TQFT de Turaev-Viro associée à une catégorie sphérique de fusion
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Introduction

It is well known that there are strong interactions between knot theory and the study of compact orientable 3–dimensional manifolds due, in particular, to classical theorems in low–dimensional topology such as the Lickorish–Wallace theorem that says: every compact orientable 3–manifold can be obtained by doing a surgery on a link in the 3–sphere. Thus, two interesting (closely related) problems are:

- How to separate 3–manifolds? That is, given two 3–manifolds how can we distinguish them?
- How to separate knots? That is, given two knots (or links) how can we distinguish them?

A natural answer is to construct topological invariants, that is, a kind of “assignment” of an algebraic quantity to each topological object under consideration, with the property that two equivalent objects (homeomorphic in the case of 3–manifolds, and isotopic in the case of knots) have the same “assignment”. This is a very fruitful approach which has permeated not only topology but almost all mathematical branches.

Some of these invariants are for example the fundamental group, homology groups, cohomology rings, Reidemeister torsion, among others in the case of 3–manifolds and for example linking number, fundamental group of the exterior, bridge number, Alexander polynomial, among others in the case of knots. These kind of invariants are currently known as classical invariants in contrast with a new kind of invariants called quantum invariants, which go beyond “classical topology” (homology theory, homotopy theory, covering spaces, etc.).

The context of these quantum invariants is the so called “quantum topology”, a mathematical branch which is a confluence of both mathematical and physical ideas. Notably Hopf algebras, quantum groups, topology of knots and 3–manifolds, among others on the mathematics side, and quantum field theory, statistical mechanics and conformal field theory on the physics side.

There are two remarkable facts in this subject, first the discovery of the Jones polynomial in 1984 (see [J87]) in the theory of operator algebras, which revolutionized the study of knots and 3–manifolds. Second the seminal works of Witten (see [W88, W89]) who proposed new invariants of 3–manifolds, gave a new interpretation of the Jones polynomial and introduced the idea of topological quantum field theory (or TQFT), a notion that later would be axiomatized by Atiyah [A88].

There are several approaches to these quantum invariants; the original Chern-Simons path integral approach [W88] (which only exists at a physical level of rigor), conformal field theory [Koh98] and also combinatorial approaches [RT91, TuVi92]. These combinatorial approaches have a culmination point in the Turaev–Viro TQFT and the Reshetikhin–Turaev TQFT, the first based in triangulations of 3–manifolds and the second in surgery presentation of 3–manifolds along links. However, these two TQFT’s are closely related (see [BK, TuV13]). In this dissertation we focus on Turaev–Viro invariants following [BW96, TuVi92, TuV13]. An important property of these invariants is that they are very powerful (used together with homology groups) to distinguish 3–manifolds of low complexity (see [Mat]).
The idea behind Turaev–Viro invariants is as follows.

- Pick a combinatorial presentation of 3–manifolds given by, a triangulation, a Heegaard splitting or more generally a skeleton.
- Endow this combinatorial presentation with some additional information (coloring or weight) coming from some categorical data (spherical fusion categories).
- Calculate a number (or a vector) from the previous step.
- Sum over all possible colorings or weights.

Schematically,

\[
\text{combinatorial presentation of 3–manifolds} \quad + \quad \text{categorical data} \quad \mapsto \quad \text{topological invariants.}
\]

The main goal of this dissertation is to survey the construction of Turaev–Viro invariants for closed 3–manifolds associated to a spherical fusion category and also the corresponding TQFT. We introduce roughly the necessary ingredients to this purpose:

- **Spherical fusion categories.** These are a kind of monoidal categories with a rich structure (duality, pivotal structure, direct sums, semi-simplicity) which allow us to interpret them graphically. More precisely, we have a category \( \mathcal{C} \) with a functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \), a distinguished object \( \mathbb{1} \) playing the role of a unit, and for each object \( X \) of \( \mathcal{C} \) a dual object \( X^\ast \) in \( \mathcal{C} \) and four morphisms \( ev_X : X^\ast \otimes X \to \mathbb{1}, \ \widetilde{ev}_X : X \otimes X^\ast \to \mathbb{1}, \ coev_X : \mathbb{1} \to X \otimes X^\ast \) and \( \widetilde{coev}_X : \mathbb{1} \to X^\ast \otimes X \) satisfying certain constraints.

- **Combinatorial presentations.** These are nice ways of presenting manifolds by means of objects more workable as simplices and polyhedra.

- **\( n \)–Cobordisms.** A \( n \)–cobordism between two closed oriented \((n - 1)\)–manifolds \( \Sigma \) and \( \Sigma' \) is a compact oriented \( n \)–manifold \( M \) such that \( \partial M = (-\Sigma) \sqcup \Sigma, \) where \(-\Sigma\) denotes \( \Sigma \) with the opposite orientation.

- **\( n \)–TQFT.** This a procedure that assigns to each closed oriented \((n - 1)\)–manifold \( \Sigma \) a finite dimensional vector space \( Z(\Sigma) \) and to every cobordism \( M \) between two closed oriented \((n - 1)\)–manifolds \( \Sigma \) and \( \Sigma' \) a homomorphism \( Z(\Sigma) \to Z(\Sigma') \).

This dissertation is organized as follows. Chapter 1 introduces the basic notions in category theory together with their graphical interpretation necessary for the other chapters. Through this chapter we give some interesting examples both in algebra and topology to enlighten the definitions. Chapter 2 is the heart of all the dissertation, here we move from category theory to topology. We begin by studying combinatorial presentations of 3–manifolds, then we introduce topological invariants for colored graphs and finally, we use them to construct the Turaev–Viro invariants from a triangulation and more generally from a skeleton.

Chapter 3 deals with the generalization of the notions introduced in chapter 2 to the case of 3–manifolds with boundary. Finally, in chapter 4 we study the notion of TQFT.
together with some basic examples in dimension 1 and 2 and we finish the chapter by showing the construction of a 3–dimensional TQFT from a spherical fusion category, this allows us to make contact with all the work made in the previous chapters.

To conclude, we emphasize that this is only an expository work and that nothing contained here is new or original, except for some examples of calculations based on Heegaard splittings. Everything can be found in more detail in the given bibliography, principally [BW96, TuVi92, TuV13]. Also we remark that we use the word manifold to refer to a compact oriented manifold and, for such a manifold with boundary, we orient its boundary following “outward normal first” convention. Some notations used through the text are the following.

\[ \mathbb{K}^\times \] The group of invertible elements of \( \mathbb{K} \).
\[ \mathbb{N} \] The monoid of non-negative integers.
\[ \eta : F \Rightarrow F' \] A natural transformation from \( F \) to \( F' \).
\[ \text{id}_C \] The identity functor of the category \( C \).
\[ \text{Sets} \] The category of sets.
\[ \text{Groups} \] The category of groups.
\[ \text{Rings} \] The category of rings.
\[ \text{Top} \] The category of topological spaces.
\[ \text{Cob}_n \] The category of \( n \)–cobordisms.
\[ \mathbb{K} \text{-Mod} \] The category of modules over a commutative ring \( \mathbb{K} \).
\[ \text{Rep}_k(G) \] The category of representations (over a field \( k \)) of a group \( G \).
\[ \text{Tang} \] The category of tangles.
\[ \text{OTang} \] The category of oriented tangles.
1 Categorical Preliminaries

The goal of this chapter is to introduce the categorical preliminaries necessary for the other chapters. Sections 1.1 and 1.2 deal with the notion of monoidal category or tensor category. Sections 1.3, 1.4 and 1.5 are the heart of the chapter, here we introduce the notions of pivotal category, $K$–category, spherical and fusion categories. We give some interesting examples both in algebra and topology to enlighten the definitions. These kind of categories have a structure that is rich enough for having a nice way of interpretation by using the graphical calculus (or Penrose calculus) which we introduce through all the chapter.

In the graphical language of a category $\mathcal{C}$, we represent the information by planar diagrams in which objects are depicted as oriented arcs (or edges) and the morphisms are depicted as boxes (or nodes). We adopt the convention [Tu10, Kas95] of reading the diagrams from the bottom to the top. For another convention we refer to [Sel09].

Let $X, Y, Z \in \text{Ob}(\mathcal{C})$ and $f : X \to Y$ and $g : Y \to Z$ be morphisms, we show the graphical language in Table 1.

<table>
<thead>
<tr>
<th>Graphical Language</th>
<th>Diagram</th>
</tr>
</thead>
</table>
| $id_X : X \to X$   | \[
\begin{array}{c}
X \\
\downarrow
\end{array}
\]
| $f : X \to Y$      | \[
\begin{array}{c}
Y \\
\downarrow
\end{array}
\]
| $g \circ f$       | \[
\begin{array}{c}
Z \\
\downarrow
\end{array}
\]

Table 1: Graphical language in a category

1.1 Strict Monoidal Categories

Let $\mathcal{C}$ be a category. A tensor product in $\mathcal{C}$ is a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We denote by $U \otimes V$ the image by $\otimes$ of an object $(U, V)$ in the product category $\mathcal{C} \times \mathcal{C}$ and by $f \otimes g$ the image by $\otimes$ of a morphism $f \times g$ in $\mathcal{C} \times \mathcal{C}$. Saying that $\otimes$ is a functor implies that we have the following identities

$$ (f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g), \quad (1.1) $$
$$ id_U \otimes id_V = id_{U \otimes V}. \quad (1.2) $$

For example, if $\mathbb{K}$ is a commutative ring, then the usual tensor product in the category $\mathbb{K}$-Mod of modules over $\mathbb{K}$ is a tensor product in the sense above.

**Definition 1.1.** A strict monoidal category (or strict tensor category) is a triple $(\mathcal{C}, \otimes, \mathbb{1})$, where $\mathcal{C}$ is a category, $\mathbb{1}$ is a distinguished object in $\mathcal{C}$, called unit object and $\otimes$ is a tensor product in $\mathcal{C}$ satisfying

$$ U \otimes \mathbb{1} = \mathbb{1} \otimes U = U, \quad (1.3) $$
$$ (U \otimes V) \otimes W = U \otimes (V \otimes W), \quad (1.4) $$
$$ f \otimes id_{\mathbb{1}} = id_{\mathbb{1}} \otimes f = f, \quad (1.5) $$
$$ (f \otimes g) \otimes h = f \otimes (g \otimes h), \quad (1.6) $$
for any objects $U, V$ and $W$ and for any morphisms $f, g$ and $h$ in the category.

If we consider the category $\mathbb{K}$-$\text{Mod}$ together with the usual tensor product of modules and we think the unit object $\mathbb{I}$ as the ground ring $\mathbb{K}$ (considered as a $\mathbb{K}$-module over itself) we note that instead of the identities (1.3) and (1.4) we have canonical isomorphisms

\[ U \otimes \mathbb{I} \cong \mathbb{I} \otimes U = U, \]
\[ (U \otimes V) \otimes W \cong U \otimes (V \otimes W). \]

For this reason ($\mathbb{K}$-$\text{Mod}$, $\otimes$, $\mathbb{K}$) is not a strict monoidal category, but it is in fact a monoidal category (see Example 1.7).

**Example 1.2.** Let $C$ be a small category. Let us consider the category of endofunctors $\text{End}(C)$ of $C$, i.e., the objects of $\text{End}(C)$ are the functors $F : C \rightarrow C$, and if $F, G : C \rightarrow C$ are functors then $\text{Hom}_{\text{End}(C)}(F, G)$ is the collection of natural transformations $\eta : F \Rightarrow G$ of $F$ in $G$. For $F, G$ objects of $\text{End}(C)$ let us define $F \otimes G := F \circ G$, and if $\eta : F \Rightarrow F'$, and $\mu : G \Rightarrow G'$ are morphisms, we define the natural transformation $\eta \otimes \mu : F \otimes G \Rightarrow F' \otimes G'$ by $(\eta \otimes \mu)_C := \eta_{G'(C)} \circ F(\mu_C)$ for all object $C$ of $C$. Then $(\text{End}(C), \otimes, id_C)$, where $id_C$ is the identity functor, is a strict monoidal category.

**Example 1.3.** Let $G$ be a monoid with neutral element $e$. We consider the category $\mathcal{C}(G)$ whose objects are the set $G$ itself and such that for $a, b \in G$

\[ \text{Hom}_{\mathcal{C}(G)}(a, b) = \begin{cases} \{id_a\} & \text{if } a = b, \\ \emptyset & \text{if } a \neq b. \end{cases} \]

We define $a \otimes b := ab$ and $id_a \otimes id_b := id_{ab}$ for all $a, b \in G$. Then $(\mathcal{C}(G), \otimes, e)$ is a strict monoidal category.

**Example 1.4.** Let $(G_n)_{n \in \mathbb{N}}$ be a family of monoids such that $G_0 = 1$. We consider the category $\mathcal{G}$ whose objects are the set $\mathbb{N}$ and such that for $m, n \in \mathbb{N}$

\[ \text{Hom}_{\mathcal{G}}(m, n) = \begin{cases} G_m & \text{if } m = n, \\ \emptyset & \text{if } m \neq n. \end{cases} \]

The composition on $\text{Hom}_{\mathcal{G}}(n, n)$ is given by the product on $G_n$. The identity morphism on the object $n$ is the neutral element of $G_n$. Suppose that for any pair $(m, n)$ there exists a homomorphism $\rho_{m,n} : G_m \times G_n \rightarrow G_{m+n}$ satisfying

\[ \rho_{m+n,l} \circ (\rho_{m,n} \times id_{G_l}) = \rho_{m,n+l} \circ (id_{G_m} \times \rho_{n,l}) \tag{1.7} \]

for all $m, n, l \in \mathbb{N}$. We define $m \otimes n := m + n$ for all $m, n$ objects of $\mathcal{G}$, and $f \otimes g := \rho_{m,n}((f, g))$ for all morphisms $f \in G_m$ and $g \in G_n$. Then $(\mathcal{G}, \otimes, 0)$ is a strict monoidal category. In particular, if we take the family of permutations groups $(S_n)_{n \in \mathbb{N}}$ together with the maps $\rho_{m,n} : S_m \times S_n \rightarrow S_{m+n}$ such that for $\sigma \in S_m$ and $\tau \in S_n$

\[ \rho_{m,n}(\sigma, \tau)(k) = \begin{cases} \sigma(k) & \text{if } 1 \leq k \leq m, \\ \tau(k-m) & \text{if } m+1 \leq k \leq m+n, \end{cases} \]
for all $m, n \in \mathbb{N}$. We can easily check the relation (1.7). Then we obtain a strict monoidal category associated to the family $(S_n)_{n \in \mathbb{N}}$. We note this category by $\mathbb{S}$ and call it the symmetric category. The same construction can be made with the family of braid groups $(B_n)_{n \in \mathbb{N}}$ and we obtain another strict monoidal category, for details and for another similar example we refer to [KRT, Kas95].

We finish this section extending the notions of functors and natural transformation to the case of a strict monoidal category. If $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$ are strict monoidal categories, a strict monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ preserving the monoidal structure, i.e., $F(U \otimes V) = F(U) \otimes' F(V)$ for all objects $U$ and $V$ of $\mathcal{C}$ and $F(\mathbb{1}) = \mathbb{1}'$ and the similar conditions for morphisms. If $F, F' : \mathcal{C} \to \mathcal{C}'$ are strict monoidal functors, a natural transformation $\eta : F \Rightarrow F'$ is monoidal if it satisfies $\eta_{\mathbb{1}} = id_{\mathbb{1}'}$ and $\eta_{U \otimes V} = \eta_U \otimes' \eta_V$ for all objects $U$ and $V$ of $\mathcal{C}$.

### 1.2 Monoidal Categories

Although we will be concerned mainly with strict monoidal categories, for the sake of completeness and clarity, we briefly mention the notion of monoidal category and give some examples. At the end of the section we mention the MacLane’s coherence theorem and we give references for a proof. These theorems show that working only with strict monoidal categories does not lead to a loss of generality.

