KNOTS

HEEGAARD FLOER HOMOLOGY

CONTACT GEOMETRY

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(joint work with ANDRÁS STIPSICZ)

3-MANIFOLDS

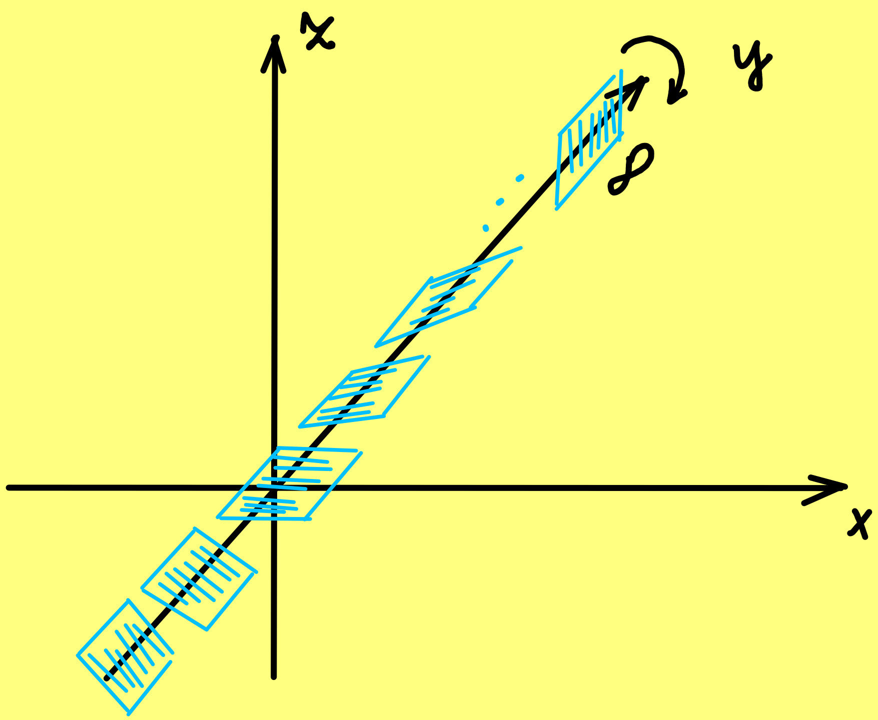
CONTACT STRUCTURES

a totally non-integrable planefield on a 3-manifold

$\iff \xi = \ker \alpha \quad \& \quad \alpha \wedge d\alpha > 0$

standard contact structure on $\mathbb{R}^3$: $\xi_{st} = \ker(dz - y\,dx)$

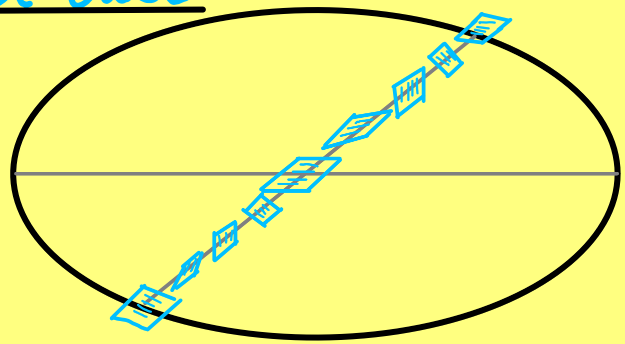
$$\overset{\shortparallel}{\left\langle \frac{\partial}{\partial y},\ y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle}$$



Thm (Darboux) Every contact structure is locally isotopic to ξ_{st}.

Some Contact Structures

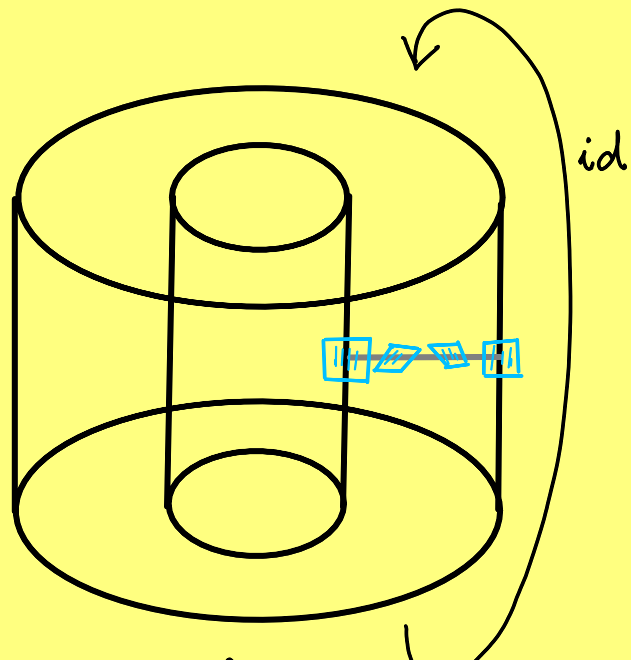
overtwisted disc



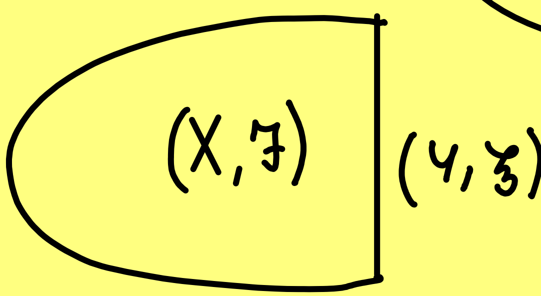
$$\xi_{OT} = \ker\left(\cos r\, dx + r \sin r\, d\vartheta\right)$$

Giroux torsion on $T^2 \times [0,1]$

$$\xi_{GT} = \ker\left(\cos(2\pi t)\, dx - \sin(2\pi t)\, dy\right)$$



Def ξ is **Stein fillable**



$$Y \xrightarrow[\iota]{\text{hol}} \mathbb{C}^N$$

$$\iota^{-1}(S^{N-1}) = Y$$

$$\xi = TY \cap J(TY)$$

Stein fillable $\Rightarrow$ not OT (**tight**)

