N_0 \simeq N_1$, $f|_{\mathring{N}_0}$ is a causal future oriented map and $f|_{\partial N_0} = g$ quotiented by composition with diffeomorphisms of N_0. The set $J(g)$ endowed with the uniform topology is called the causal diamond of g.

Theorem 2.5. *Let M be a globally hyperbolic space. Then the causal diamonds of M are either empty or compact.*

Unlike in the Lorentzian case, the converse is not true; counter examples emerge from the works of Collier, Tholozan and Toulisse ([CTT19]). In the pseudo-Riemannian spaces where all causal diamonds are compact, which now include globally hyperbolic spaces, this compacity can be used to deduce a result regarding the Plateau problem which should be considered an analogue to the existence of causal geodesics in the Lorentzian setting. Let $f : N_0 \to M$ be a causal future oriented map. Since f is smooth almost everywhere, we may define the q-dimensional volume $Vol(f)$ of f as the volume of N_0 with the pullback metric of M by f.

Theorem 2.6. *Let M be a space with compact diamonds and let $g : N_1 \to M$ be a causal immersion with N_1 a smooth closed manifold of dimension $q-1$. Then there exists a map $f \in J(g)$ such that $-Vol(f)$ is maximal.*

3. Anosov subgroups in $\mathrm{SO}_0(p,q)$

In this section I will discuss a joint work with Clarence Kineider. The goal of our paper ([KT24] has been to count the number of connected components in the space of triples of transverse flags for any parabolic subgroup P_Θ and to use this count to obtain rigidity results regarding the P_Θ-Anosov subgroups of $\mathrm{SO}_0(p,q)$.

Let $\mathcal{F}_0, \mathcal{F}_\infty$ be two transverse points in $\mathrm{SO}_0(p,q)/P_\Theta$ and let Ω be the set of point $\mathcal{F}$ which are transverse to both $\mathcal{F}_0$ and $\mathcal{F}_\infty$. The space Ω usually contains multiple connected components.

Theorem 3.1 (with C. Kineider)**.** *The number of connected component of $\Omega \subset \mathrm{SO}_0(p,q)/P_\Theta$ is know for all p,q and all Θ. Furthermore, we give a local parametrisation of those connected components and a global parametrisation of the Θ-positive ones.*

By using results from Dey, Greenberg and Riestenberg (see [Dey24], [DGR23]) we then prove the following rigidity results :

Theorem 3.2 (with C. Kineider)**.**
- *When $p = q+1$, $q = 1$ or $q = 2$ mod 4 and Θ contains the last root, any P_Θ-Anosov subgroup of $\mathrm{SO}_0(q+1,q)$ is virtually isomorphic to either a free group or a surface group.*
- *When $p = q$, $q = 2$ mod 4 and Θ contains any of the last two roots, any P_Θ-Anosov subgroup of $\mathrm{SO}_0(q,q)$ is virtually isomorphic to either a free group or a surface group.*

We build counter examples to some of the cases not covered by the theorem using a combination result by Dey and Kapovitch ([DK23]):

Theorem 3.3 (with C. Kineider)**.** *There exists a subgroup Γ of $\mathrm{SO}_0(p,q)$ which is $P_{1,\dots,q-2}$-Anosov and is isomorphic to the free product of a surface group and a cyclic group.*

4. Future research

4.1. Θ-positive subgroups in $\mathrm{SO}_0(q+1,q)$ which are Borel Anosov are Hitchin. In his paper [Dav24], Davalo proves that any subgroup Γ of $\mathrm{SO}_0(3,2)$ which is both maximal and Borel Anosov is Hitchin. A natural generalization of this result would be to prove that any subgroup Γ of $\mathrm{SO}_0(q+1,q)$ which is both Θ-positive for $\Theta=\{1,...,q-1\}$ and Borel Anosov has to be Hitchin, i.e Θ-positive for $\Theta=\{1,...,q\}$. I would like to try and combine the knowledge obtained from my study of the double Schubert cells of $\mathrm{SO}_0(q+1,q)$ and the methods employed in [Dav24] to prove this result.

4.2. P_1-Anosov representations in $\mathrm{SO}_0(p,q)$ as holonomies of $\mathbb{H}^{p,q-1}$-spaces. Let Λ be a positive $p-1$-dimensional sphere in $\mathrm{Ein}^{p-1,q-1}$ seen as the boundary of $\mathbb{H}^{p,q-1}$. Let $\Omega(\Lambda)$ be the set of points in $\mathbb{H}^{p,q-1}$ which are not causally related to any point of Λ. Finally, let $\rho:\Gamma\to\mathrm{SO}_0(p,2)$ be a representation preserving Λ. It is know that when $\Omega(\Lambda)$ is non-empty, ρ acts properly discontinuously on $\Omega(\Lambda)$.