**Definition 1.5.** A monoidal category is a sextuplet $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ where $\mathcal{C}$ is a category, $\mathbb{1}$ is a unit object, $\otimes$ is a tensor product in $\mathcal{C}$ and such that the the following axioms hold.

**Triangle Axiom.** For all objects $U$ and $V$ of $\mathcal{C}$ the following diagram is commutative.

$$
\begin{array}{ccc}
(U \otimes \mathbb{1}) \otimes V & \xrightarrow{\alpha_{U,1,V}} & U \otimes (\mathbb{1} \otimes V) \\
\rho_U \otimes id_V & & id_U \otimes \lambda_V \\
U \otimes V & \xrightarrow{id_U \otimes \lambda_V} & U \otimes V \\
\end{array}
$$

**Pentagon Axiom.** For all objects $U, V, W$ and $X$ of $\mathcal{C}$ the following diagram is commutative.
Even though the definition of monoidal category looks a little bit abstract, it is in fact very natural in the sense that there are many mathematical structures that satisfy this definition. For example every strict monoidal category is a monoidal category. The following are some examples of non-strict monoidal categories.

**Example 1.6.** Any additive category $\mathcal{A}$ is monoidal, with the product as the tensor product and the null object as the unit object.

**Example 1.7.** The category $(\mathbb{K}-\text{Mod}, \otimes, \mathbb{K})$ is monoidal.

**Example 1.8.** The category $\text{Sets}$ together with the cartesian product as tensor product and with unit object any singleton. This example can be widely generalized to a concrete category, like $\text{Groups}$, $\text{Rings}$ or $\text{Top}$.

**Example 1.9.** The categories $\text{Sets}$ and $\mathbb{K}-\text{Mod}$ together with the coproduct as tensor product.

**Example 1.10.** Let $G$ be a group and $k$ be a field. The category $\text{Rep}_k(G)$ of all the representations of $G$ over $k$ together with the tensor product of representations as tensor product and the trivial representation as the unit object.

As we have seen there are many examples of monoidal categories. For other interesting examples we refer to [EGNO, Kas95, Mü10].

**Definition 1.11.** Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $\mathcal{C}' = (\mathcal{C}', \otimes', \mathbb{1}', \alpha', \lambda', \rho')$ be monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a triple $(F, e, \{ d_{X,Y} \}_{X,Y \in \text{Ob}(\mathcal{C})})$ where

(i) $F : \mathcal{C} \to \mathcal{C}'$ is a functor,

(ii) $e : F(\mathbb{1}) \to \mathbb{1}'$ is an isomorphism and

(iii) $d_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)$ is a family of natural isomorphisms indexed by $X, Y \in \text{Ob}(\mathcal{C})$,

such that the following diagrams

\[
\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{d_{X,Y} \otimes' \text{id}_{F(Z)}} & F(X \otimes Y) \otimes' F(Z) \\
\xrightarrow{\alpha'_{F(X), F(Y), F(Z)}} & & \xrightarrow{d_{X\otimes Y, Z}} F((X \otimes Y) \otimes Z) \\
F(X) \otimes' (F(Y) \otimes' F(Z)) & \xrightarrow{\text{id}_{F(X)} \otimes' \alpha'_{F(Y), F(Z)}} & F(X) \otimes' F(Y \otimes Z) \\
\xrightarrow{\text{id}_{F(X)} \otimes' \text{id}_{F(Y) \otimes Z}} & & \xrightarrow{d_{X,Y \otimes Z}} F(X \otimes (Y \otimes Z))
\end{array}
\]
\[ F(X) \otimes F(\mathbb{1}) \xrightarrow{id_{F(X)} \otimes \iota} F(X) \otimes X' \]
\[ d_{X,1} \]
\[ F(X \otimes \mathbb{1}) \xrightarrow{\rho_{F(X)}} F(X) \]
\[ F(\mathbb{1} \otimes X) \xrightarrow{\lambda_{F(X)}} F(X) \]

are commutative for all \( X, Y, Z \in \text{Ob}(\mathcal{C}) \).

**Definition 1.12.** Let \( \mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho) \) and \( \mathcal{C}' = (\mathcal{C}', \otimes', \mathbb{1}', \alpha', \lambda', \rho') \) be monoidal categories and \((F, e, d), (F', e', d') : \mathcal{C} \to \mathcal{C}' \) be monoidal functors. A *monoidal natural transformation* between \((F, e, d)\) and \((F', e', d')\) is a natural transformation \( \eta : F \Rightarrow F' \) such that the following diagram is commutative:

\[ F(X) \otimes F(Y) \xrightarrow{d_{X,Y}} F(X \otimes Y) \]
\[ d_{X,Y} \]
\[ F'(X) \otimes F'(Y) \xrightarrow{d_{X,Y}} F'(X \otimes Y) \]

Finally, if \( \mathcal{C} \) and \( \mathcal{C}' \) are monoidal categories, a monoidal functor \( F : \mathcal{C} \to \mathcal{C}' \) is a *monoidal equivalence* if there exist a monoidal functor \( F : \mathcal{C}' \to \mathcal{C} \) and natural monoidal isomorphisms \( \eta : F \circ G \Rightarrow \text{id}_{\mathcal{C}} \) and \( \mu : F \circ G \Rightarrow \text{id}_{\mathcal{C}'} \). In this case we say that \( \mathcal{C} \) and \( \mathcal{C}' \) are *monoidally equivalent*. Now we can state the principal result of this section.

**Theorem 1.13** (MacLane’s coherence theorem). *Any monoidal category is monoidally equivalent to a strict monoidal category.*

For a proof we refer to [EGNO, JS93, Kas95]. As mentioned at the beginning of the section, this theorem allows us to assume that all our monoidal categories are strict. We finish this section showing the graphical calculus for monoidal categories. Let \((\mathcal{C}, \otimes, \mathbb{1})\) be a monoidal category, \( X, X', Y, Y', A_1, \ldots, A_n, B_1, \ldots, B_m \in \text{Ob}(\mathcal{C}) \) and \( f : X \to X', \ g : Y \to Y', \ h : A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m \) be morphisms, the graphical calculus in \( \mathcal{C} \) is showed in Table 2.

\[
\begin{array}{c}
\text{id}_{\mathbb{1}} : \mathbb{1} \to \mathbb{1} = \text{(empty)} \\
f \otimes g = \begin{array}{c}
X' \\
X \\
Y \\
Y' \\
h = \begin{array}{c}
B_1 \\
\cdots \\
B_m \\
A_1 \\
\cdots \\
A_n \\
\end{array}
\end{array}
\end{array}
\]

Table 2: Graphical language in a monoidal category

### 1.3 Pivotal Categories

The aim of this section is to introduce the notion of *pivotal category* together with its graphical language.
Definition 1.14 ([TuV13]). Let \( \mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}) \) be a strict monoidal category, we say that \( \mathcal{C} \) is pivotal if for each \( X \in \text{Ob}(\mathcal{C}) \) there exists a dual object \( X^* \in \text{Ob}(\mathcal{C}) \), whose morphism \( id_{X^*} \) is graphically represented by

\[
\begin{array}{c}
\xymatrix{
& X^* \ar[dl]_{id_{X^*}} \ar[dr]^{id_{X^*}} \\
X & & X
}
\end{array}
\]

and four morphisms, called duality morphisms, whose graphical representation are given by

\[
\begin{align*}
ev_X &: X^* \otimes X \to \mathbb{1} \\
\tilde{ev}_X &: X \otimes X^* \to \mathbb{1} \\
coev_X &: \mathbb{1} \to X \otimes X^* \\
\tilde{coev}_X &: X^* \otimes X \to \mathbb{1}
\end{align*}
\]

such that the following conditions hold.

(PV1) For every \( X \in \text{Ob}(\mathcal{C}) \) the pair \( (ev_X, coev_X) \) is a left duality for \( X \), i.e.,

\[
(id_X \otimes ev_X) \circ (coev_X \otimes id_X) = id_X, \text{ graphically}
\]

and

\[
(ev_X \otimes id_{X^*}) \circ (id_{X^*} \otimes coev_X) = id_{X^*}, \text{ graphically}
\]

(PV2) For every \( X \in \text{Ob}(\mathcal{C}) \) the pair \( (\tilde{ev}_X, \tilde{coev}_X) \) is a right duality for \( X \), i.e.,

\[
(id_{X^*} \otimes \tilde{ev}_X) \circ (\tilde{coev}_X \otimes id_X) = id_{X^*}, \text{ graphically}
\]

and

\[
(\tilde{ev}_X \otimes id_X) \circ (id_X \otimes \tilde{coev}_X) = id_X, \text{ graphically}
\]

(PV3) For every morphism \( f : X \to Y \) in \( \mathcal{C} \), the left dual

\[
f^* = (ev_Y \otimes id_{X^*}) \circ (id_{Y^*} \otimes f) \circ id_{X^*} \circ (id_{Y^*} \otimes coev_X) : Y^* \to X^*
\]

and the right dual

\[
f^* = (id_{X^*} \otimes \tilde{ev}_Y) \circ (id_{X^*} \otimes f \otimes id_{Y^*}) \circ (\tilde{coev}_X \otimes id_{Y^*}) : Y^* \to X^*
\]

are equal. Graphically,
(PV4) For all $X, Y \in Ob(C)$ the left monoidal constraint

$$(ev_X \otimes id_{Y \otimes X}) \circ (id_X \otimes ev_Y \otimes id_{X \otimes (Y \otimes X)}) \circ (id_X \otimes Y^* \otimes coev_{Y \otimes X}) : X^* \otimes Y^* \to (Y \otimes X)^*$$

is equal to the right monoidal constraint

$$(id_{(Y \otimes X)^*} \otimes \tilde{ev}_Y) \circ (id_{(Y \otimes X)^*} \otimes \tilde{ev}_X \otimes id_Y) \circ (\tilde{coev}_{Y \otimes X} \otimes id_{X \otimes Y^*}) : X^* \otimes Y^* \to (Y \otimes X)^*.$$ Graphically,

where

$$(Y \otimes X^*) = (id_{Y \otimes X})$$

(PV5) The morphisms $ev_\mathbb{1} : \mathbb{1}^* \otimes \mathbb{1} = \mathbb{1}^* \to \mathbb{1}$ and $\tilde{ev}_\mathbb{1} : \mathbb{1} \otimes \mathbb{1}^* = \mathbb{1}^* \to \mathbb{1}$ are equal. Or equivalently $coev_\mathbb{1} = \tilde{coev}_\mathbb{1} : \mathbb{1} \to \mathbb{1}^*$.

As immediate consequences of the definition we obtain that the morphisms $ev_\mathbb{1}$ and $coev_\mathbb{1}$ are mutually inverse because by (PV1) the pair $(ev_\mathbb{1}, coev_\mathbb{1})$ is a left duality for $\mathbb{1}$. Similarly $\tilde{ev}_\mathbb{1}$ and $\tilde{coev}_\mathbb{1}$ are mutually inverse. The morphisms $ev_X, \tilde{ev}_X$ and $coev_X, \tilde{coev}_X$ are called evaluation and coevaluation, respectively.

The axiom (PV1) (resp. (PV2)) says that each object $X$ of $\mathcal{C}$ has a left dual $(X^*, ev_X, coev_X)$ (respectively right dual $(X^*, \tilde{ev}_X, \tilde{coev}_X)$). In the terminology of [FY89, FY92, JS93, Mal95, Sel09] the family $\{(X^*, ev_X, coev_X)\}_{X \in \mathcal{C}}$ (respectively the family $\{(X^*, \tilde{ev}_X, \tilde{coev}_X)\}_{X \in \mathcal{C}}$) is called a left autonomous structure (respectively right autonomous structure) on $\mathcal{C}$. A monoidal category together with a left autonomous structure (respectively left autonomous structure) is called left autonomous (respectively right autonomous). Finally a monoidal category is autonomous if it is at the
same time right and left autonomous. Another common word in the literature for autonomous is rigid.

With the terminology above, we say that a left autonomous category \( C \) with left autonomous structure \( \{ (X^*, ev_X, coev_X) \} \) has a pivotal structure if there exist a natural monoidal isomorphism \( \eta : id_C \Longrightarrow ( )^{**} \). The standard definition of pivotal category [FY89, FY92, JS93, Mal95, Sel09] is a left autonomous category together with pivotal structure. Although this definition is equivalent to the Definition 1.14. In fact if \( C \) is a pivotal category (in the sense of Definition 1.14) we can obtain a pivotal structure in \( C \) defining \( \eta : id_C \Longrightarrow ( )^{**} \) by \( \eta_X = (\tilde{ev}_X \otimes id_X^{**}) \circ (id_X \otimes coev_X^{**}) \). Reciprocally if \( C \) is a pivotal category (in the sense of pivotal structures) with pivotal structure \( \eta : id_C \Longrightarrow ( )^{**} \), it is enough to define \( \tilde{ev}_X = ev_X \circ (\eta_X \otimes id_X^{**}) \) and \( \tilde{coev}_X = (id_X \otimes \eta_X^{-1}) \circ coev_X^{**} \). We will use the definition in terms of pivotal structures in some examples.

In a pivotal category the morphisms represented by the diagrams are invariant under isotopies of the diagrams in the plan keeping fixed the bottom and the top endpoints. Moreover an equation between morphisms follows from the axioms if and only if it follows in the graphical language up to isotopy. Finally every monoidal functor preserves dual objects and the MacLane’s coherence Theorem 1.13 has a generalization to the case of a pivotal category, i.e., any pivotal category is equivalent to a strict pivotal category [BW93].

**Example 1.15.** We give a short description of he category of tangles \( \text{Tang} \) for details we refer to [FY89, Kas95, Tu10]. The objects of \( \text{Tang} \) are the non-negative integers \( \mathbb{N} \) and if \( k, l \in \mathbb{N} \) a morphism \( k \rightarrow l \) is the isotopy class of a tangle of type \((k, l) \). For instance in the following figure we show tangles \( A \) and \( B \) of types \((2, 4)\) and \((2, 2)\), respectively.

The composition \( A \circ B \) in \( \text{Tang} \) of morphisms \( B : k \rightarrow l \) and \( A : l \rightarrow k \) is given by the isotopy class of the tangle obtained by gluing the top of \( B \) (more precisely, the top of a representative of the class \( B \)) with the bottom of \( A \). For instance for the morphisms \( B : 2 \rightarrow 2 \) and \( A : 2 \rightarrow 4 \) as above, \( A \circ B : 2 \rightarrow 4 \) is showed in the following figure (up to isotopy).

The tensor product in \( \text{Tang} \) is defined by \( k \otimes l := k+l \) for objects \( k, l \in \mathbb{N} \) and if \( A : k \rightarrow l \) and \( B : m \rightarrow n \) are morphisms, \( A \otimes B \) is the isotopy class of the tangle obtained by
horizontal concatenation of the tangles $A$ and $B$ (more precisely, representative tangles of the classes $A$ and $B$). For instance for the morphisms $A : 2 \to 4$ and $B : 2 \to 2$ as above, $A \otimes B : 2 \otimes 2 = 4 \to 4 \otimes 2 = 6$ is showed in the following figure (up to isotopy):

$$A \otimes B =$$

For the time being we have that $(\text{Tang}, \otimes, 0)$ is a strict monoidal category and it only remains to define the duals and the pivotal structure. For $k \in \mathbb{N}$ the dual $k^*$ is $k$ itself and the morphism $ev_k : k^* \otimes k = 2k \to 0$ is given by the tangle connecting pairs of points beginning in the center and working outwards, the morphism $coev_k : 0 \to k \otimes k^* = 2k$ is defined similarly. In the following figure we show these morphisms (up to isotopy):

$$ev_k : 2k \to 0 \quad \quad \quad \quad coev_k : 0 \to 2k$$

Since $(k^*)^* = k$ for all $k \in \mathbb{N}$, we have that the functors $(\ )^{**} : \text{Tang} \to \text{Tang}$ and $id_{\text{Tang}} : \text{Tang} \to \text{Tang}$ are equal. Then the pivotal structure is given by the identity natural monoidal isomorphism $\eta : id_{\text{Tang}} \Longrightarrow (\ )^{**}$, i.e., $\eta_k = id_k$ for all $k \in \mathbb{N}$. To sum up the category $\text{Tang}$ is a strict pivotal category.