OT $\Rightarrow$ contains Giroux torsion

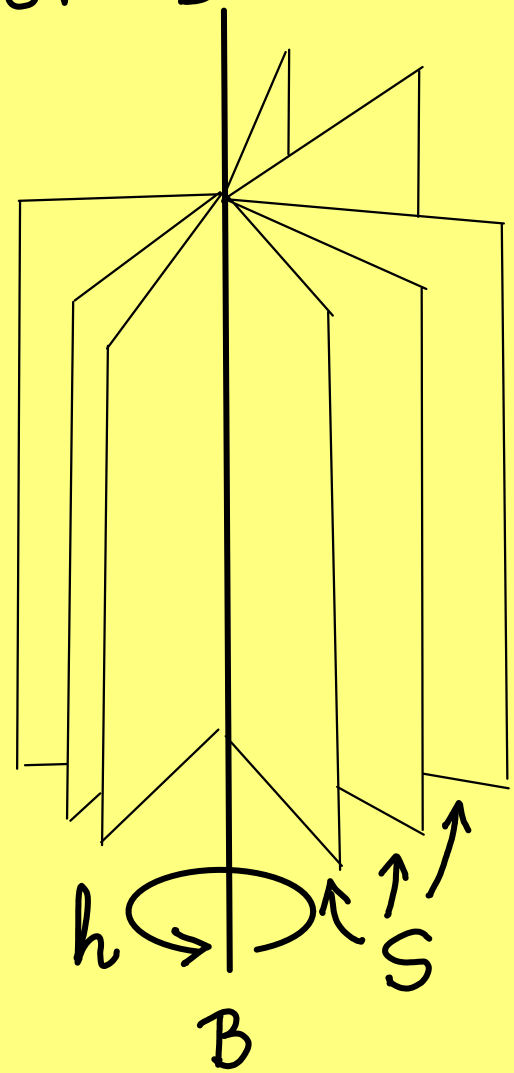
OPEN BOOKS

a link in Y = the binding of the OB

Def $\quad Y - B$

$\qquad \downarrow S$ ← the page of the OB $\quad \partial F = B$

$\qquad S^1$



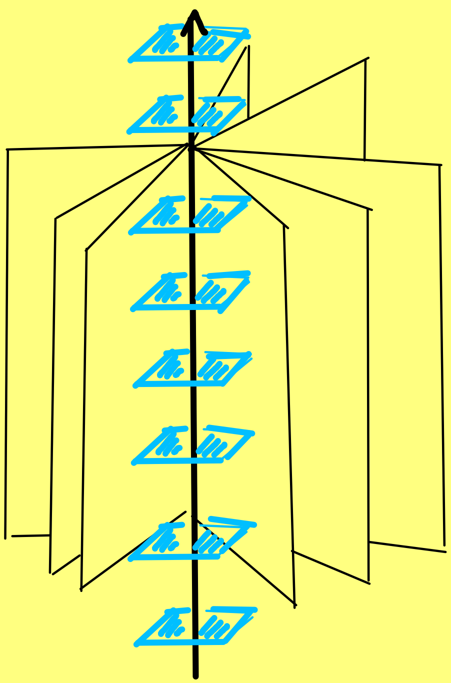
in the neighborhood of the binding:

going around the base S^1 once gives an identification of S with itself: $S \circlearrowleft h$
a diffeomorphism

Thm (Alexander) Every 3-manifold admits an OB decomposition

Open Books & Contact Structures

Def an OB decomposition is compatible with ξ if:



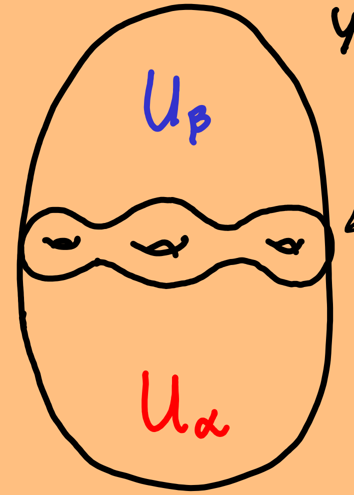
- $\xi \pitchfork B$ $(\Leftrightarrow \alpha > 0 \text{ on } B)$
- $d\alpha$ is a volume form on S $(\Leftrightarrow d\alpha > 0 \text{ on } S)$

Thm (Thurston-Winkelnkemper, Giroux)

- every ctct str admits a compatible OB decomposition
- every OB decomposition admits a compatible ctct structure
- contact structures of Y / isotopy $\Leftrightarrow$ OB decompositions of Y / stabilisation

Heegaard Decompositions

Fact every 3-manifold decomposes as the union of two handlebodies: *Heegaard decomposition* of Y



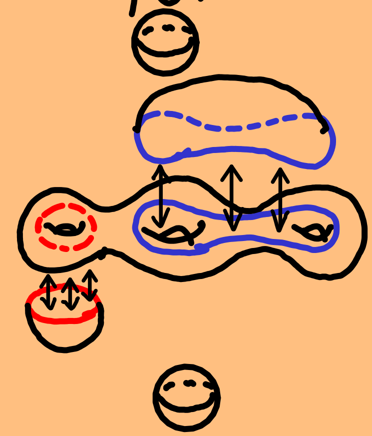
Heegaard surface Σ

a HD can be described by two g-tuples of curves: $\underline{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_g\}$

$\underline{\beta} = \{\beta_1, \beta_2, \dots, \beta_g\}$

s.t.:
- $\alpha_1, \dots, \alpha_g$ bound discs in U_α
- $\Sigma - \cup \underline{\alpha}$ is connected
- $\beta_1, \dots, \beta_g$ bound discs in U_β
- $\Sigma - \cup \underline{\beta}$ is connected

$(\Sigma, \underline{\alpha}, \underline{\beta})$ is a *Heegaard diagram* for Y

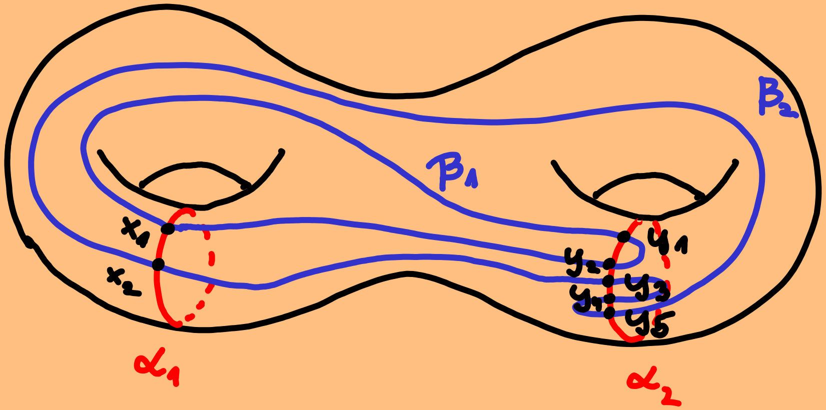


Note Y can be rebuilt from $(\Sigma, \underline{\alpha}, \underline{\beta})$: by attaching 1-handles along the α- & 2-handles along the β-curves

HEEGAARD FLOER HOMOLOGIES

For a (Σ, α, β) HD generators of $\widehat{CF}(\Sigma, \alpha, \beta)$ are g-tuples of intersections of $\cup\alpha \cap \cup\beta$ such that there is exactly one point on each α & β - curve

e.g.