I would like to use the notion of global hyperbolicity I introduced in [Tro24] to generalize a result by Mess, proven in [Mes07] for $p=2$ and in [And+07] for $p\geqslant 3$:

Theorem 4.1 (Mess)**.** *Let M be a maximal globally hyperbolic Lorentzian space locally modeled on $\mathbb{H}^{p,1}$ with a complete Cauchy surface. Then there exists a representation $\rho:\pi_1(M)\to\mathrm{SO}_0(p,2)$ which preserves a positive $(p-1)$-sphere Λ in $\partial\mathbb{H}^{p,1}$ such that the quotient $\Omega(\Lambda)/\rho$ is isometric to M. Inversely, any such quotient is globally hyperbolic with a complete Cauchy surface.*

A pseudo-Riemannian manifold (M,g) will be said to be *causally convex* if for each inextendible immersion $f:N_0\to M$ which is locally totally geodesic, any two points in (N_0,f^*g) have a geodesic coming from one to the other.

Conjecture 4.2. *Let M be a maximal causally convex pseudo-Riemannian space locally modeled on $\mathbb{H}^{p,q-1}$ with a complete Cauchy surface. Then there exists a representation $\rho:\pi_1(M)\to\mathrm{SO}_0(p,q)$ which preserves a positive $(p-1)$-sphere Λ in $\partial\mathbb{H}^{p,q-1}$ such that the quotient $\Omega(\Lambda)/\rho$ is isometric to M. Inversely, any such quotient is causally convex with a complete Cauchy surface.*

4.3. Θ-causality. Let $\Theta=\{1,...,q-1\}$. The notion of Θ-positivity gives us a nice notion of future oriented and past oriented paths in $\mathrm{SO}(p,q)/P_{1,...,q-1}$. Let M be a conformally flat pseudo-Riemannian manifold of signature $(p-1,q-1)$ and let $\mathrm{Pho}_{0,...,q-2}(M)$ be the space of sub-maximal flags of photons in M, $\{x\}\subset Q_1\subset ...\subset Q_{q-2}$. Since M is a $(\mathrm{SO}_0(p,q),\mathrm{SO}_0(p,q)/P_1)$-structure, $\mathrm{Pho}_{0,...,q-2}$ must then be a $(\mathrm{SO}_0(p,q),\mathrm{SO}_0(p,q)/P_{1,...,q-1})$-structure, which endows $\mathrm{Pho}_{0,...,q-2}$ with a notion of future oriented and past-oriented paths. I would be interested in studying the nature of this "Θ-causality" structure on M, if it can be related to the notions introduced in [Tro24], what some reasonable Θ-causality conditions on M may tell us on its geometric properties. I would also like to generalize this notion to non-conformally flat manifolds by considering the fiber bundle of sub-maximal photon flags in tangent spaces of M, $\mathrm{Pho}_{0,...,q-2}(M)=\{(x,Q_1,...,Q_{q-2}),x\in M,Q_1\subset ...\subset Q_{q-2}\subset T_xM\}$ which should be endowed with a Cartan geometry modeled on $(\mathrm{SO}_0(p,q),\mathrm{SO}_0(p,q)/P_{1,...,q-1})$ inhertited from M.

4.4. Proper open space associated to a Θ-positive representation in $\mathrm{SO}_0(p,q)$. Let $\rho:\Gamma\to\mathrm{SO}_0(p,2)$ be a Θ-positive representation. The method used by Collier and Tholozan, then by Barbot and Danciger-Guéritaud-Kassel for higher dimensional lattices (see [Bar15], [CTT19], [DGK17]), is to embbed $\mathrm{SO}_0(p,2)$ in $\mathrm{SO}_0(p,3)$ diagonally and to consider the set of points transverse to every point in the image of the Anosov map $\xi:\mathbb{S}^1\to\mathrm{Ein}^{p-1,1}\subset\mathrm{Ein}^{p-1,2}$. This gives us two proper connected components, one of which is the set $\Omega(\xi(\Gamma))\subset\mathbb{H}^{2,p-1}$ discussed before. I would like to adapt those methods by considering a Θ-positive representation ρ in $\mathrm{SO}_0(p,q)$, embedding it in $\mathrm{SO}_0(p,q+1)$ and considering the set of points transverse to every point of the image of ξ in $\mathrm{SO}_0(p,q+1)/P_{1,\dots,q-1}$. This should yield a union of 2^{q-1} connected components, all of which are proper in $\mathrm{SO}_0(p,q+1)/P_{1,\dots,q-1}$. I would like to study those spaces and their quotient by the action of ρ.

4.5. **Splitting theorem in higher signature.** The Cheeger–Gromoll splitting theorem is a well known result in Riemannian geometry giving sufficient conditions for a space to be isometric to a splitting $(N \times \mathbb{R}, g_N + dt^2)$. A Lorentzian analogue has been proven by Eschenburg in [Esc88] :

Theorem 4.3 (Eschenburg)**.** *Let (M,g) be a connected, time orientable Lorentzian manifold admitting a complete timelike geodesic $\gamma : \mathbb{R} \to M$ and such that $Ric(v,v) \geqslant 0$ for each timelike vector v. Then M is isometric to $(S \times \mathbb{R}, g_S - dt^2)$ where (S, g_s) is a complete Riemannian manifold and the factor $(\mathbb{R}, -dt^2)$ is represented by γ.*

I would like to see if the notion of global hyperbolicity I introduced in [Tro24] could be used to give a nice translation of this theorem and of others into the pseudo-Riemannian setting.

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Roméo Troubat, Institut de Recherche Mathématique Avancée, Université de Strasbourg, Strasbourg, France