The Example 1.15 can be easily generalized to the categories $\text{OTang}$ of oriented tangles. For more details we refer to [FY89].

**Example 1.16.** We give a short description of the Temperley-Lieb category $\mathcal{TL}(\tau)$, for more details we refer to [GH02, M"{u}10]. Let $F$ be a field and $\tau$ a non-zero element in $F$. The objects of the category $\mathcal{TL}(\tau)$ are the non-negative integers $\mathbb{N}$ and if $k, l \in \mathbb{N}$, a morphism $k \to l$ is a $F$-linear combination of the isotopy classes of a $\mathcal{TL}$-diagrams of type $(k,l)$. By a $\mathcal{TL}$-diagram of type $(k,l)$ we mean a planar diagram (as in the case of a tangle) with $k$ points in the bottom and $l$ points in the top connected by polygonal lines without intersections between them. For instance in the following figure we show some $\mathcal{TL}$-diagrams $A$ and $B$ of types $(6,4)$ and $(2,6)$, respectively:

$$A = \quad \quad \quad \quad B =$$
As in the case of the category of tangles, the tensor product of objects \( k, l \in \mathbb{N} \) is defined by \( k \otimes l := k + l \) and if \( A : k \to l \) and \( B : m \to n \) are basic morphisms, \( A \otimes B \) is the isotopy class of the TL-diagram obtained by horizontal concatenation of the TL-diagrams \( A \) and \( B \) and we extend this definition linearly. For instance for the morphisms \( A : 6 \to 4 \) and \( B : 2 \to 6 \) as above, \( A \otimes B : 6 \otimes 2 = 8 \to 4 \otimes 6 = 10 \) is showed in the following figure (up to isotopy):

\[
A \otimes B =
\]

The morphisms sets have a structure of \( \mathbb{F} \)-modules. The composition in \( \mathcal{TL}(\tau) \) is defined as in the category of tangles but additionally we have to remove all bounded regions originated by the gluing and then multiplying by a factor of \( \tau \) for each bounded region removed. Then we extend this definition linearly. The composition is \( \mathbb{F} \)-linear. For instance for the morphisms \( B : 2 \to 6 \) and \( A : 6 \to 4 \) as above, the morphism \( A \circ B : 2 \to 4 \) is showed in the following figure (up to isotopy):

\[
A \circ B =
\]

\[
= \tau^2
\]

The category \( \mathcal{TL}(\tau) \) is endowed with duals and a pivotal structure in the same way that the category of tangles. To sum up, \( \mathcal{TL}(\tau) \) is a strict pivotal category.

A important characteristic of a pivotal category is that we can define a notion of “trace” and “dimension”.

**Definition 1.17.** Let \( \mathcal{C} \) be a pivotal category, \( X \) be an object of \( \mathcal{C} \) and \( f \) be an endomorphism of \( X \).

(i) The left trace of \( f \) is \( tr_l(f) = ev_X \circ (id_X \otimes f) \circ coev_X \in \text{End}_\mathcal{C}(I) \), graphically,

\[
tr_l(f) =
\]
(ii) The right trace of $f$ is $tr_r(f) = ev_X \circ (f \otimes id_{X^*}) \circ coev_X \in \text{End}_C(\mathbb{1})$, graphically,

\[
tr_r(f) = \begin{array}{c}
\text{X} \\
\text{f} \\
\text{X}
\end{array}
\]

(iii) The left and right dimension of $X$ are defined as $dim_l(X) = tr_l(id_X)$ and $dim_r(X) = tr_r(id_X)$, respectively. Graphically,

\[
\begin{array}{c}
\text{dim}_r(X) = \begin{array}{c}
\text{X}
\end{array} \\
\text{dim}_l(X) = \begin{array}{c}
\text{X}
\end{array}
\end{array}
\]

**Proposition 1.18.** Let $\mathcal{C}$ be a pivotal category satisfying

\[
\alpha \otimes id_X = id_X \otimes \alpha
\]

for all $\alpha \in \text{End}_\mathcal{C}(\mathbb{1})$ and all object $X$ of $\mathcal{C}$. Then

(i) $tr_l(f \circ g) = tr_l(g \circ f)$ for any $f : X \to Y$ and $g : Y \to X$ morphisms in $\mathcal{C}$,

(ii) $tr_l(f \otimes g) = tr_l(f) tr_l(g)$ for any endomorphisms $f$ and $g$ in $\mathcal{C}$,

(iii) $dim_l(X \otimes Y) = dim_l(X) dim_l(Y)$ for any objects $X, Y$ of $\mathcal{C}$,

and the same equalities hold for the right trace and right dimension.

**Proof.** We give graphical proofs for the left case (the right being analogous) and from these we can recover an algebraic proof using the axioms of a pivotal category.

(i) Let $f : X \to Y$ and $g = Y \to X$ be morphisms in $\mathcal{C}$, then
(ii) Let \( f : X \to X \) and \( g = Y \to Y \) be morphisms in \( C \), then

\[
tr_l(f \circ g) = Y
\]
Let $C$ be a pivotal category. We say that $C$ is spherical if the left and right traces are equal for any endomorphism $f$ of an object in $C$, that is, $tr_l(f) = tr_r(f)$. Graphically,

For spherical categories the graphical calculus is carried out in the 2–sphere $S^2 = \mathbb{R} \cup \{\infty\}$ and an equation between morphisms follows from the axioms if and only if it follows in the graphical language up to isotopy in $S^2$. 

(iii) This follows immediately from (ii).
1.4 Pivotal \( \mathbb{K} \)-categories

In this section \( \mathbb{K} \) is a commutative ring. A monoidal category \( \mathcal{C} \) is said to be a \( \mathbb{K} \)-category if for all \( X, Y \in \text{Ob}(\mathcal{C}) \) the set \( \text{Hom}_\mathcal{C}(X, Y) \) is a \( \mathbb{K} \)-module, the composition and the tensor product of morphisms are \( \mathbb{K} \)-bilinear and \( \text{End}_\mathcal{C}(1) \) is isomorphic to \( \mathbb{K} \) (see Examples 1.7, 1.9 and 1.10).

A pivotal \( \mathbb{K} \)-category verifies the relation (1.8) so that the left and right traces and dimensions are multiplicative \( \mathbb{K} \)-linear maps.

From now and until the end of this section let \( \mathcal{C} \) be a pivotal \( \mathbb{K} \)-category. A signed object of \( \mathcal{C} \) is a pair \((X, \epsilon)\) with \( X \in \text{Ob}(\mathcal{C}) \) and \( \epsilon \in \{+, -\} \). For a signed object \((X, \epsilon)\) of \( \mathcal{C} \), set \( X^\epsilon = X \) if \( \epsilon = + \) and \( X^\epsilon = X^* \) if \( \epsilon = - \). A \( \mathcal{C} \)-cyclic set is a triple \( E = (E, c, \epsilon) \), where \( E \) is a totally cyclically ordered finite set, \( c \) is a map \( E \to \text{Ob}(\mathcal{C}) \) (the coloring map) and \( \epsilon \) is a map \( E \to \{+, -\} \) (the sign map). The aim of this section is assign a \( \mathbb{K} \)-module \( H(E) \) to a \( \mathcal{C} \)-cyclic set \( E \). For this purpose we will need the followings maps (the permutation maps).

For two objects \( X, Y \) of \( \mathcal{C} \) let

\[
\sigma_{X,Y} : \text{Hom}_\mathcal{C}(1, X \otimes Y) \to \text{Hom}_\mathcal{C}(1, Y \otimes X)
\]

be the map defined by

\[
\sigma_{X,Y}(\alpha) = (ev_X \otimes id_Y \otimes X) \circ (id_X \otimes \alpha \otimes id_Y) \circ coev_X
\]

for all \( \alpha \in \text{Hom}_\mathcal{C}(1, X \otimes Y) \). Graphically,

\[
\begin{array}{c}
\xymatrix{
X \ar[r] & Y \\
X \ar[ru] & \alpha \ar[u] \\
Y \ar[ru] & \ar[l] X
}
\end{array}
\]

**Proposition 1.19.** For all \( X, Y, Z \in \text{Ob}(\mathcal{C}) \),

(i) \( \sigma_{X,1} = \sigma_{1,X} = id_{\text{Hom}_\mathcal{C}(1,X)} \);

(ii) \( \sigma_{X,Z} = \sigma_Y Z \circ \sigma_{X,Y} Z \) and \( \sigma_{X,Y} Z = \sigma_Z X \circ \sigma_{X,Z} Y \);

(iii) \( \sigma_{X,Y} \) is an isomorphism and \( \sigma_{X,Y}^{-1} = \sigma_{Y,X} \).

**Proof.** First notice that \((ev_1 \otimes id_\mathbb{1}) \circ coev_\mathbb{1} = ev_1 \circ coev_\mathbb{1} = \bar{ev}_1 \circ coev_\mathbb{1} = id_\mathbb{1}\), here we have used (PV5). This equality can be expressed graphically as follows

\[
\begin{array}{c}
\xymatrix{
1 \ar[r] & 1 \\
1 \ar[ru] & \ar[l] 1
}
\end{array}
\]

(1.10)
(i) Let $\alpha \in \text{Hom}_C(\mathbb{1}, X)$, then

$$\sigma_{X,\mathbb{1}}(\alpha) = \frac{X}{\alpha} \xrightarrow{1} X \overset{1}{\xrightarrow{\alpha}} X \overset{1}{\xrightarrow{\sigma_{X,\mathbb{1}}(\alpha)}} \overset{(PV1)}{=} X \overset{1}{\xrightarrow{\alpha}} X \overset{1}{\xrightarrow{\sigma_{X,\mathbb{1}}(\alpha)}} \overset{(PV3)}{=} \overset{(1.10)}{=} X \overset{1}{\xrightarrow{\alpha}} X \overset{1}{\xrightarrow{\sigma_{X,\mathbb{1}}(\alpha)}} \overset{(PV1)}{=} \overset{(PV3)}{=}. $$

(ii) Let $\alpha \in \text{Hom}_C(\mathbb{1}, X \otimes Y \otimes Z)$, then

$$\sigma_{X \otimes Y, Z}(\alpha) = \frac{X \otimes Y}{\alpha} \xrightarrow{x_{\otimes Y}} Z \overset{1}{\xrightarrow{\sigma_{X \otimes Y, Z}(\alpha)}} \overset{(PV2)}{=} \overset{(PV1)}{=} \overset{(PV1)}{=}. $$

$$= \sigma_{Y \otimes Z, X} \circ \sigma_{X, Y \otimes Z}(\alpha).$$
Similarly, \( \sigma_{X,Y \otimes Z} = \sigma_{Z \otimes X,Y} \circ \sigma_{X \otimes Y,Z} \).

(iii) Let \( X,Y \in \text{Ob}(\mathcal{C}) \), then by (i) \( \sigma_{X \otimes Y, \mathbf{1}} = \text{id}_{\text{Hom}_C(\mathbf{1},X \otimes Y)} \) and by (ii) \( \sigma_{X \otimes Y, \mathbf{1}} = \sigma_{Y, \mathbf{1} \otimes X} \circ \sigma_{X \otimes Y, \mathbf{1}} = \sigma_{Y, \mathbf{1} \otimes X} \circ \sigma_{X,Y} \). Hence \( \sigma_{X,Y}^{-1} = \sigma_{Y,X} \).

\( \square \)

Let \( (E,c,\epsilon) \) be a cyclic \( \mathcal{C} \)-set with \( \#E = n \). The construction of the \( \mathbb{K} \)-module \( H(E) \) is as follows.

**Step 1.** For \( e \in E \), set

\[
Z_e = c(e_1)^{\epsilon(e_1)} \otimes \cdots \otimes c(e_n)^{\epsilon(e_n)} \quad \text{and} \quad H_e = \text{Hom}_\mathbb{C}(\mathbf{1}, Z_e),
\]

where \( e = e_1 < e_2 < \cdots < e_n \) are the elements of \( E \) ordered linearly starting from \( e \).

**Step 2.** For \( e,f \in E \), fix the order \( e = e_1 < e_2 < \cdots < e_n \) as above. Let us say that \( f = e_k \) for \( k \in \{1, \ldots, n\} \). Set

\[
X = c(e_1)^{\epsilon(e_1)} \otimes \cdots \otimes c(e_{k-1})^{\epsilon(e_{k-1})} \quad \text{and} \quad Y = c(e_k)^{\epsilon(e_k)} \otimes \cdots \otimes c(e_n)^{\epsilon(e_n)}.
\]

Thus \( Z_e = X \otimes Y \) and \( Z_f = Y \otimes X \). We define a map \( p_{e,f} : H_e \to H_f \) by \( p_{e,f}(\alpha) = \sigma_{X,Y}(\alpha) \) for all \( \alpha \in H_e \).

**Step 3.** The family of \( \mathbb{K} \)-modules \( \{H_e\}_{e \in E} \) together with the family of isomorphisms \( \{p_{e,f}\}_{e,f \in E} \) form a projective system. Set

\[
H(E) = \lim_{\overset{\rightarrow}{e \in E}} H_e.
\]

This \( \mathbb{K} \)-module only depends on \( E \), it is called the symmetrized multiplicity module. The \( \mathbb{K} \)-module \( H(E) \) is equipped with a family of isomorphisms \( \tau_e : H(E) \to H_e \) for \( e \in E \), called the cone isomorphisms.

For a tuple \( S = (X_1 \epsilon_1, \ldots, X_n \epsilon_n) \) of signed objects of \( \mathcal{C} \). Set \( X_S = X_1^{\epsilon_1} \otimes \cdots \otimes X_n^{\epsilon_n} \) and \( S^* = (X_n(-\epsilon_n), \ldots, X_1(-\epsilon_1)) \). The tuple \( S \) determines a cyclic \( \mathcal{C} \)-set \( (E_S, c_S, \epsilon_S) \) where \( E_S = \{1, \ldots, n\} \) with the obvious cyclic order, \( c_S(i) = X_i \) and \( \epsilon_S(i) = \epsilon_i \) for \( i = 1, \ldots, n \). Denote by \( H(S) \) the \( \mathbb{K} \)-module \( H(E_S) \). We define a morphism \( Ev_S : X_S^* \otimes X_S \to \mathbf{1} \) recursively by

\[
Ev_{(X,+)} = ev_X, \quad Ev_{(X,-)} = \epsilon ev_X, \quad \text{and} \quad Ev_{ST} = Ev_T \circ (id_{X_T^*} \otimes Ev_S \otimes id_{X_T})
\]

where \( ST \) denotes the concatenation of the tuples \( S \) and \( T \). This morphism induces a \( \mathbb{K} \)-bilinear form

\[
\omega_S : \text{Hom}_\mathbb{C}(\mathbf{1}, X_S^*) \otimes \text{Hom}_\mathbb{C}(\mathbf{1}, X_S) \to \text{End}_\mathbb{C}(\mathbf{1}) = \mathbb{K}
\]

defined by \( \omega_S(\alpha \otimes \beta) = Ev_S \circ (\alpha \otimes \beta) \) for all \( \alpha \in \text{Hom}_\mathbb{C}(\mathbf{1}, X_S^*) \) and \( \beta \in \text{Hom}_\mathbb{C}(\mathbf{1}, X_S) \).