$\cap$	β_1	β_2
α_1	x_1	x_2
α_2	y_1, y_2	y_3, y_4, y_5

the generators are: $x_1y_3,\ x_1y_4,\ x_1y_5,\ x_2y_1,\ x_2y_2$

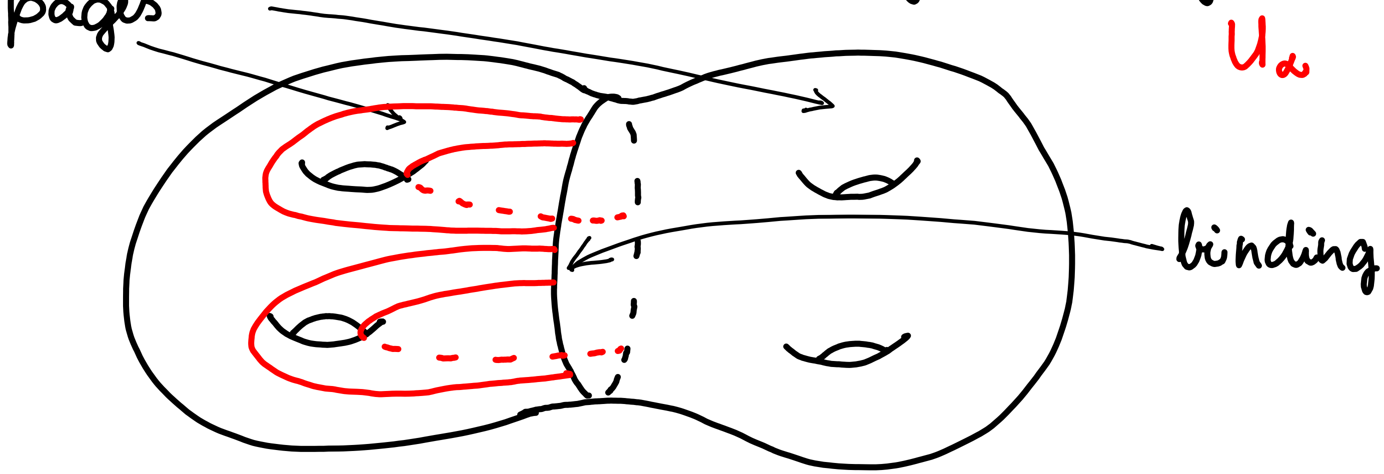
boundary map $\partial x = \sum_y m(x,y)\, y$, where $m(x,y)$ is a count of holomorphic discs missing a point

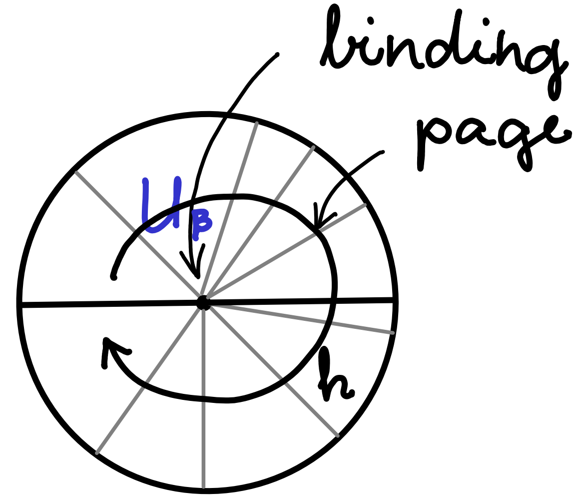
Thm (Ozsváth-Szabó) the homology $\widehat{HF}(Y)$ is a 3-manifold invariant.

Heegaard Diagrams & Open Books

an OB decomposition defines a HD

the union of two pages →

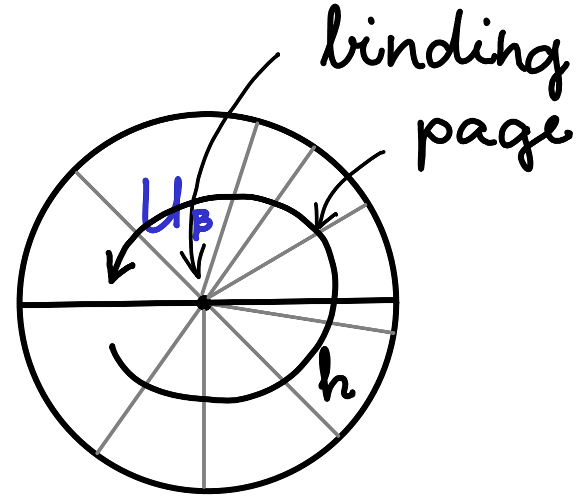




choose arcs that cut up the page into a disc
rotating them they sweep out discs both in the lower and upper halfs

Heegaard Diagrams & Open Books

an OB decomposition defines a HD



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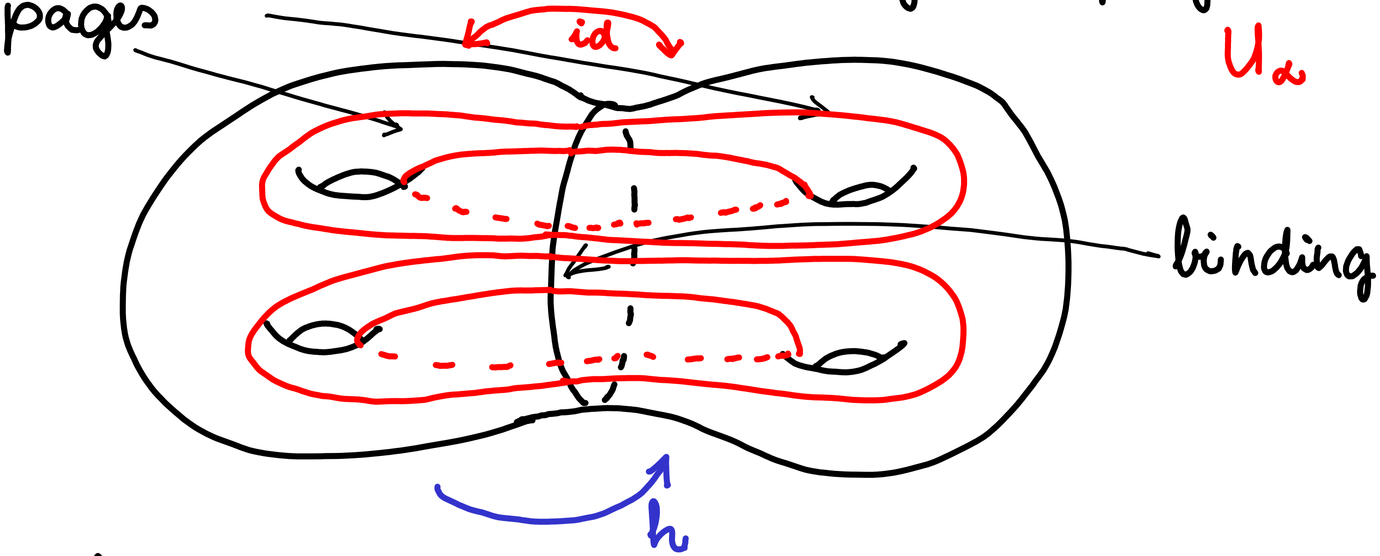