Now, \( \alpha \in \text{Hom}_\mathbb{C}(\mathbf{1}, X_S^*) \cong H(S^*) \) and \( \text{Hom}_\mathbb{C}(\mathbf{1}, X_S) \cong H(S) \) by the cone isomorphisms. If \( \mathcal{C} \) is spherical, then the form \( \omega_S \) induces a pairing, also denoted \( \omega_S \) (see [TuV13], Lemma 2.2).

\[
\omega_S : H(S^*) \otimes H(S) \to \mathbb{K}.
\]

A duality between cyclic \( \mathcal{C} \)-sets \( E \) and \( E' \) is a bijection \( \phi : E^{op} \to E' \) (where \( E^{op} \) is the same set \( E \) but with the opposite cyclic order) preserving the cyclic order
and commuting with the coloring and sign maps. Such a $\phi$ induces a $\mathbb{K}$–isomorphism $H(\phi) : H(E') \to H(E)$ and a pairing $H(E') \otimes H(E) \to \mathbb{K}$ (see [TuV13, §2.2] for more details).

Recall that a bilinear form $\omega : M \otimes N \to \mathbb{K}$ (where $M$ and $N$ are $\mathbb{K}$–modules) is non-degenerate if there is a $\mathbb{K}$–linear map $\Omega : \mathbb{K} \to N \otimes M$ such that $(id_N \otimes \omega) \circ (\Omega \otimes id_N) = id_N$ and $(\omega \otimes id_M) \circ (id_M \otimes \Omega) = id_M$. If $\omega$ is non-degenerate, we call the element $\Omega(1) \in N \otimes M$ the inverse of $\omega$ and the adjoint map $\Omega^* : M^* \otimes N^* \to \mathbb{K}$ the contraction associated to $\omega$.

We say that a pivotal $\mathbb{K}$–category $C$ is non-degenerate if the bilinear form $\omega_X : \text{Hom}_C(\mathbb{1}, X^*) \otimes \text{Hom}_C(\mathbb{1}, X) \to \mathbb{K}$ defined by $\omega_X(\alpha \otimes \beta) = ev_X(\alpha \otimes \beta)$ for $\alpha \in \text{Hom}_C(\mathbb{1}, X^*)$ and $\beta \in \text{Hom}_C(\mathbb{1}, X)$ is non-degenerate for all object $X \in \text{Ob}(C)$. If $C$ is non-degenerate, the the bilinear form (1.12) is non-degenerate for all tuple $S$ of signed objects (see [TuV13], Lemma 2.3).

### 1.5 Pre-fusion and fusion categories

Let $C$ be a monoidal $\mathbb{K}$–category over a commutative ring $\mathbb{K}$. An object $X$ of $C$ is said to be simple if $\text{End}_C(X) \cong \mathbb{K}$.

**Definition 1.20.** A pre–fusion category (over $\mathbb{K}$) is a pivotal $\mathbb{K}$–category $C$ verifying:

- (PF1) (Coproduct) Any pair of objects $X$ and $Y$ of $C$ has a direct sum;
- (PF2) (Semisimplicity) Any object of $C$ is a finite direct sum of simple objects;
- (PF3) (Schur) If $i, j \in \text{Ob}(C)$ are non–isomorphic simple objects , then $\text{Hom}_C(i, j) = 0$.

Notice that conditions (PF2) and (PF3) imply that for any pair of objects $X$ and $Y$ of $C$, the $\mathbb{K}$–module $\text{Hom}_C(X, Y)$ is a free $\mathbb{K}$–module of finite rank. A set $I$ of simple objects of a pre–fusion category $C$ is said to be representative if $\mathbb{1} \in I$ and any simple object of $C$ is isomorphic to a unique element of $I$. A pre–fusion category is always non-degenerate ([TuV13], Lemma 4.1). If a pre–fusion category $C$ satisfies the following condition:

- (PF4) The set of isomorphism classes of simple objects of $C$ is finite,

then we say that $C$ is a fusion category. We define the dimension of a fusion category $C$ as:

$$\dim(C) = \sum_{i \in I} \dim(i)\dim_r(i).$$

There are two families of examples of fusion categories; the first family consists of algebraic examples as the category of finite-rank representations (over $\mathbb{K}$) of a finite group whose order is relatively prime to the characteristic of $\mathbb{K}$ and the category of representations of an involutory finite dimensional Hopf algebra over a field of characteristic zero (by involutory we mean that the antipode map is its own inverse). The second family comes from the Temperley–Lieb categories (see Example 1.16) over the fraction field of the ring of complex polynomials in a formal parameter $q$. The last family is closely related to quantum groups (see [C14]). Finally, other source of examples are the modular categories, because any modular category is a fusion category (see [Mü12]).
2 Turaev- Viro for closed 3-manifolds

The main goal of this chapter is to show the construction of the Turaev-Viro topological invariants for closed oriented 3-manifolds from a spherical fusion category $\mathcal{C}$ over a commutative ring $K$ following [TuV13]. For this purpose we need a combinatorial way of presenting 3–manifolds, Sections 2.1 and 2.4 deal with these topological preliminaries. Section 2.2 shows one of the principal constructions in this dissertation and after that we will use it for the construction of the Turaev–Viro invariant in Sections 2.3 and 2.5. Finally in Section 2.6 we introduce a nice way to obtain a combinatorial presentation (a skeleton) of a 3–manifold from a Heegaard splitting.

2.1 Topological Preliminaires I

We introduce some terminology from piecewise-linear topology, for details we refer to [RS72]. Let $K$ be a simplicial complex and $S$ a collection of simplices in $K$. The star of $S$, denoted by $St(S)$, is the set of all simplices $\sigma$ in $K$ for which there is an element $\tau \in S$ such that $\tau$ is a face of $\sigma$. The closure of $S$, denoted by $Cl(S)$, is the smallest simplicial subcomplex of $K$ that contains each simplex in $S$. The link of a 0-simplex $x$ in $K$, denoted by $lk(x)$, is the simplicial complex $Cl(St\{x\}) \setminus St\{x\}$. See the following figure.

Recall that a triangulation of a manifold $M$ is a pair $(K, h)$ where $K$ is a simplicial complex and $h : K \to M$ is a homeomorphism. We have the following famous theorem (see [Mo52] for a proof).
Theorem 2.1 (Bing-Moise, 1950). Any 3-manifold admits a triangulation.

Let $M$ be a closed 3-manifold, $t$ a triangulation and $x$ a vertex of $t$. The simplicial complex $Cl(St(\{x\}))$ is a closed 3-ball $B \subset M$ and the simplicies of $t$ contained in $\partial B$ form a triangulation $t_B$ of $\partial B$. Let $\Gamma \subset \partial B$ the 1-skeleton of $t_B$. We call the pair $(\partial B, \Gamma)$ the link of $x$ in $(M, t)$. The link $(\partial B, \Gamma)$ can be identified with a pair $(\partial B_x, \Gamma_x)$ where $B_x \subset M$ is a closed 3-ball that meets every simplex of $t$ incident to $x$ and $\Gamma_x$ is a graph on $\partial B_x$ obtained from the intersection of $\partial B_x$ with the 2-skeleton of $t$. Then for each vertex $x$ of $t$ we have a graph $\Gamma_x$ in $\partial B_x$, so an idea to construct a topological invariant for closed 3-manifolds is to first construct an invariant for graphs on the sphere $\mathbb{S}^2$ and then try to use this invariant to obtain an invariant of $M$. We will see later that for generalizing the invariant for 3-manifolds with boundary we will need invariants of graphs on surfaces.

2.2 Invariant of colored graphs

Let $C$ a pivotal $\mathbb{K}$-category. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively. We say that $G$ is $C$-colored if $G$ is an oriented graph and if there is a function $c : E(G) \to C$. If $e \in E(G)$, we call the object $c(e)$ of $C$ the color of $e$ and the map $c$ coloring map.

Let $\Sigma$ be an oriented surface and $G$ a $C$-colored graph in $\Sigma$. Each vertex $v \in V(G)$ determines a cyclic $C$-set $(E_v, c_v, \epsilon_v)$ as follows: $E_v$ is the set of half-edges of $G$ incident to $v$ with cyclic order induced by the opposite orientation of $\Sigma$; the map $c_v : E_v \to C$ assigns to $e \in E_v$ its color and the map $\epsilon_v : E_v \to \{+, -\}$ assigns to $e \in E_v$ the sign $+$ if $e$ is oriented towards $v$ and $-$ otherwise. In this way we obtain a tuple of signed objects $(X_1c_1, \ldots, X_nc_n)$ of $C$. Then we can associate to each vertex $v \in V(G)$ a $\mathbb{K}$-module $H(E_v)$ (see 1.4) that we denote by $H_v(G)$ or $H_v(G; \Sigma)$. In the following figure we represent this procedure, where we have denoted $c_v(e_i) = X_i \in Ob(C)$:

Thus, if we consider all vertices of $G$ we can obtain a $\mathbb{K}$-module associated to the pair $(G, \Sigma)$, by

$$H(G; \Sigma) = \otimes_{v \in V(G)} H_v(G)$$

(2.1)

Given two $C$-colored graphs $G$ and $G'$ in $\Sigma$, an isotopy of $G$ into $G'$ is an isotopy of $G$ into $G'$ in $\Sigma$ preserving the vertices, the edges, the orientation and the color of the edges. An isotopy $\iota : G \to G'$ induces an isomorphism of cyclic $C$-sets $E_v \to E_{\iota(v)}$, for each $v \in V(G)$. This induces $\mathbb{K}$-isomorphisms $H_v(\iota) : H_v(G) \to H_{\iota(v)}(G')$ and $H(\iota) = \otimes_v H_v(\iota) : H(G; \Sigma) \to H(G'; \Sigma)$.
A duality between two vertices \( u, v \in V(G) \) is a duality between the \( C \)-sets \( E_u \) and \( E_v \). Such duality induces a pairing \( \omega_{u,v} : H_u(G) \otimes H_v(G) \to \mathbb{K} \) and, if the category \( C \) is non-degenerate, a contraction homomorphism

\[
*_{u,v} : H_u(G)^* \otimes H_v(G)^* \to \mathbb{K}.
\] (2.2)

A \( C \)-colored graph \( G \subset \Sigma \) determines a \( C \)-colored graph \( G^{\text{op}} \) in \( -\Sigma \) obtained by reversing the orientation in all edges of \( G \) and in \( \Sigma \) while keeping the colors of the edges. The cyclic \( C \)-sets determined by a vertex \( v \) of \( G \) and \( G^{\text{op}} \) are dual. If \( C \) is non-degenerate, then we can conclude that

\[
H_v(G^{\text{op}}; -\Sigma) = H_v(G; \Sigma)^* \quad \text{and} \quad H(G^{\text{op}}; -\Sigma) = H(G; \Sigma)^*.
\]

Let \( G \) be a \( C \)-colored graph in \( \mathbb{R}^2 \) (always oriented counterclockwise). For each vertex \( v \) of \( G \), we choose a half-edge \( e_v \in E_v \) and isotope \( G \) near \( v \) so that the half-edges incident to \( v \) lie above \( v \) with respect to the second coordinate on \( \mathbb{R}^2 \) and \( e_v \) is the leftmost of them. Pick any \( \alpha_v \in H_v(G) \) and replace \( v \) by a box colored with \( \tau_{e_v}^v(\alpha_v) \) as depicted below, where \( \tau^v \) is the universal cone of \( H_v(G) \).

![Diagram](image1)

This transforms \( G \) into a planar diagram which determines, by the Penrose calculus, an element of \( \text{End}_C(\mathbb{1}) = \mathbb{K} \), denoted \( F_C(G)(\otimes e_v \alpha_v) \in \mathbb{K} \). By linear extension, this procedure defines a vector \( F_C(G) \in H(G)^* = \text{Hom}_\mathbb{K}(H(G), \mathbb{K}) \). This vector is an isotopy invariant of \( G \) (for details see [TuV13, §2.2]).

If in addition we suppose that the category \( C \) is spherical, then we can extend the invariant \( F_C \) defined above for graphs in \( \mathbb{R}^2 \) to an invariant of graphs in the 2-sphere \( \mathbb{S}^2 = \mathbb{R}^2 \cup \infty \) in a natural way, since a graph in \( \mathbb{S}^2 \) can be pushed away from \( \infty \) obtaining a graph in \( \mathbb{R}^2 \).
By definition, it follows that for two disjoint $\mathcal{C}$-colored graphs $G, G' \subset S^2$ we have $H(G \amalg G') = H(G) \otimes H(G')$ and $F_{\mathcal{C}}(G \amalg G') = F_{\mathcal{C}}(G) \otimes F_{\mathcal{C}}(G')$.

The following theorem shows some local relations verified by $F_{\mathcal{C}}$.

**Lemma 2.2.** Let $\mathcal{C}$ be a spherical pre-fusion category and let $I$ be a representative set of simple objects of $\mathcal{C}$.

(i) For any $i, j \in I$,

\[
 F_{\mathcal{C}}\left( \begin{array}{c} j \\ i \end{array} \right) = \delta_{i,j} F_{\mathcal{C}} \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right)
\]

(ii) For any $i, j \in I$

\[
 F_{\mathcal{C}}\left( \begin{array}{c} j \\ i \end{array} \right) = \delta_{i,j} (dim(i))^{-1} F_{\mathcal{C}}\left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) \otimes F_{\mathcal{C}}\left( \begin{array}{c} \vdots \\ \vdots \end{array} \right)
\]

(iii)

\[
 F_{\mathcal{C}}\left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) = \sum_{i \in I} dim(i)*_{u,v} F_{\mathcal{C}}\left( \begin{array}{c} \vdots \\ \vdots \end{array} \right)
\]

(iv)

\[
 F_{\mathcal{C}}\left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) = *_{u,v} F_{\mathcal{C}}\left( \begin{array}{c} \vdots \\ \vdots \end{array} \right)
\]

In (ii) and (iv) the empty rectangles stand for pieces of $\mathcal{C}$-colored graphs sitting inside the rectangles.

**Proof.** See [TuV13], Lemma 4.2.

We illustrate the usage of this lemma in the proof of Theorem 2.3 below and then we will see its applications also to calculate the invariant in examples 2.10 and 2.11. Let $\mathcal{C}$ be a spherical pre-fusion category, let $i, j, k, l, m, n \in Ob(\mathcal{C})$ and consider the graph $\Gamma$ given in the figure.
seen as a graph on $S^2$. Let us denote by $\begin{vmatrix} i^+ & j^+ & k^+ & l^+ & m^+ & n^+ \\ i & j & k & l & m & n \end{vmatrix}$ the vector $F_C(\Gamma) \in H(\Gamma)^*$. For a 6-tuple $T = (i\varepsilon_i, j\varepsilon_j, k\varepsilon_k, l\varepsilon_l, m\varepsilon_m, n\varepsilon_n)$ of signed objects in $C$, let $\Gamma_T$ be the $C$-colored graph in $S^2$ obtained from the graph $\Gamma$ above by reversing the orientation on all edges colored by $x \in \{i, j, k, l, m, n\}$ with $\varepsilon_x = -$. We set $\begin{vmatrix} i\varepsilon_i & j\varepsilon_j & k\varepsilon_k & l\varepsilon_l & m\varepsilon_m & n\varepsilon_n \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \mathbb{F}_C(\Gamma_T) \in H(\Gamma_T)^*$. This vector is called the 6-j-symbol determined by $T$. For example if $T = (i-, j+, k-, l-, m-, n+)$ then $\Gamma_T$ is the graph

Given $a, b, c \in \text{Ob}(C)$ and signs $\varepsilon, \mu, \nu \in \{+, -\}$ we have a canonical (non-degenerate) bilinear form $\omega_{a\varepsilon, b\mu, c\nu} : H(a\varepsilon, b\mu, c\nu) \otimes H(a(-\varepsilon), b(-\mu), c(-\nu)) \to \mathbb{K}$ and then (as in (2.2)) we have a contraction $*_{a\varepsilon, b\mu, c\nu} : H(a\varepsilon, b\mu, c\nu)^* \otimes H(a(-\varepsilon), b(-\mu), c(-\nu))^* \to \mathbb{K}$.

**Theorem 2.3** (The Biedenharn-Elliott identity). Let $C$ be a spherical pre-fusion category and $I$ be a representative set of simple objects in $C$. Then for any $a, b, c, i, j, k, l, m, n \in \text{Ob}(C)$ we have

$$\sum_{z \in I} \text{dim}(z) *_{m\varepsilon, k\varepsilon, z\varepsilon} *_{a\varepsilon, b\varepsilon, c\varepsilon} \begin{vmatrix} i\varepsilon_i & z\varepsilon_i & b\varepsilon_i & l\varepsilon_i & m\varepsilon_i & k\varepsilon_i \\ z\varepsilon_i & l\varepsilon_i & m\varepsilon_i & k\varepsilon_i & 1 & 1 \end{vmatrix} \otimes \begin{vmatrix} i\varepsilon_i & j\varepsilon_j & c\varepsilon_k & l\varepsilon_l & m\varepsilon_m & k\varepsilon_m \\ z\varepsilon_j & l\varepsilon_l & m\varepsilon_m & k\varepsilon_m & 1 & 1 \end{vmatrix}$$

$$= \sum_{z \in I} \text{dim}(z) *_{m\varepsilon, k\varepsilon, z\varepsilon} *_{c\varepsilon, j\varepsilon, i\varepsilon} \begin{vmatrix} i\varepsilon_i & z\varepsilon_i & b\varepsilon_i & l\varepsilon_l & m\varepsilon_l & k\varepsilon_l \\ l\varepsilon_i & n\varepsilon_n & a\varepsilon_l & c\varepsilon_c & b\varepsilon_b & m\varepsilon_m \end{vmatrix}$$

**Proof.** We do the proof for the upper choice of signs, the other case being similar.

$$\sum_{z \in I} \text{dim}(z) *_{m-\varepsilon, k+\varepsilon, z+} *_{a-\varepsilon, b+\varepsilon, c+} \begin{vmatrix} i\varepsilon_i & z- & b+ & l+ & m+ & k+ \\ z+ & j- & c+ & l+ & m+ & k+ \end{vmatrix} \otimes \begin{vmatrix} i\varepsilon_i & j- & a- & c- & b+ & z- \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \ldots$$
\[
= \sum_{z \in I} \dim(z) \ast_{m-,k+,z+} \ast_{c-,j-,z+} \ast_{b+,i-,z+} \left( \mathcal{F}_C \left( \begin{array}{c}
\text{Graph 1}
\end{array} \right) \right) \otimes \mathcal{F}_C \left( \begin{array}{c}
\text{Graph 2}
\end{array} \right)
\]
\[
= \sum_{z \in I} \dim(z) \ast_{m-,k+,z+} \ast_{c-,j-,z+} \ast_{b+,i-,z+} \left( \mathcal{F}_C \left( \begin{array}{c}
\text{Graph 3}
\end{array} \right) \right)
\]
\[
= \text{Lemma 2.2(iv)} \sum_{z \in I} \dim(z) \ast_{m-,k+,z+} \ast_{c-,j-,z+} \left( \mathcal{F}_C \left( \begin{array}{c}
\text{Graph 4}
\end{array} \right) \right)
\]
\[
= \text{Lemma 2.2(iv)} \sum_{z \in I} \dim(z) \ast_{m-,k+,z+} \left( \mathcal{F}_C \left( \begin{array}{c}
\text{Graph 5}
\end{array} \right) \right)
\]
\[
= \text{isotopy} \sum_{z \in I} \dim(z) \ast_{m-,k+,z+} \left( \mathcal{F}_C \left( \begin{array}{c}
\text{Graph 6}
\end{array} \right) \right)
\]
Let $C$ be a spherical fusion category over a commutative ring $K$ such that $\text{dim}(C) \in K^\times$. Fix a representative set $I$ of simple objects of $C$. Given a closed oriented 3-manifold $M$ we would like to construct a topological invariant $|M|_C \in K$: we proceed as follows. Let $t$ be a triangulation of $M$, fix an orientation on the 2-skeleton $t^{(2)}$ (i.e., an orientation on each face) and let $c : \text{Reg}(t) \to I$ be a map (coloring map) where $\text{Reg}(t)$ denotes the set of all faces of $t$.

**Step 1.** For each oriented edge $e$ of $t$ we can associate a $K$-module $H_c(e)$ in the following way. Let $t_e$ be the set of all faces of $t$ containing $e$. The orientations of $M$ and $e$ induce an orientation on a small loop encircling $e$ and this allows us to define a cyclic order on $t_e$. See below:
For each $r \in t_e$ we assign the color $c(r) \in I$ and a sign $\epsilon(r) \in \{+, -\}$ (we assign $+$ if the orientation of $r$ and $e$ are compatible, $-$ otherwise). Thus the set $t_e$ becomes a cyclic $C$-set. Set $H_c(e) = H(t_e)$. If we consider the edge $e$ with the opposite orientation, denoted by $e^{op}$, then the sets $t_e$ and $t_{e^{op}}$ verify $(t_e)^{op} = t_{e^{op}}$, i.e. are in duality.

Thus there is a canonical pairing $\omega_e : H_c(e) \otimes H_c(e^{op}) \to \mathbb{K}$ and, since $C$ is a fusion category (then non-degenerate), we have a contraction $\ast_e : H_c(e) \otimes H_c(e^{op}) \to \mathbb{K}$. Note that $\ast_e$ and $\ast_{e^{op}}$ are equal up to permutation of the tensor factors. We consider the tensor product of all the contractions $\ast_e$ where $e$ runs over all the oriented edges of $t$, and we denote this tensor product by $\ast_t$.

$$\ast_t : \otimes_e H_c(e)^* \to \mathbb{K}. \quad (2.3)$$

**Step 2.** Let $x$ be a vertex of $t$. Let $t_x$ be the set of all edges incident to $x$ oriented away from $x$. Let us consider the link $(\partial B_x, \Gamma_x)$ of $x$. We want to $C$-color the graph $\Gamma_x$ on $\partial B_x$, we proceed as follows. If $\alpha$ is an edge of $\Gamma_x$ there is a unique face $r_\alpha \in t_x$ such that $\alpha \subset r_\alpha$. We assign to $\alpha$ the color $c(r_\alpha)$ and we consider $\alpha$ with the orientation induced by $r_\alpha \setminus \partial B_x$. The orientation of $\partial B_x$ is that induced by the orientation of $M$. Then $\Gamma_x$ is a $C$-colored graph in the oriented surface $\partial B_x$.

From Section 2.2 we have the $\mathbb{K}$-module $H(\Gamma_x)$. Notice that the vertices of $\Gamma_x$ are in bijection with the elements of $t_x$ and if $v$ is a vertex of $\Gamma_x$ and $e \in t_x$ are such that $e \cap \partial B_x = \{v\}$, then $H_0(\Gamma_x) = H_c(e)$. Thus we have $H(\Gamma_x) = \otimes_{v \in V(\Gamma_x)} H_0(\Gamma_x) = \otimes_{e \in t_x} H_c(e)$. Therefore the invariant $F_C(\Gamma_x)$ belongs to $H(\Gamma_x)^* = \otimes_{e \in t_x} H_c(e)^*$. Hence the tensor product of $F_C(\Gamma_x)$ over all vertices $x$ of $t$ is an element of the tensor product of $H_c(e)^*$ over all oriented edges $e$ of $t$:

$$\otimes_x F_C(\Gamma_x) \in \otimes_e H_c(e)^*.$$

**Step 3.** Set

\[
\]
\[ |M|_\mathcal{C} = (\text{dim}(\mathcal{C}))^{-|t|} \sum_c \left( \prod_{r \in \text{Reg}(t)} \text{dim}(c(r)) \right) *_{t} \left( \otimes \mathbb{F}_c(\Gamma_x) \right) \in \mathbb{K}, \quad (2.4) \]

where $|t|$ denotes the number of tetrahedra in the triangulation $t$, and $c$ runs over all the maps $c : \text{Reg}(t) \to I$. Notice that for steps 1 and 2 above, the right-hand side of the equality (2.4) has a sense.

**Theorem 2.4** ([TuV13] Theorem 5.1). The scalar $|M|_\mathcal{C} \in \mathbb{K}$ is a topological invariant of the closed 3–manifold $M$. More precisely, the definition of $|M|_\mathcal{C}$ given in equation (2.4) is independent of the choice of a triangulation $t$ and an orientation of $t^{(2)}$. Also $|M|_\mathcal{C}$ is independent of the choice of the representative set $I$ of simple objects of $\mathcal{C}$.

See the comments about the proof after the Theorem 2.6. The definition of the Turaev–Viro invariant from a triangulation is very simple and beautiful but its deficiency is in the moment of doing particular calculations. It is for this reason that we will introduce a more general definition from the one given above. This more general definition is based in the concept of skeleton of a manifold.

### 2.4 Topological Preliminaires II

As we have seen in the previous section to define the Turaev–Viro invariant for a closed 3–manifold $M$ we used a triangulation of $M$, this is a combinatorial way of presenting $M$. In this section we study a more general (more flexible) combinatorial presentation of a closed 3-manifold.

By a 2-polyhedron we mean a compact topological space that can be triangulated by using simplices of dimension $\leq 2$. Let $P$ be a 2-polyhedron, an arc in $P$ is the image of a continuous map $\alpha : [0, 1] \to P$ such that $\alpha_{|[0,1]}$ is embedding, we denote the arc by the same letter $\alpha$. The points $\alpha(0)$ and $\alpha(1)$ are called the endpoints of $\alpha$ and $\alpha((0,1))$ its interior. Set a finite set $E$ of arcs in $P$ such that

(i) if $\alpha$ and $\beta$ are two different arcs, then $\alpha \cap \beta$ is the empty set or consists of endpoints,

(ii) $P \setminus \bigcup_{e \in E} e \subset \text{Int}(P) := \{x \in P \mid \text{there is an open set } U \subset P \text{ with } x \in U \cong \mathbb{R}^2 \}$

and $P \setminus \bigcup_{e \in E} e$ is dense in $P$.

The elements of $E$ and their endpoints are called edges and vertices of $P$, respectively. Set $P^{(1)}$ the graph in $P$ formed by the edges and vertices of $P$ and $\hat{P}$ the compact surface obtained by cutting along $P^{(1)}$ (that is, by removing a regular neighborhood of $P^{(1)}$ from $P$) and let $p : \partial \hat{P} \to P^{(1)}$ the natural projection. We say that the pair $(P, E)$ is a stratified 2-polyhedron if $S := p^{-1}(V(P))$ is a finite set (where $V(P)$ is the set of vertices of $P$) and the complement of $S$ in each component of $\partial \hat{P}$ is mapped homeomorphically onto the interior of an edge of $P$. The connected components of $\hat{P}$ are called regions and we denote by $\text{Reg}(P)$ the set of regions of $P$. We say that $P$ is orientable (respectively oriented) if all its regions are orientable (respectively oriented).

For example in the following figure we can see a 2-polyhedron $P$ with four edges $e_1$ (red), $e_2$ (blue), $e_3$ (purple) and $e_4$ (green) and two vertices $u$ and $v$. Together with the compact surface $\hat{P}$.
Notice that if \( P \) is a 2-polyhedron such that \( \text{Int}(P) \) is dense in \( P \) then \( P \) admits an stratification (that is, as set \( E \) as above) given by the edges of any triangulation. Also we have that a closed surface is a stratified 2-polyhedron with an empty set of edges.

Let \((P, E)\) be a stratified 2-polyhedron, \( x \in V(P) \) and \( e \in E \). A branch of \( P \) at \( x \) is a set
\[
\left\{ \gamma : [0,1] \to P \mid \gamma \text{ is a path, } \gamma(0) = x, \text{ and } \gamma((0,1]) \subset P \setminus P^{(1)} \right\},
\]
and a branch of \( P \) at \( e \) is a set
\[
\left\{ \gamma : [0,1] \to P \mid \gamma \text{ is a path, } \gamma(0) \in \text{int}(e), \text{ and } \gamma((0,1]) \subset P \setminus P^{(1)} \right\}.
\]

Intuitively the branches of \( P \) at \( x \), or at \( e \), are the interior of the regions of \( P \) adjacent to \( x \) or to \( e \). Denote by \( P_e \) the set of branches of \( P \) at the edge \( e \), we call the cardinality of this set the valence of \( e \) and we define the boundary of \( P \) as the graph \( \{ e \in E \mid \#P_e = 1 \} \) and it is denoted by \( \partial P \). For example in the 2-polyhedron considered in the figure above we can see that the vertices \( u \) and \( v \) have two branches and \( \#P_{e_1} = \#P_{e_2} = 1 \), \( \#P_{e_3} = \#P_{e_4} = 2 \) so that \( \partial P = e_1 \cup e_2 \).

**Definition 2.5.** Let \( M \) be a closed 3–manifold. A skeleton of \( M \) is an oriented stratified 2-polyhedron \((P, E)\) such that \( P \subset M \), \( \partial P = \emptyset \) and \( M \setminus P \) is a disjoint union of open 3-balls.

Let \( t \) be a triangulation of \( M \). The 2-skeleton \( t^{(2)} \) with an arbitrary orientation of the faces together with the set \( E \) of edges of \( t \) is a skeleton of \( M \), that is, the notion of skeleton is more general than the notion of triangulation. As a consequence of the Moise-Bing Theorem 2.1 we have that all closed 3-manifolds admit a skeleton.

Now we want to generalize the notion of link given in Section 2.1 for a vertex of a triangulation of \( M \). Let \((P, E)\) be a skeleton of \( M \) and \( x \) a vertex of \( P \), there is a sufficiently small closed 3-ball \( B_x \) centered at \( x \) such that \( B_x \) meets all the branches of \( P \) at \( x \) and all the edges of \( P \) incident at \( x \) in its interior. In this way we obtain a graph \( \Gamma_x \) on \( \partial B_x \). The vertices of \( \Gamma_x \) correspond to the intersection of \( \partial B_x \) with the half-edges of \( P \) incident to \( x \) and the edges of \( \Gamma_x \) correspond to the intersection of \( \partial B_x \) with the branches of \( P \) at \( x \). The pair \((\partial B_x, \Gamma_x)\) is called the link of \( x \) in \((M, P)\).
2.5 Turaev-Viro from a skeleton

Let \( C \) be a spherical fusion category over a commutative ring \( \mathbb{K} \) such that \( \text{dim}(C) \in \mathbb{K}^\times \). Fix a representative set \( I \) of simple objects of \( C \). Given a closed oriented 3–manifold \( M \) we want to calculate the topological invariant \( |M|_C \in \mathbb{K} \) defined in Section 2.3 not only from a triangulation but from an arbitrary skeleton. Let \((P,E)\) be a skeleton of \( M \) and consider a map \( c: \text{Reg}(P) \to I \) (coloring map).

**Step 1.** To each oriented edge \( e \) of \( P \) we associate the \( \mathbb{K} \)-module \( H_c(e) = H(P_e) \) where \( P_e \) is the set of all branches of \( P \) at \( e \) seen as a \( C \)-cyclic set as in Section 2.3. If we consider the edge \( e \) with opposite orientation, denoted by \( e^{op} \), we have that \( P_e^{op} = (P_e)^{op} \) so that we have a pairing \( \omega_e: H_c(e) \otimes H_c(e^{op}) \to \mathbb{K} \) and thus a contraction \( *_c: H_c(e)^* \otimes H_c(e^{op})^* \to \mathbb{K} \). We denote by \( *_P \) the tensor product of the maps \( *_c \) over all the oriented edges \( e \) of \( P \).