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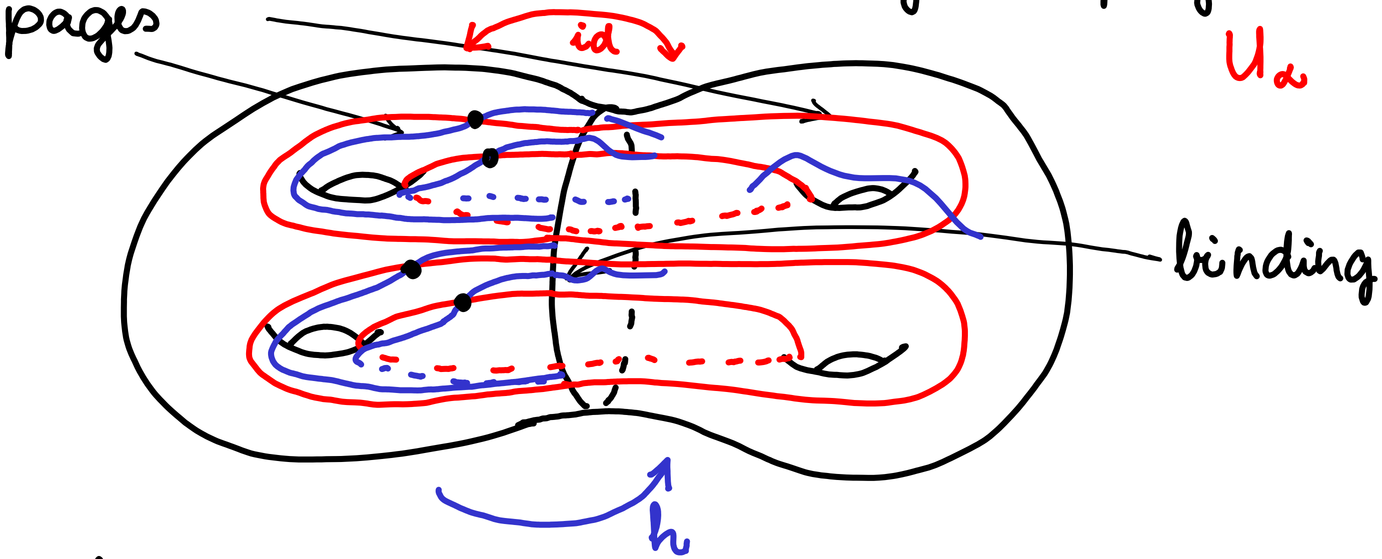

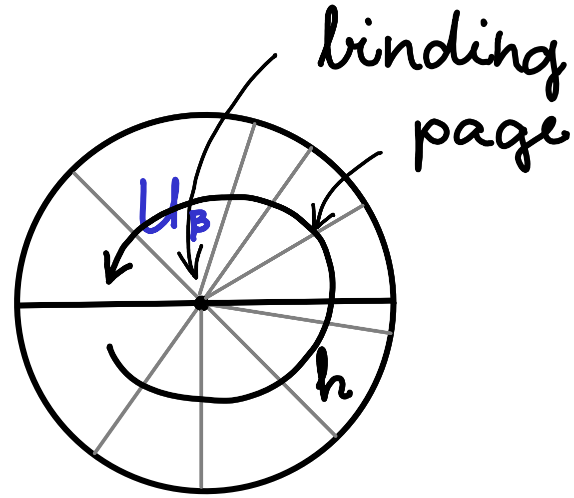

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rotating them they sweep out discs both in the lower and upper halfs

<u>Notice</u>: There is exactly one generator of $\widehat{CF}(-Y)$ on the LHS.

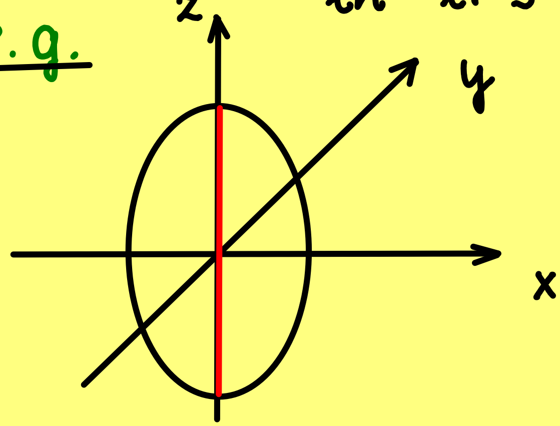
<u>Thm</u> (Ozsváth - Szabó) its homology class is an invariant of the contact structure: $c(\xi) \in \widehat{HF}(Y)$ <u>contact invariant</u>

3-MANIFOLDS WITH BOUNDARY

Convex Surfaces

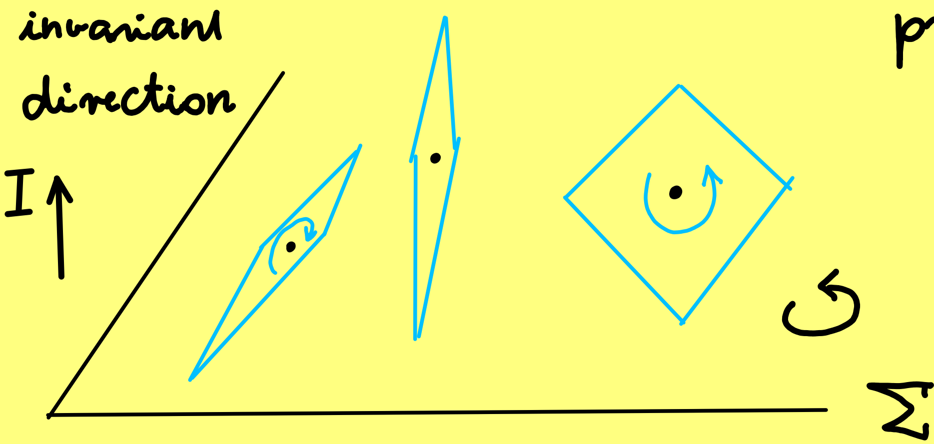
$\Sigma \hookrightarrow (Y,\xi)$ is convex if the contact structure is I-invariant in it's neighborhood

e.g.