**Step 2.** Let \( x \) be a vertex of \( P \) and \((\partial B_x, \Gamma_x)\) its link. Then \( \Gamma_x \) becomes a \( C \)-colored graph as in Section 2.3. Thus we obtain a vector

\[
\bigotimes_x \mathbb{F}_C(\Gamma_x) \in \bigotimes_e H_c(e)^*
\]

where \( x \) runs over all vertices of \( P \) and \( e \) runs over all oriented edges of \( P \).

**Theorem 2.6** ( [TuV13] Theorem 6.1). For any skeleton \((P,E)\) of \( M \), we have

\[
|M|_C = (\text{dim}(C))^{-|P|} \sum_C \left( \prod_{r \in \text{Reg}(P)} \text{dim}(c(r))^{\chi(r)} \right) *_P \left( \bigotimes_x \mathbb{F}_C(\Gamma_x) \right) \in \mathbb{K},
\]

where \(|P|\) denotes the number of components of \( M \setminus P \), \( c \) runs over all maps \( \text{Reg}(P) \to I \) and \( \chi(r) \) is the Euler characteristic of \( r \).

Notice that for steps 1 and 2 above, the right-hand side of the equality (2.5) has a sense and this formula generalizes the formula (2.4) because the latter is obtained from the former by taking as \( P \) the oriented 2-skeleton \( t(2) \) of a triangulation \( t \) of \( M \). Therefore Theorem 2.4 follows from Theorem 2.6.

**Remark.** When \( P \) is the oriented 2–skeleton of the cellular subdivision of \( M \) dual to a triangulation \( t \), formula (2.5) is the formula given in [BW96, TuVi92]. Although in [TuVi92] is given for a particular category.

**Comments about the proof of theorems 2.4 and 2.6.** The definition of \( |M|_C \) is a combinatorial definition. Its invariance is showed essentially following three steps.

- Definition of certain local transformations on skeletons of \( M \), called primary moves.

- Showing that any two skeletons of \( M \) can be related by these primary moves.

- Showing the invariance of the combinatorial definition under the primary moves.

We will define below the primary moves on skeletons of a compact 3–manifold \( M \). Denote by \( \#v, \#e \) and \( \#r \) the number of vertices, edges and regions of a skeleton \((P,E)\) of \( M \), respectively.

**Phantom move** \( T_1 \).
This move preserves $\#v$, increases $\#e$ by 1 and either preserves $\#r$ or increases it by 1.

**Contraction move** $T_2$.

This move is allowed only when at least one endpoint of the contracted edge is the endpoint of some other edge. This move decreases $\#v$ and $\#e$ by 1 and preserves $\#r$.

**Percolation move** $T_3$.

This move preserves $\#v$ and increases both $\#e$ and $\#r$ by 1.

**Bubble move** $T_4$.

This move increases $\#v$ and $\#e$ by 1 and $\#r$ by 2.

To show that any two skeletons of $M$ can be related by the primary moves it is useful to define some additional moves among these the *Matveev–Piergallini* (or *MP*) moves (see [Mat]). Finally the invariance under these moves is based in Lemma 2.2 (see [TuV13, §7] for all the details).
2.6 Skeleton from a Heegaard splitting

We have seen in the Section 2.5 that to calculate $|M|_{\mathcal{C}}$ of a closed oriented 3-manifold $M$ we need a skeleton of $M$. There is an interesting way to construct a skeleton of $M$ based in the so called theory of Heegaard splittings. In this section we introduce briefly this method.

By a handlebody of genus $g$ we mean the compact orientable 3-manifold with boundary obtained from a closed 3-ball by attaching $g$ handles (a handle is a topological space homeomorphic to $D \times [0,1]$). Notice that a handlebody is totally determined by its genus. If $H_g$ is a handlebody of genus $g$, then the boundary of $H_g$ is a closed oriented surface of genus $g$. We denote by $\Sigma_g$ the “standard” closed oriented of genus $g$. See the figure below:

Consider two handlebodies of the same genus, $H_g^1$ and $H_g^2$. Fix an orientation-preserving homeomorphism $h : \partial H_g^1 = \Sigma_g \rightarrow \Sigma_g = \partial H_g^2$. Let $M_h = H_g^1 \cup_h (-H_g^2)$, that is, $M_h$ is the 3-manifold obtained by attaching the boundaries $\partial H_g^1$ and $\partial H_g^2$ by means of $h$. An important fact is that the homeomorphism type of $M_h$ only depends of the isotopy class of $h$. This leads to the study of the mapping class group $\mathcal{M}(\Sigma_g)$ of a closed surface $\Sigma_g$, which is precisely defined as the group of orientation-preserving homeomorphisms $h : \Sigma_g \rightarrow \Sigma_g$ up to isotopy (see [FM12, I02, M09]).

A relevant consequence of the fact that all 3-manifold has a triangulation is that all 3-manifold can be obtained as in the construction above, that is, if $M$ is a 3-manifold, there is a positive integer $g$ and a orientation-preserving homeomorphism $h : \partial H_g^1 \rightarrow \partial H_g^2$ such that $M = H_g^1 \cup_h (-H_g^2)$. The triple $(H_g^1, H_g^2, h)$ is called a Heegaard splitting of $M$. For details we refer to [R00].

Let us assume that $M = H_g^1 \cup_h (-H_g^2)$ with $[h] \in \mathcal{M}(\Sigma_g)$. Let $D_1, \ldots D_g$ the $g$ disks of $H_g^1$ corresponding to the central meridional disks $D^1_i \times \{1/2\}$ of the $g$ handles $D^1_i \times [0,1]$ of $H_g^1$. (This collection of disks is not unique.)
Let $C_i^1 = \partial D_i^1$ and $h(C_i^1)$ their images on $\partial H^2_g$ by the attaching map $h$. The family of curves $\{C_1^2, \ldots, C_g^2, h(C_1^1), \ldots, h(C_g^1)\}$ on $\partial H^2_g$ are called a Heegaard diagram of genus $g$. The curves $\{C_1^2, \ldots, C_g^2\}$ determine completely the homeomorphism type of $M$.

**Example 2.7.** Let $M = S^3$ the 3-sphere. A Heegaard diagram of genus 1 for $S^3$ is the following

**Example 2.8.** Let $M = S^1 \times S^2$. A Heegaard diagram of genus 1 for $S^1 \times S^2$ is the following

Let $M$ be a 3-manifold and let $H^3_1 \cup_h (-H^2_2)$ be a Heegaard splitting. We can obtain a skeleton of $M$ by attaching $2g$ disks along the $2g$ curves determined by the Heegaard splitting (see [Mat]). We will use this kind of skeleton in the examples of the following section.

### 2.7 Examples

All along this section, let $\mathcal{C}$ be a spherical fusion category over a commutative ring $K$ such that $dim(\mathcal{C}) \in K^\times$. 
Example 2.9. Let $M = S^3$ be the 3-sphere. A skeleton of $S^3$ is given by the pair $(P, E = \emptyset)$ where $P$ is an oriented 2-sphere embedded in $S^3$. This skeleton corresponds to the skeleton associated with the Heegaard diagram of $S^3$ of genus 0. The skeleton $P$ has a unique region $P$ and does not have vertices or edges. Then we have $|P| = 2$, $\chi(P) = 2$ and by formula 2.5

$$|S^3| = \dim(C)^{-2} \sum_{i \in I} (\dim(i))^2 = \dim(C)^{-2} \dim(C) = \dim(C)^{-1}.$$ 

Example 2.10. Let $M$ be the closed 3-manifold $S^1 \times S^2$. Let us fix a point $x \in S^1$ and a circle $l \subset S^2$. We know a Heegaard diagram of $M$ by Example 2.8. Set $P = (S^1 \times \{l\}) \cup (\{x\} \times S^2)$ and $E = \{\{x\} \times l\}$ where $\{x\} \times l$ is seen as a loop, then the pair $(P, E)$ is a skeleton of $S^1 \times S^2$.

![Diagram](image)

Notice that $(S^1 \times S^2) \setminus P$ is the disjoint union of two open 3-balls, then $|P| = 2$. The skeleton $P$ has one edge $e := \{x\} \times l$, one vertex $x$ and three regions: a cylinder $r_1$ and two disks $r_2$ and $r_3$. We fix an arbitrary orientation of the regions (for example, as in the figure above). Now $\chi(r_1) = 0$ and $\chi(r_2) = \chi(r_3) = 1$ and a coloring map $c : \text{Reg}(P) \to I$ is given by a triple of objects $c(r_1) = i, c(r_2) = j, c(r_3) = k \in I$. We represent the colors $i$ by green, $j$ by blue and $k$ by orange.

![Coloring](image)

By formula $(2.5)$, we have

$$|S^1 \times S^2| = (\dim(C))^{-2} \sum_c \dim(c(r_1))^\chi(r_1) \dim(c(r_2))^\chi(r_2) \dim(c(r_3))^\chi(r_3) \ast_P \left( \mathbb{F}_C(\Gamma_x) \right)$$

$$= (\dim(C))^{-2} \sum_{i,j,k} \dim(j) \dim(k) \ast_P \left( \mathbb{F}_C(\Gamma_x) \right).$$

Now we want to determine the $C$-colored graph $\Gamma_x$. For this we represent $S^2$ as $\mathbb{R}^2 \cup \{\infty\}$ using the stereographic projection from the south pole. With this representation the skeleton $P$ can be visualized as in the figure.
where the top and the bottom of the cylinder are identified. Thus the graph $\Gamma_x$ can be determined using the above figure by taking a small closed 3-ball $B_x$ centered at $x$. See the figure below:

We consider the graph $\Gamma_x$ as a planar graph, and recall that $i$ is represented by green, $j$ by blue and $k$ by orange. Then $\Gamma_x$ is

\[
\Gamma_x = \begin{array}{c}
  \begin{array}{c}
    k \\
    i \\
    j \\
    i \\
  \end{array} \\
  \begin{array}{c}
    v \\
    j \\
    u \\
  \end{array}
\end{array}
\]

So that
\[ |S^1 \times S^2|_C = \left( \dim(C) \right)^{-2} \sum_{i,j,k} \dim(j) \dim(k) \ast_P \left( F_C(\Gamma_x) \right) \]

\[ = \left( \dim(C) \right)^{-2} \sum_{i,j,k} \dim(k) \dim(j) \ast_{u,v} \left( F_C \left( \begin{array}{c} k \\ j \\ i \\ u \\ v \end{array} \right) \right) \]

\[ = \text{isotopy in } S^2 \left( \dim(C) \right)^{-2} \sum_{i,j,k} \dim(k) \dim(j) \ast_{u,v} \left( F_C \left( \begin{array}{c} \cdots \end{array} \right) \right) \]

\[ = \text{Lemma 2.2(iii)} \left( \dim(C) \right)^{-2} \sum_{i,k} \dim(k) \left( F_C \left( \begin{array}{c} \cdots \end{array} \right) \right) \]

\[ = \left( \dim(C) \right)^{-2} \sum_{i,k} \dim(k) \dim(i) \dim(k) \dim(i^*) \]

\[ = \left( \dim(C) \right)^{-2} \sum_{i,k} \dim(k)^2 \dim(i)^2 \]

\[ = \left( \dim(C) \right)^{-2} (\dim(C))^2 = 1. \]

We have used the fact \( \dim(i) = \dim(i^*) \) which can be easily verified by using graphical calculus.

**Example 2.11.** Let \( M = S^3 \) be the 3-sphere. Let us calculate \( |S^3|_C \) by using the skeleton associated to the Heegaard diagram of genus 1 (see Example 2.7). The skeleton is

\[ P = \]

\[ P \] is a skeleton with two edges \( e_1, e_2 \), one vertex \( x \) and three regions \( r_1, r_2, r_3 \) which are disks. We have that \( M \setminus P \) is the disjoint union of two open 3-balls, then \( |P| = 2 \) and
\( \chi(r_1) = \chi(r_2) = \chi(r_3) = 1 \). A coloring map \( c : \text{Reg}(P) \to I \) is given by a triple of objects \( c(r_1) = i, c(r_2) = j, c(r_3) = k \in I \). We represent the colors \( i \) by green, \( j \) by blue and \( k \) by orange.

By formula (2.5), we have

\[
|S^3|_C = (\dim(C))^{-2} \sum_c \dim(c(r_1))^{\chi(r_1)} \dim(c(r_2))^{\chi(r_2)} \dim(c(r_3))^{\chi(r_3)} \ast_P (\mathbb{F}_C(\Gamma_x))
\]

\[
= (\dim(C))^{-2} \sum_{i,j,k} \dim(i) \dim(j) \dim(k) \ast_P (\mathbb{F}_C(\Gamma_x)).
\]

Thus the graph \( \Gamma_x \) can be determined using the above figure by taking a small closed 3-ball \( B_x \) centered at \( x \). See the figure below.

We consider the graph \( \Gamma_x \) as a planar graph, and recall that \( i \) is represented by green, \( j \) by blue and \( k \) by orange. Then \( \Gamma_x \) is

\[
\Gamma_x = \quad i \quad k \quad i
\]

\[
\quad v \quad \quad i \quad j
\]

Thus

\[
|S^3|_C = \quad (\dim(C))^{-2} \sum_{i,j,k} \dim(i) \dim(j) \dim(k) \ast_P (\mathbb{F}_C(\Gamma_x))
\]

\[
= (\dim(C))^{-2} \sum_{i,j,k} \dim(i) \dim(j) \dim(k) \ast_P \left( \mathbb{F}_C \left( \begin{array}{c}
\gamma \\
\end{array} \right) \right)
\]
\[
\begin{align*}
&= (\dim(C))^{-2} \sum_{i,j} \dim(i) \dim(j) \ast_{z,w} \left( \sum_k \dim(k) \ast_{u,v} \left( \mathbb{F}_C \begin{pmatrix}
& & u \\
& v & \\
& & w
\end{pmatrix} \right) \right) \\
&= \text{Lemma 2.2(iii)} (\dim(C))^{-2} \sum_{i,j} \dim(i) \dim(j) \ast_{z,w} \left( \mathbb{F}_C \begin{pmatrix}
& & i \\
& i & \\
& & i
\end{pmatrix} \right) \\
&= \text{Lemma 2.2(iii)} (\dim(C))^{-2} \sum_i \dim(i) \mathbb{F}_C \begin{pmatrix}
& & i \\
& i & \\
& & i
\end{pmatrix} \\
&= (\dim(C))^{-2} \sum_i \dim(i)^2 \\
&= (\dim(C))^{-2} (\dim(C)) = (\dim(C))^{-1}.
\end{align*}
\]

As expected that we have obtained the same result as in Example 2.9. This suggests another proof of Theorem 2.6 if one specializes this theorem to skeletons defined by Heegaard diagrams. In the theory of Heegaard splittings there are some transformation on a Heegaard diagram of a 3–manifold, called stabilization and handleslide. Formally if \((\Sigma_g, \{C_1^1, \ldots, C_g^1, C_g^2, \ldots, C_g^2\})\) is a Heegaard diagram of \(M\), the stabilization of \((\Sigma_g, \{C_1^1, \ldots, C_g^1, C_g^2, \ldots, C_g^2\})\) is the Heegaard diagram of \(M\) obtained by doing the connected sum of \((\Sigma_g, \{C_1^1, \ldots, C_g^1, C_g^2, \ldots, C_g^2\})\) with \((\Sigma_1, \{C_1^1, C_1^2\})\), where \((\Sigma_1, \{C_1^1, C_1^2\})\) is the genus one Heegaard diagram of \(S^3\) (see Example 2.7).
For instance, the Heegaard diagram of genus 1 of $\mathbb{S}^3$ given in Example 2.7 is obtained by stabilization on the Heegaard diagram of genus zero of $\mathbb{S}^3$ used in Example 2.9. The *Reidemeister–Singer theorem* says that given two Heegaard splittings $\mathcal{D}_1$ and $\mathcal{D}_2$ of a 3–manifold $M$, there is a third Heegaard splitting $\mathcal{D}$ of $M$ which is the stabilization (applying possibly several times the procedure) of both $\mathcal{D}_1$ and $\mathcal{D}_2$ (see [S99]). Thus, examples 2.9 and 2.11 show that the value of $|\mathbb{S}^3|_C$ is equal if we calculate it using the skeleton associated to the Heegaard splitting of genus 0 or the skeleton associated to the Heegaard splitting of genus 1 obtained by stabilization on the Heegaard splitting of genus 0.
3 Turaev-Viro for compact 3-manifolds with boundary

In this chapter $\mathcal{C}$ denotes a spherical fusion category over a commutative ring $\mathbb{K}$ such that $\text{dim}(\mathcal{C}) \in \mathbb{K}^\times$. The objective in this chapter is to extend the definition of $| \cdot |_{\mathcal{C}}$ given for closed 3–manifolds in chapter 2 to the case of 3–manifolds with boundary.

Let $M$ be a compact oriented 3–manifold with boundary. Let us consider an oriented graph $G = (V(G), E(G))$ in $\partial M$ such that every $v \in V(G)$ has valence $\geq 2$ (i.e., the number of edges incident to $v$ is $\geq 2$). Notice that if $G$ is a $\mathcal{C}$–colored graph then by Section 2.2 we have the $\mathbb{K}$-module $H(G^{\text{op}}, -\partial M)$ associated to the pair $(G^{\text{op}}, -\partial M)$, where the orientation of $\partial M$ is induced by that of $M$.

Similarly as we have done with closed 3-manifolds we want to define a topological invariant $|\cdot| : \mathcal{C} \rightarrow H(G^{\text{op}}, -\partial M)^*$. For this purpose we need to extend the notion of skeleton. By skeleton of the pair $(M, G)$ we mean a stratified 2-polyhedron $(P, E)$ in $M$ verifying

(i) $P \cap \partial M = \partial P = G$.

(ii) For all $v \in V(G)$ there is a unique edge $d_v \in E$ such that $v$ is an endpoint of $d_v$, moreover $d_v \cap \partial M = \{v\}$.

(iii) $E(G) \subset E$ and for all $a \in E(G)$, $\# P_a = 1$, where $P_a$ is the set of branches of $P$ at $a$. The only region $D_a$ adjacent to $a$ satisfies $D_a \cap \partial M = a$ and the orientation of $D_a$ is compatible with that of $a$.

(iv) $M \setminus P$ is the disjoint union of a finite collection of open 3-balls and a 3–manifold homeomorphic to $(\partial M \setminus G) \times [0, 1)$.

The primary moves on skeletons of closed manifolds extend to skeletons of pairs $(M, G)$ consisting of a compact oriented 3–manifold $M$ with boundary and an oriented graph $G \subset \partial M$ such that every vertex has valence $\geq 2$. It is not evident but true that every pair $(M, G)$ has a skeleton and also that any pair of skeletons are related by primary moves (see [TuV13] Lemma 8.1). Let us assume that $G$ is a $\mathcal{C}$-colored graph colored only by elements of a representative set $I$ of simple objects of $\mathcal{C}$ (we call this kind of graphs $I$-colored). We define $|\cdot| : \mathcal{C} \rightarrow H(G^{\text{op}}, -\partial M)^*$ as follows. Let $(P, E)$ be a skeleton of $(M, G)$ and $c : \text{Reg}(P) \rightarrow I$ be a coloring map extending the coloring.
of $G$, that is, for every edge $a \in E(G)$ the value $c(D_a)$ of the only region $D_a$ adjacent to $a$ is the $C$-color of $a$.

Step 1. To each oriented edge $e$ of $P$ we associate the $\mathbb{K}$-module $H_c(e) = H(P_e)$ where $P_e$ is the set of all branches of $P$ at $e$ seen as a $C$-cyclic set as in Section 2.3. Let $E_0$ be the set of edges of $P$ with both endpoints in $int(M)$ and let $E_\partial$ be the set of edges of $P$ with exactly one endpoint in $\partial M$ oriented towards this endpoint. By definition of a skeleton of $(M, G)$ each edge $e \in E_\partial$ is incident to a unique vertex $v \in V(G)$, we denote this unique edge by $e_v$. Thus $H_c(e_v) = H_v(G^{op}, -\partial M)$ and therefore

$\bigotimes_{e \in E_0} H_c(e)^* = \bigotimes_{v \in V(G)} H_v(G^{op}, -\partial M)^* = H(G^{op}, -\partial M)^*$.

Now, if we consider the edge $e \in E_0$ with opposite orientation, denoted by $e^{op}$, we have that $P_{e^{op}} = (P_e)^{op}$ so that we have a pairing $\omega_e : H_c(e) \otimes H_c(e^{op}) \to \mathbb{K}$ and thus a contraction $*: H_c(e)^* \otimes H_c(e^{op})^* \to \mathbb{K}$. We denote by $*P$ the tensor product of the maps $*_e$ over all the oriented edges $e \in E_0$ (the edges in $E_0$ considered with the two possible orientations). Thus we have

$*P : \bigotimes_{e \in E_0 \cup E_\partial} H_c(e)^* \to \bigotimes_{e \in E_0} H_c(e)^* = H(G^{op}, -\partial M)^*$.

Step 2. Let $x$ be a vertex in the interior of $P$ and $(\partial B_x, \Gamma_x)$ its link. Then $\Gamma_x$ becomes a $C$-colored graph as in Section 2.3. Thus we obtain a vector

$\bigotimes_x \mathbb{F}_C(\Gamma_x) \in \bigotimes_{e \in E_0 \cup E_\partial} H_c(e)^*$

where $x$ runs over all the vertices in the interior of $P$.

Step 3. Set

$|(M, G)_C = (\dim(C))^{-|P|} \sum_c \left( \prod_{r \in \text{Reg}(P)} \dim(c(r)) \chi(r) \right) *P \left( \bigotimes_x \mathbb{F}_C(\Gamma_x) \right)$, \hspace{1cm} (3.1)

where $|P|$ denotes the number of components of $M \setminus P$, $c$ runs over all maps $\text{Reg}(P) \to I$ extending the coloring of $G$, $\chi(r)$ is the Euler characteristic of $r$ and $x$ runs over all the vertices in the interior of $P$.

Theorem 3.1 ([TuV13] Theorem 9.1). The vector $|(M, G)_C \in H(G^{op}, -\partial M)$ is a topological invariant of the pair $(M, G)$, that is, $|(M, G)_C$ does not depend on the choice of the skeleton $P$.

Notice that if we take $G$ as the empty graph in equation (3.1) we obtain the topological invariant $|M|_C$ defined in Section 2.5.

Comments about the proof of Theorem 3.1. Since the primary moves are local transformations on the skeleton, then the invariance of $|(M, G)_C$ under the primary moves is verified in the same way that for Theorem 2.6 (see [TuV13, §7]).
4 Topological quantum field theory

In this chapter we introduce the central notion of topological quantum field theory. Section 4.1 deals with the basic definitions. In Section 4.2 we illustrate the definitions with some examples. Finally, Section 4.3 is the culmination of all the dissertation, here we use all the constructions made in the precedent chapters to show the construction of a 3–dimensional TQFT from a spherical fusion category. For the history and the physical interpretation of the subject we refer to [A90, Koh98].

4.1 Generalities

Let \( n \in \{1,2,3\} \). We define a strict monoidal category called the category of \( n \)-cobordisms, denoted by \( \text{Cob}_n \). The objects of \( \text{Cob}_n \) are closed oriented \((n-1)\)-manifolds. If \( \Sigma_0 \) and \( \Sigma_1 \) are objects of \( \text{Cob}_n \), a morphism \( \Sigma_0 \to \Sigma_1 \) in \( \text{Cob}_n \) is the class of a pair \((M,h)\), where \( M \) is a compact oriented \( n \)-manifold and \( h \) is an orientation-preserving homeomorphism \( h : (-\Sigma_0) \sqcup \Sigma_1 \to \partial M \), and two of such pairs \((M,h)\) and \((M',h')\) are equivalent if there is an orientation-preserving homeomorphism \( F : M \to M' \) such that \( h' = F \circ h \).

The identity morphism of an object \( \Sigma \) in \( \text{Cob}_n \) is represented by the cylinder \( \Sigma \times [0,1] \) with the product orientation and the tautological homeomorphism \( h : (-\Sigma) \sqcup \Sigma \to \partial(\Sigma \times [0,1]) \).

The composition of morphisms in \( \text{Cob}_n \) is defined through gluing of cobordisms, more precisely, if \((M_0,h_0) : \Sigma_0 \to \Sigma_1\) and \((M_1,h_1) : \Sigma_1 \to \Sigma_2\) are representatives of morphisms in \( \text{Cob}_n \), then their composition is represented by \((M,h)\), where \( M \) is the resulting of gluing \( M_0 \) to \( M_1 \) along \( h_1 \circ h_0^{-1} : h_0(\Sigma_1) \to h_1(\Sigma_1) \) and \( h = (h_0)|_{\Sigma_0 \sqcup (h_1)|_{\Sigma_2} : (-\Sigma_0) \sqcup \Sigma_1 \to \partial M \). We denote the composition of \((M_0,h_0)\) and \((M_1,h_1)\) by \( M = M_0 \sqcup_{\Sigma_1} M_1 \).
If $\Sigma_0$ and $\Sigma_1$ are objects in $\text{Cob}_n$ and $M$ is a compact $n$-manifold such that $\partial M = (-\Sigma_0) \sqcup \Sigma_1$, then we say that the triple $(M, \Sigma_0, \Sigma_1)$ is an $n$-cobordism.

**Remark.** The above definition of the category $\text{Cob}_n$ for $n \in \{1, 2, 3\}$ can be generalized for all positive integers in the same way, but for $n \geq 4$ we require that the manifolds are smooth and for the composition of cobordisms we demand a condition of compatibility of the smooth structures for doing the gluing. We will only work in the case $n \in \{1, 2, 3\}$, principally $n = 3$, where there is no problem with these technicalities.

We consider the empty set $\emptyset$ as a manifold of dimension $n$ for all $n \in \mathbb{N}$ and with a unique orientation. The category $\text{Cob}_n$ is endowed with a symmetric monoidal structure, the tensor product is given by the disjoint union and the unit object is the empty manifold $\emptyset$.

Let $\mathbb{K}$ be a commutative ring (usually a field, and more often $\mathbb{C}$). Denote by $\text{vect}_\mathbb{K}$ the symmetric monoidal category of finitely generated projective $\mathbb{K}$-modules.

**Definition 4.1** ([A88]). An $n$-dimensional TQFT (topological quantum field theory or shorter $n$–TQFT) is a symmetric monoidal functor $Z : \text{Cob}_n \to \text{vect}_\mathbb{K}$.

Let us unwind the above definition. The monoidality of $Z$ says:

(i) If we consider the empty manifold $(n - 1)$-manifold, then $Z(\emptyset) = \mathbb{K}$.

(ii) For $\Sigma_0$ and $\Sigma_1$ objects in $\text{Cob}_n$, $Z(\Sigma_0 \sqcup \Sigma_1) = Z(\Sigma_0) \otimes Z(\Sigma_1)$ and the similar condition for cobordisms.

(iii) For $(M_0, h_0) : \Sigma_0 \to \Sigma_1$ and $(M_1, h_1) : \Sigma_1 \to \Sigma_2$ morphisms in $\text{Cob}_n$, $Z(M_0 \sqcup M_1) = Z(M_1) \circ Z(M_0)$.

Some important consequences of the Definition 4.1 are the following:

(a) Let $\Sigma$ and $\Sigma'$ be objects in $\text{Cob}_n$ and let $f : \Sigma \to \Sigma'$ be an orientation-preserving homeomorphism. The homeomorphism $f$ determines a morphism $\Sigma \to \Sigma'$ in $\text{Cob}_n$ represented by $(C_f = \Sigma' \times [0, 1], h : (-\Sigma) \sqcup \Sigma' \to \partial C_f)$, where $h(x) = (f(x), 0)$ for $x \in \Sigma$ and $h(x') = (x', 1)$ for $x' \in \Sigma'$. This morphism is called cylinder cobordism and it only depends on the isotopy class of $f$. In particular if $\Sigma = \Sigma'$, then a $n$-dimensional TQFT determines an action by automorphisms of the mapping class group $\mathcal{M}(\Sigma)$ on $Z(\Sigma)$. 

\[ (M, h) \]

\[ \Sigma_2 \]

\[ (M_1, h_1) \]

\[ \Sigma_1 \]

\[ (M_0, h_0) \]

\[ \Sigma_0 \]
(b) If \( M \) is a closed oriented \( n \)-manifold, then we can consider it as a morphism \( \emptyset \to \emptyset \) in \( \text{Cob}_n \), therefore, by the functoriality of \( Z \) we get a morphism \( Z(M) : \mathbb{K} \to \mathbb{K} \) which is a multiplication by an element of \( \mathbb{K} \). This scalar is a topological invariant of \( M \). Thus, a \( n \)-dimensional TQFT gives topological invariants for closed oriented \( n \)-manifolds.

The topological invariant \( Z(M) \) for a closed oriented \( n \)-manifold \( M \) can be determined from a decomposition of \( M \) by using the functoriality. We illustrate informally this statement with one example.

**Example 4.2.** Let \( M \) be a closed oriented surface of genus 2, seen as a morphism \( \emptyset \to \emptyset \) in \( \text{Cob}_2 \). Take an oriented circle \( C \) in \( M \) and cut \( M \) along \( C \) for obtaining two compact oriented surfaces \( M_0 \) and \( M_1 \). In this way we decompose the morphism \( \emptyset \xrightarrow{M} \emptyset \) into the composition of the two morphisms \( \emptyset \xrightarrow{M_0} C \) and \( C \xrightarrow{M_1} \emptyset \).

4.2 1–dimensional and 2–dimensional TQFT’s

In this section we sketch a classification of 1–dimensional and 2–dimensional TQFT’s, for details we refer to [Ko03]. Let \( \mathbb{K} \) be a field.