$D = \{y^2+z^2 \leq 1\} \times \{x_0\}$

is invariant in the $\xrightarrow{\frac{\partial}{\partial x}}$ direction



project ξ_x to $T\Sigma$

- at some pts is not onto
  $\parallel$
  1-dimensional submanifold of Σ
  Γ_z = dividing curve

- if it is onto
  - → the orientation of ξ_x and of Σ agree Σ_+
  - → the orientation of ξ_x and of Σ disagree Σ_-

$\rightsquigarrow \Sigma - \Gamma_z = \Sigma_+ \cup^* \Sigma_-$

Contact Manifolds with Boundary

Thm (Giroux)
- convex surfaces are generic
- The isotopy class of the dividing curve Γ_Σ only depends on the isotopy class of the contact structure near Σ
- Γ_Σ determines the isotopy class of contact structure near Σ.

Thus it is easy to glue contact structures along convex surfaces

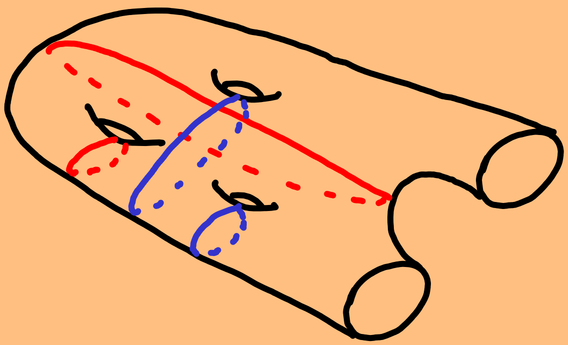
$\Rightarrow$ We require the boundary of a contact 3-manifold to be convex : (Y, ξ, Γ)

- Partial open books are generalisations of OBs for (Y, ξ, Γ)
- a weaker version of the Giroux correspondence can be generalised for these settings

Sutured Floer Homology

a generalization of $\widehat{HF}$ for 3-manifolds with boundary:

Remember: could rebuild a closed 3-manifold from (Σ, α, β)



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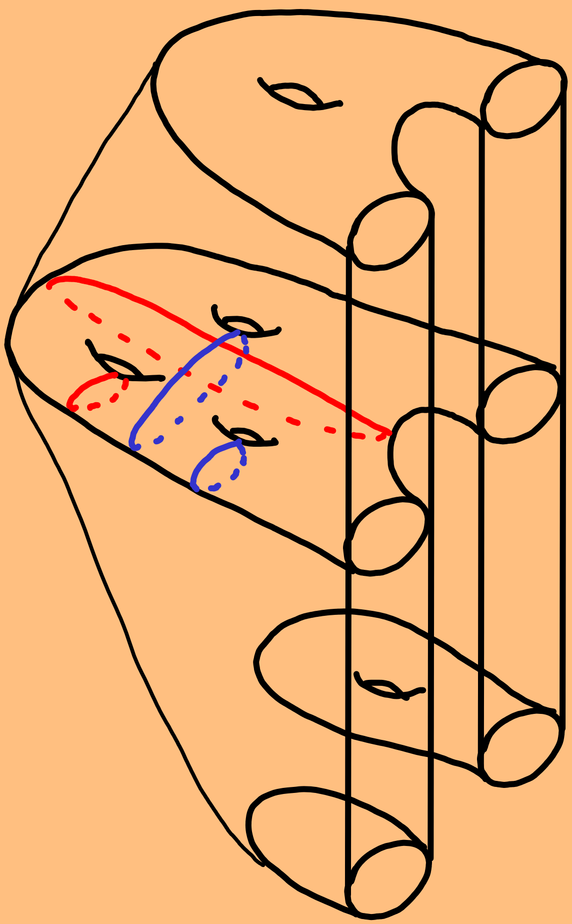

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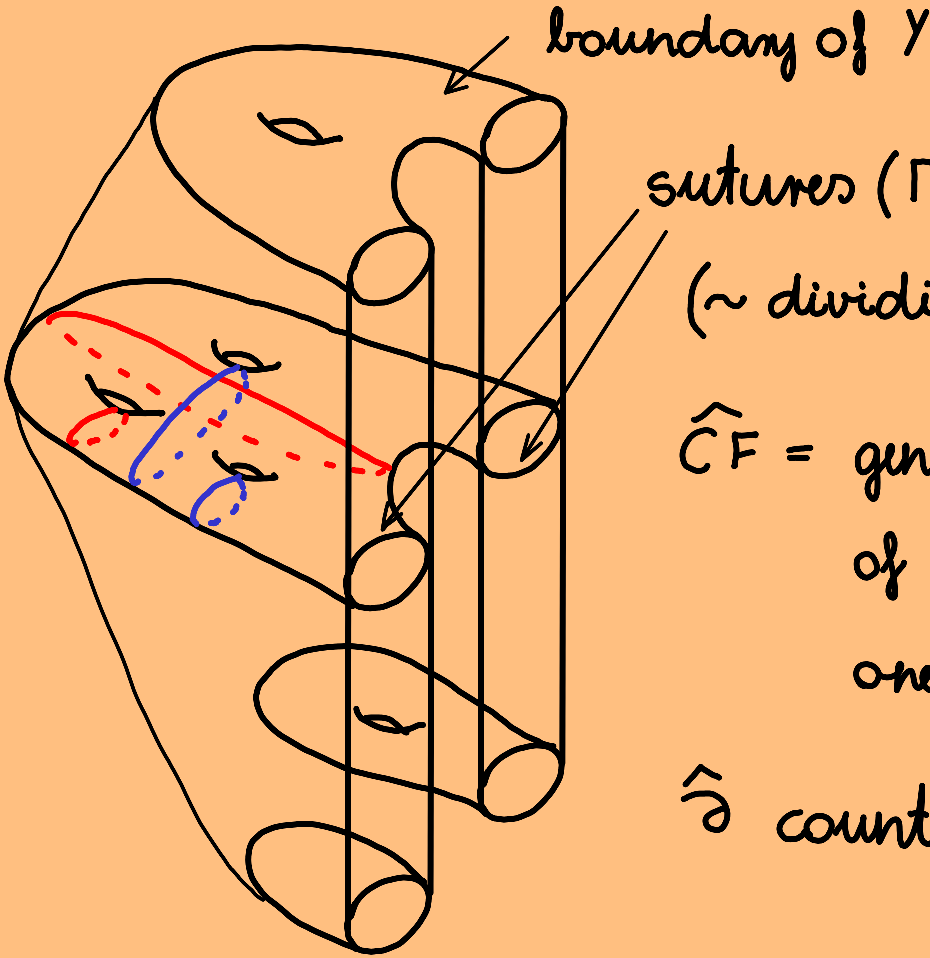


sutures (Γ) marking on ∂Y

($\sim$ dividing curves of ∂Y)

$\widehat{CF}$ = generated by g-tuples of intersections of $\cup \alpha \cap \cup \beta$ such that there is exactly one point on each α_i & β_j

$\hat{\partial}$ counts holomorphic discs missing Γ

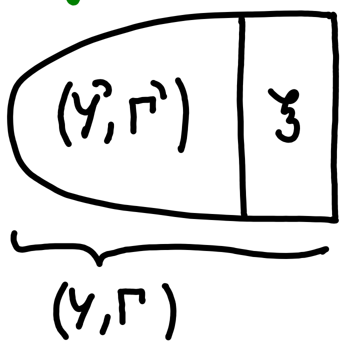
Thm (Juhász) Its homology is an invariant of (Y, Γ): $\widehat{HF}(Y, \Gamma)$

Thm (Honda-Kazez-Matić) $\exists$ invariant $c(\xi) \in \widehat{HF}(-Y, \Gamma)$

Properties of the Contact Invariant

Thm (Ozsváth - Szabó) ξ Stein-fillable $\Rightarrow c(\xi) \neq 0$

Gluing Thm (Honda - Kazez - Matić)



$\exists\ \varphi_\xi : \widehat{HF}(-Y',\Gamma') \longrightarrow \widehat{HF}(-Y,\Gamma)$

moreover if (Y',Γ') comes from (Y',ξ') then

$$\varphi_\xi : c(\xi') \longmapsto c(\xi' \cup \xi)$$

$c(\text{OT-disc}) = 0$

$c(\text{Giroux torsion}) = 0$

$\Rightarrow$ **Thm** (OSz) ξ OT $\Rightarrow c(\xi) = 0$

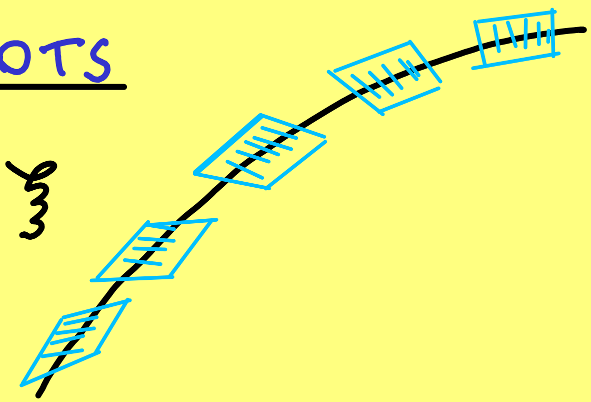
Thm (Ghiggini - Honda - Van Horn - Morris)

ξ contains Giroux torsion $\Rightarrow c(\xi) = 0$

KNOTS

Legendrian and Transverse Knots

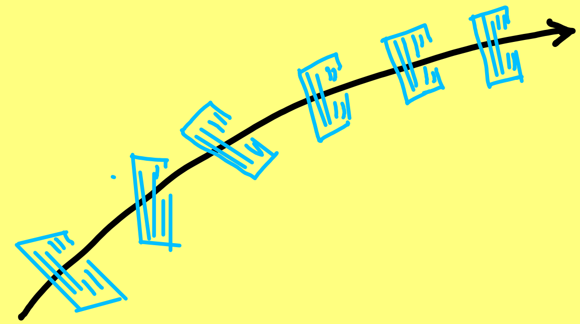
a knot L is <u>Legendrian</u> if $TL \in \xi$



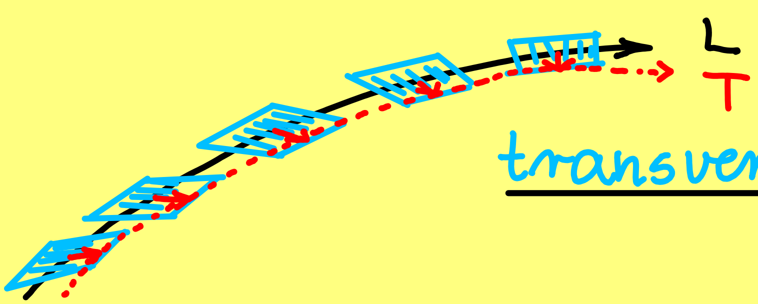
<u>Fact</u> Every knot can be put in Legendrian position

a knot T is <u>transverse</u> if $TT \pitchfork \xi$

<u>Rmk</u> bindings of OBs are transverse



Equivalence: isotopy through Legendrian / transverse knots: <u>Legendrian / transverse isotopy</u>



<u>transverse push off</u> of L

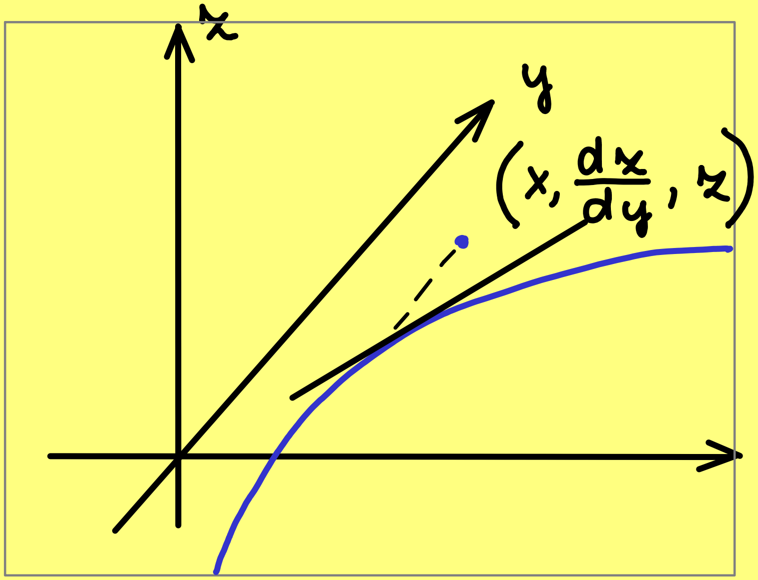
transverse knots have <u>Legendrian approximations</u>, but these are not unique

STABILIZATION

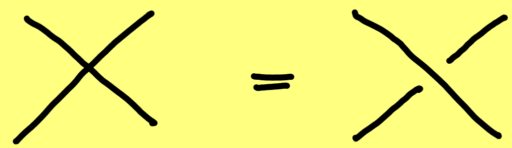
a local operation on L $\Rightarrow$ can be described in $(\mathbb{R}^3, \xi_{st})$

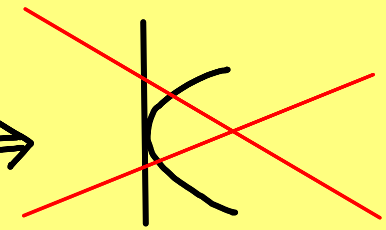
$$TL \in \xi_{st} = \ker(dz - y\,dx) \iff y = \frac{dz}{dx}$$