1–Dimensional case

A compact 0–dimensional manifold is simply a finite set of points. We have two connected oriented 0–dimensional manifolds given by a point together with an orientation positive or negative. Denote these two manifolds by \( +\cdot \) and \( -\cdot \). Thus a 1–dimensional topological quantum field theory \( Z : \text{Cob}_1 \to \text{vect}_\mathbb{K} \) is completely determined, on the objects, by the image of \( +\cdot \) and \( -\cdot \). Let \( V = Z(+\cdot) \) and \( W = Z(-\cdot) \). There are two particular morphisms \( M \) and \( N \) in \( \text{Cob}_1 \):

\[
\begin{align*}
\emptyset &\xrightarrow{M} \mathbb{K} \\
(+\cdot)\sqcup(-\cdot) &\xrightarrow{(\cdot)\sqcup(+\cdot)} \\
\emptyset &\xrightarrow{N} \mathbb{K}
\end{align*}
\]

which determines two morphisms \( Z(M) : V \otimes W \to \mathbb{K} \) and \( Z(N) : \mathbb{K} \to W \otimes V \) in \( \text{vect}_\mathbb{K} \), respectively. The two following equalities in \( \text{Cob}_1 \)
show that $V$ and $W$ are in duality, that is $V^* \cong W$. The maps $Z(M)$ and $Z(N)$ are called evaluation and coevaluation, respectively. To sum up, a 1-dimensional TQFT is determined by a finite dimensional vector space. Thus we have the following correspondence

$$
\left\{ \text{1–dimensional TQFT’s} \right\} \leftrightarrow \left\{ \text{finite dimensional vector spaces} \right\}.
$$

2–Dimensional case

The 2–dimensional case is very similar to the 1–dimensional case, because a closed 1–manifold is simply a disjoint union of finite circles. We have two compact connected oriented 1–manifolds given by a circle together with an orientation positive or negative. Denote these two manifolds by $+\circ$ and $-\circ$. Thus a 2–dimensional topological quantum field theory $Z : \text{Cob}_2 \to \text{vect}_K$ is completely determined, on the objects, by the image of $+\circ$ and $-\circ$. Let $A = Z(+\circ)$ and $B = Z(-\circ)$. Here, as in the 1–dimensional case, there are two particular morphisms $M$ and $N$ in $\text{Cob}_2$:

which determines two morphisms $Z(M) : A \otimes B \to K$ and $Z(N) : K \to B \otimes A$ in $\text{vect}_K$, respectively. The two following equalities in $\text{Cob}_2$

show that $A$ and $B$ are in duality, that is, $A^* \cong B$. The maps $Z(M)$ and $Z(N)$ are called evaluation and coevaluation, respectively. But unlike to the 1–dimensional case, in $\text{Cob}_2$ there are more morphisms, see the figure below.
which induce morphisms \( m : A \otimes A \rightarrow A \) (multiplication), \( \Delta : A \rightarrow A \otimes A \) (comultiplication), \( A \rightarrow \mathbb{K} \) and \( \mathbb{K} \rightarrow A \) and these morphisms satisfy certain relations coming from equalities in \( \text{Cob}_2 \). For example the following equality in \( \text{Cob}_2 \)

![Diagram](image)

says that the multiplication map \( m : A \otimes A \rightarrow A \) is associative. There are other relation satisfied by these maps. Thus we see that the vector space \( A \) has more algebraic structure, this algebraic structure is called a Frobenius algebra. To sum up, a 2-dimensional TQFT is determined by a finite dimensional vector space with certain algebraic structure (Frobenius algebra). Thus if we have a 2–TQFT, we have a commutative Frobenius algebra. The reciprocal is also true (see [Ko03]). Thus we have the following correspondence:

\[
\begin{align*}
\{ \text{2–dimensional TQFT}'s \} & \longleftrightarrow \{ \text{finite dimensional commutative Frobenius algebras} \}.
\end{align*}
\]

### 4.3 3–dimensional TQFT’s

As we have seen in the previous section, 1–dimensional TQFT’s and 2–dimensional TQFT’s are relatively simple and this is the case because there is a complete classification of closed 1-manifolds and 2-manifolds. We known that a \( n \)-dimensional TQFT gives topological invariants for closed \( n \)-manifolds, from this point of view 1-TQFT’s and 2-TQFT’s are not interesting. In contrast 3–TQFT’s are very interesting because they give topological invariants of 3–manifolds. Thus 3–TQFT’s allow us to classify certain families of 3–manifolds (for example by their complexity see [Mat]). However, defining a 3–TQFT is not easy and moreover after the definition the calculation is often very difficult because in general a 3–TQFT is defined indirectly by using a presentation of the closed 3–manifold. (For example a triangulation, a skeleton, a Heegaard splitting or by seeing the 3–manifold as a result of a surgery on a knot (or link) in the 3–sphere.)

In this section we show an example of a 3–TQFT coming from a spherical fusion category, which calculates the topological invariant for 3–manifolds defined in Chapter 2. Let \( \mathcal{C} \) be a spherical fusion category over a commutative ring \( \mathbb{K} \) such that \( \text{dim}(\mathcal{C}) \in \mathbb{K}^\times \) and let \( I \) be a representative set of simple objects of \( \mathcal{C} \). By a skeleton of a closed surface \( \Sigma \) we mean an oriented graph \( G \subset \Sigma \) such that all vertices of \( G \) have valence \( \geq 2 \) and all components of \( \Sigma \setminus G \) are open disks. The construction is as follows.

**Step 1.** Let \((M, \Sigma_0, \Sigma_1)\) be a 3-cobordism and \( G_i \subset \Sigma_i \) be \( I \)-colored graphs \((i=1,2)\). By formula (3.1) we get a vector
\[(M, G_0^\text{op} \cup G_1)|_c \in H(G_0 \cup G_1^\text{op}, -\partial M)^* = H(G_0, \Sigma_0)^* \otimes H(G_1^\text{op}, -\Sigma_1)^*.\]

Recall that we have an isomorphism \(H(G_1^\text{op}, -\Sigma_1)^* \cong H(G_1, \Sigma_1)\) (see Section 2.2). Let \(\gamma\) be map

\[
\gamma: H(G_0, \Sigma_0)^* \otimes H(G_1^\text{op}, -\Sigma_1)^* \longrightarrow \text{Hom}(H(G_0, \Sigma_0), H(G_1, \Sigma_1))
\]
defined by \(\gamma(f \otimes g)(\cdot) = f(\cdot)\phi(g)\) for all \(f \in H(G_0, \Sigma_0)^*\) and \(g \in H(G_1^\text{op}, -\Sigma_1)^*\). Set

\[
|M, \Sigma_0, G_0, \Sigma_1, G_1|_c = \frac{\text{dim}(C)^{|G_1|}}{\text{dim}(G_1)} \gamma(|M, G_0^\text{op} \cup G_1|_c): H(G_0, \Sigma_0) \longrightarrow H(G_1, \Sigma_1) \quad (4.1)
\]

where for a \(I\)-colored graph \(G\) in a surface \(\Sigma\), the symbol \(|G|\) denotes the number of components of \(\Sigma \setminus G\) and \(\text{dim}(G)\) denotes the product of dimensions of the objects associated to the edges of \(G\).

We can give an explicit definition of the map (4.1) as follows. Let \(\Omega \in H(G_1, \Sigma_1) \otimes H(G_1^\text{op}, -\Sigma_1)\) be the inverse of the canonical pairing \(H(G_1, \Sigma_1) \otimes H(G_1^\text{op}, -\Sigma_1) \rightarrow \mathbb{K}\). Let us express \(\Omega = \sum a_\alpha \otimes b_\alpha\) with \(a_\alpha \in H(G_1, \Sigma_1)\) and \(b_\alpha \in H(G_1^\text{op}, -\Sigma_1)\). Thus for any \(h \in H(G_0, \Sigma_0)\)

\[
|M, \Sigma_0, G_0, \Sigma_1, G_1|_c(h) = \frac{\text{dim}(C)^{|G_1|}}{\text{dim}(G_1)} \sum_{\alpha} (|M, G_0^\text{op} \cup G_1|_c)(h \otimes b_\alpha)a_\alpha. \quad (4.2)
\]

Now, let \((M_0, \Sigma_0, \Sigma_1)\) and \((M_1, \Sigma_1, \Sigma_2)\) be 3-cobordisms and \((M, \Sigma_0, \Sigma_1)\) be the 3-cobordism obtained by gluing \(M_0\) and \(M_1\) along \(\Sigma_1\). Equation (4.2) and the fact that the union of a skeleton of \((M_0, G_0^\text{op} \cup G)\) with a skeleton of \((M_1, G_1^\text{op} \cup G_2)\) is a skeleton of \((M, G_0^\text{op} \cup G_2)\) allow us to show that for any \(I\)-colored graphs \(G_0 \subset \Sigma_0, G_1 \subset \Sigma_1\) and any skeleton \(G \subset \Sigma_1\) we have

\[
|M, \Sigma_0, G_0, \Sigma_2, G_2|_c = \sum_{c} |M_1, \Sigma_1, (G, c), \Sigma_2, G_2|_c \circ |M_0, \Sigma_0, G_0, \Sigma_1, (G, c)|_c \quad (4.3)
\]

where \(c\) runs over the set of all maps from the edges of \(G\) to \(I\). We denote this set by \(\text{col}(G)\).

**Step 2.** For each pair \((G, \Sigma)\) where \(\Sigma\) is an object of \(\text{Cob}_3\) and \(G\) a skeleton of \(\Sigma\), we associate a \(\mathbb{K}\)-module by setting

\[
|G, \Sigma|^{\circ}_c = \bigoplus_{c \in \text{col}(G)} H((G, c), \Sigma) \quad (4.4)
\]

For a 3-cobordism \((M, \Sigma_0, \Sigma_1)\) and any skeletons \(G_i \subset \Sigma_i\) \((i = 0, 1)\), we define a homomorphism \(|M, \Sigma_0, G_0, \Sigma_1, G_1|^{\circ}_c : |G_0, \Sigma_0|^{\circ}_c \longrightarrow |G_1, \Sigma_1|^{\circ}_c\) by
where \(|M, \Sigma_0, (G_0, c_0), \Sigma_1, (G_1, c_1)|_C = \sum_{c_0, c_1} |M, \Sigma_0, (G_0, c_0), \Sigma_1, (G_1, c_1)|_C \) (4.5)

By formula (4.3) we have that

\(|M, \Sigma_0, G_0, \Sigma_2, G_1|_C^o = |M_1, \Sigma_1, G_1, \Sigma_2, G_1|_C^0 \circ |M_0, \Sigma_0, G_0, \Sigma_1, G_1|_C^0 \) (4.6)

where \(M\) is the 3–cobordism resulting from the gluing of two cobordisms \((M_0, \Sigma_0, \Sigma_1)\) and \((M_1, \Sigma_1, \Sigma_2)\) and any skeletons \(G_0 \subset \Sigma_0, G_1 \subset \Sigma_1\) and \(G_2 \subset \Sigma_2\).

Until now we have associated to a pair \((G, \Sigma)\) \((G\) a skeleton of \(\Sigma\), object of \(\text{Cob}_3\) a \(\mathbb{K}\)–module \(|G, \Sigma|_C^o\) and to a 3–cobordism \((M, \Sigma_0, \Sigma_1)\) together with skeletons \(G_i \subset \Sigma_i\) \((i = 0, 1)\) a homomorphism \(|M, \Sigma_0, G_0, \Sigma_1, G_1|_C^0 : |G_0, \Sigma_0|_C^0 \rightarrow |G_0, \Sigma_1|_C^0\). Now we want to get rid of the skeletons of these constructions.

**Step 3.** Let \(G_0, G_1 \subset \Sigma\) be two skeletons. By step 2 the cylinder cobordism \(M = \Sigma \times [0,1]\) induces a homomorphism

\[\rho(G_0, G_1) = |M, \Sigma \times \{0\}, G_0 \times \{0\}, \Sigma \times \{1\}, G_1 \times \{1\}|_C^\circ : |G_0, \Sigma|_C^0 \rightarrow |G_1, \Sigma|_C^0.\] (4.7)

Formula (4.6) implies that \(\rho(G_0, G_1) = \rho(G_1, G_2)\circ \rho(G_0, G_1)\) for any skeletons \(G_0, G_1, G_2\) of \(\Sigma\). In particular, \(\rho(G_0, G_0)^2 = \rho(G_0, G_0)\). Hence \(\rho(G_0, G_0)(|G_0, \Sigma|_C^0) =: |G_0, \Sigma|_C\) is a direct summand of \(|G_0, \Sigma|_C^0\). Moreover \(\rho(G_0, G_1)(|G_0, \Sigma|_C)\) is isomorphic to \(|G_1, \Sigma|_C\).

The family of \(\mathbb{K}\)–modules \(|G, \Sigma|_C\) together with the family \(\{\rho(G_0, G_1)\}_{G_0,G_1}\), where \(G, G_0\) and \(G_1\) runs over all the skeletons of \(\Sigma\), form a projective system. Set

\[|\Sigma|_C = \lim_{\longrightarrow} |G, \Sigma|_C\] (4.8)

By construction, this \(\mathbb{K}\)–module is independent of the skeletons. For any skeleton \(G \subset \Sigma\) we have an isomorphism of \(\mathbb{K}\)–modules between \(|G, \Sigma|_C\) and \(|\Sigma|_C\) given by the cone isomorphism (in the definition of projective limit).

**Step 4.** To each 3–cobordism \((M, \Sigma_0, \Sigma_1)\) we assign the homomorphism

\[|\Sigma_0|_C \cong (1) |G_0, \Sigma_0|_C \subset |G_0, \Sigma_0|_C^0 \overset{(2)}{\rightarrow} |G_1, \Sigma_1|_C^0 \overset{(3)}{\rightarrow} |G_1, \Sigma_1|_C \cong |\Sigma_1|_C\]

where (1) and (4) are the cone isomorphisms, (2) is the morphism given by formula (4.5) and (3) is given by formula (4.7). We denote this morphism by \(|M, \Sigma_0, \Sigma_1|\).

**Step 5.** If \(f : \Sigma \rightarrow \Sigma'\) is an orientation-preserving homeomorphism between two closed surfaces \(\Sigma\) and \(\Sigma'\), then \(f\) induces an isomorphism \(|f|_C : |\Sigma|_C \rightarrow |\Sigma'|_C\) defined by the composition

\[|\Sigma|_C \cong (1) |G, \Sigma|_C \cong (2) |G', \Sigma'|_C \cong (3) |\Sigma'|_C\]

where \(G\) is any skeleton of \(\Sigma\), \(G' = f(G)\). (1) and (3) are the cone isomorphisms and (2) is induced by the cylinder cobordism determined by \(f\).
**Step 6.** Let $\varphi : \Sigma_0 \to \Sigma_1$ be a morphism in $\mathbf{Cob}_3$ represented by the pair $(M, h : (\Sigma_0) \cup \Sigma_1 \to \partial M)$. Let $\Sigma'_i = h(\Sigma_i) \subset \partial M$ for $i = 0, 1$. By step 4 the 3–cobordism $(M, \Sigma'_0, \Sigma'_1)$ yields a homomorphism $|M, \Sigma'_0, \Sigma'_1| : |\Sigma'_0|_C \to |\Sigma'_1|_C$. By step 5 the homeomorphisms $h : \Sigma_i \to \Sigma'_i$ induce two isomorphisms $|\Sigma_i|_C \cong |\Sigma'_i|_C$ for $i = 1, 2$. Thus, by composing these three homomorphisms we get a homomorphism $|\varphi|_C : |\Sigma_0|_C \to |\Sigma_1|_C$.

**Definition 4.3** (Turaev–Viro 3–TQFT). Let $| \cdot |_C : \mathbf{Cob}_3 \to \mathbf{vect}_K$ be the 3–dimensional TQFT defined by

\[
\Sigma \mapsto |\Sigma|_C \\
\varphi \mapsto |\varphi|_C,
\]

where $\Sigma$ is any object of $\mathbf{Cob}_3$ and $\varphi$ is any morphism in $\mathbf{Cob}_3$. This TQFT is called the Turaev–Viro TQFT.

The Turaev–Viro TQFT yields the Turaev–Viro topological invariant defined in Chapter 2.

**Remerciements**

Pour commencer j’adresse mes remerciements à mon directeur de mémoire, Monsieur Gwénaël Massuyeau, pour me guider et partager avec moi ses connaissances et intuitions et aussi pour sa patience infinie avec mon mauvais français.

Mes sincères remerciements à Madame Claude Mitschi et Monsieur Christian Kassel pour leur conseils et soutien.

Je remercie aux professeurs de M2, notamment à Monsieur Vladimir Fock et Monsieur Massuyeau pour m’apprendre à voir les mathématiques sous une perspective vaste.

Je remercie également au Labex IRMIA, grâce aux bourses de Master j’ai eu l’occasion de poursuivre mes études.

Je remercie aussi à mes camarades de M2, spécialement à Arthur et Morad pour nos instructives conversations philosophiques et mathématiques.
Bibliography