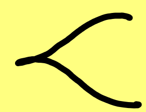
projection to the (x,z)-plane: <u>front projection</u>



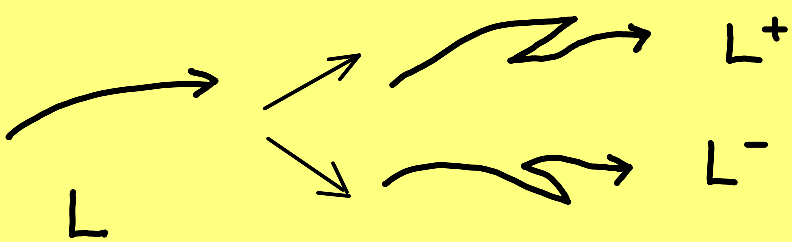
the slope determines the y-coordinate:

$\Rightarrow$ • 

• $\infty \neq y = \frac{dz}{dx}$ $\Rightarrow$ 

instead : 

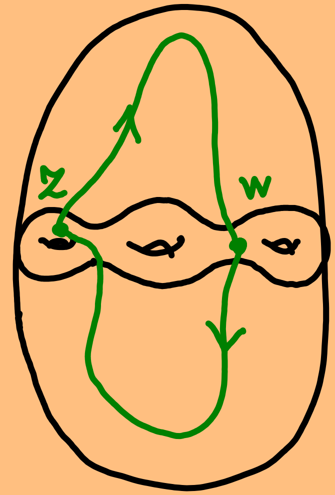
<u>Def</u>



<u>stabilisation</u> of L

<u>Fact</u> T_0 & T_1 are transverse isotopic $\iff$ their Legendrian approximations have Legendrian isotopic negativ stabilisations.

Knot Floer Homology



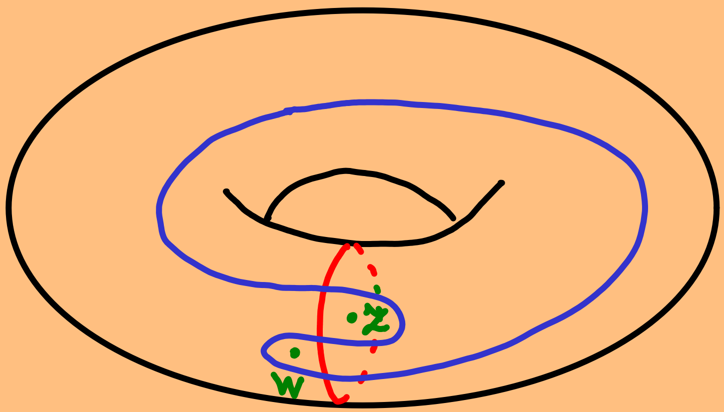
$K \hookrightarrow Y$, $\exists$ HD such that $|K \cap \Sigma| = 2$

$\rightsquigarrow$ two basepoints on Σ : $(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$

Conversely : given $(\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$

$\Rightarrow$ can recover $K \hookrightarrow Y$:

$Y = S^3$



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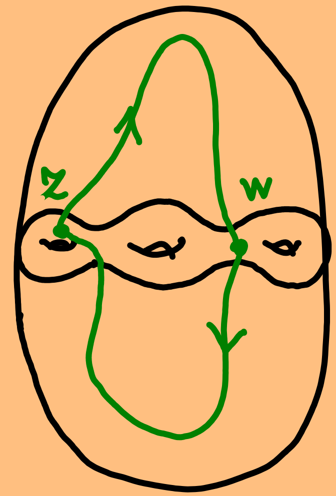

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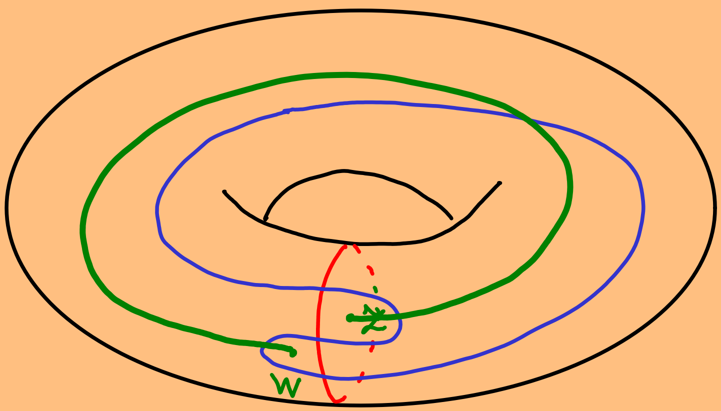

connect w to z in $\Sigma - \cup \underline{\alpha}$

KNOT FLOER HOMOLOGY

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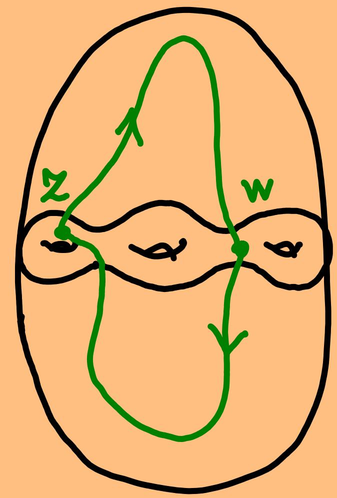

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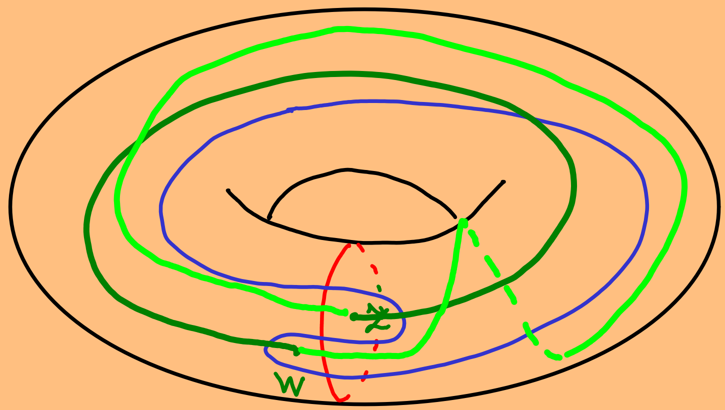

connect w to z in $\Sigma - \cup\underline{\alpha}$ connect z to w in $\Sigma - \cup\underline{\beta}$

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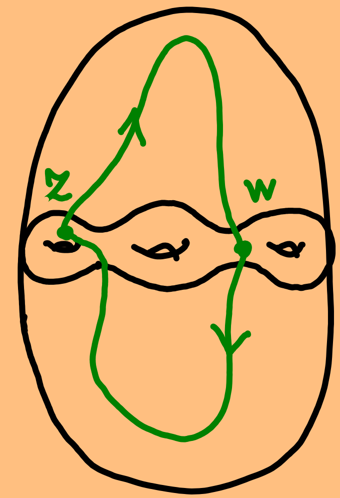

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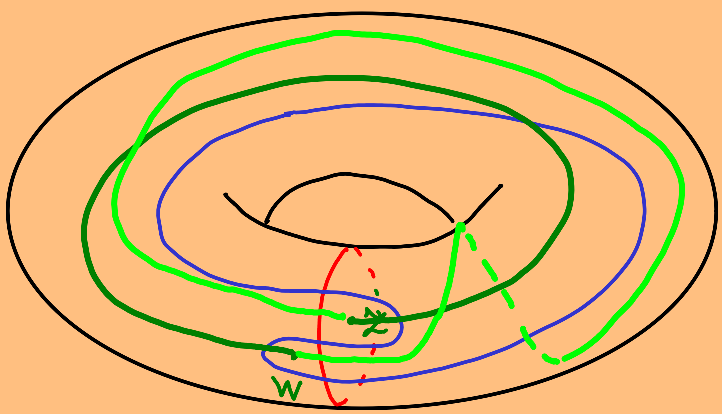
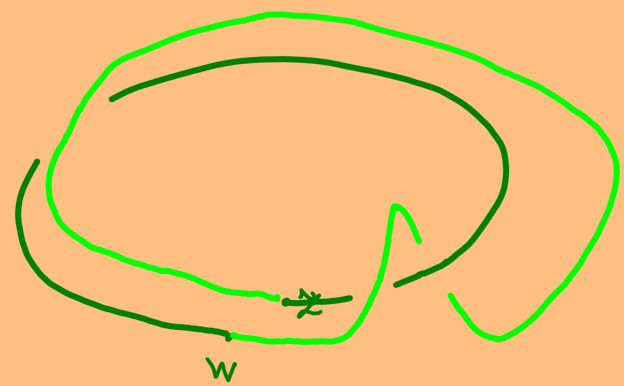


$\rightsquigarrow$



connect w to z in $\Sigma - \cup\underline{\alpha}$ connect z to w in $\Sigma - \cup\underline{\beta}$

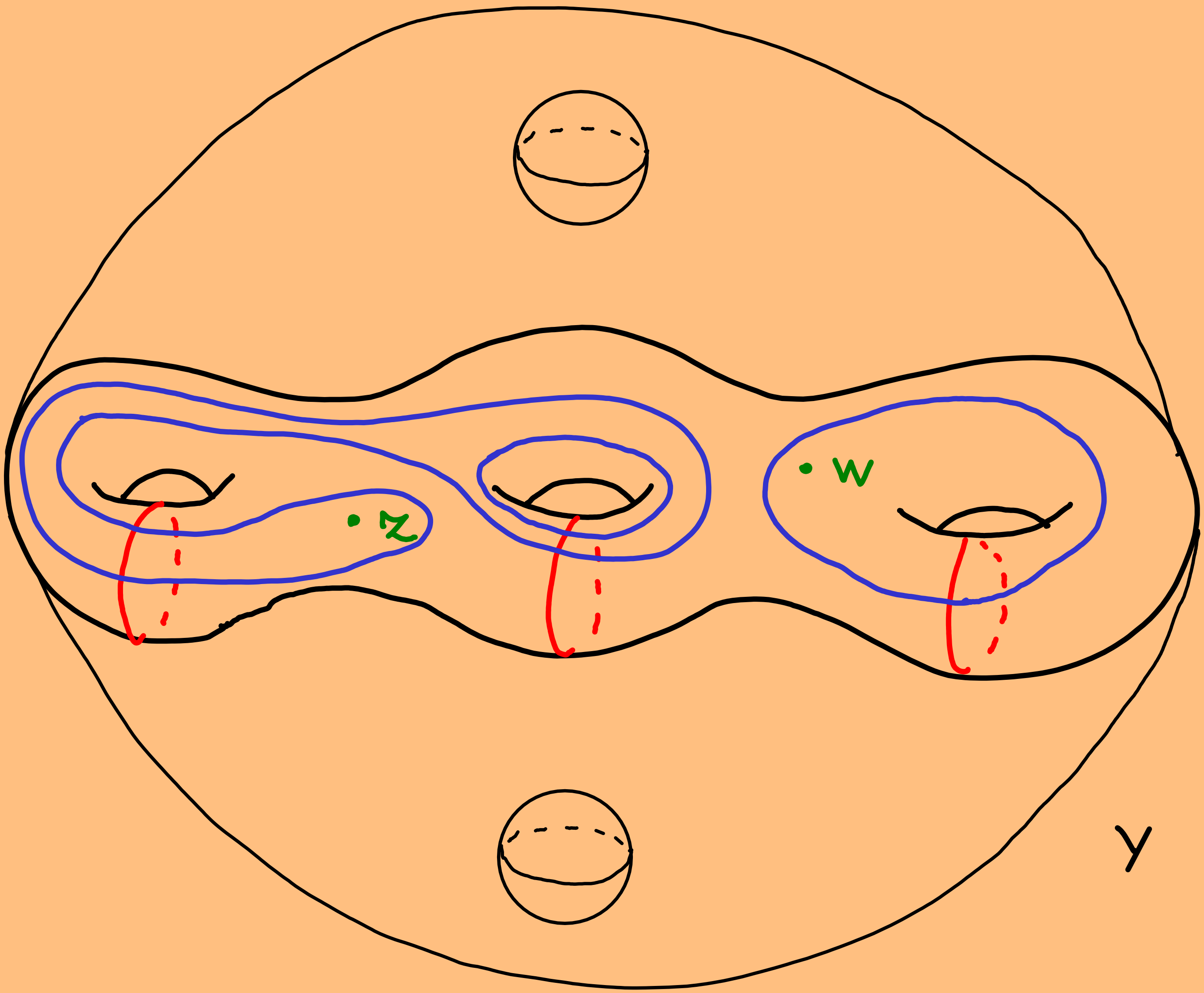
push the arc into U_α push the arc into U_β

$\widehat{CFK} = \widehat{CF}$ $\hat{\partial}_K$ counts holomorphic discs missing both z & w

<u>Thm</u> (OSz) This homology is an invariant of the isotopy class of the knot : $\widehat{HFK}(K)$ the <u>knot Floer homology</u>

Knot Floer Homology as Sutured Floer Homology

$K \hookrightarrow Y \rightsquigarrow (\Sigma, \underline{\alpha}, \underline{\beta}, z, w)$ HD



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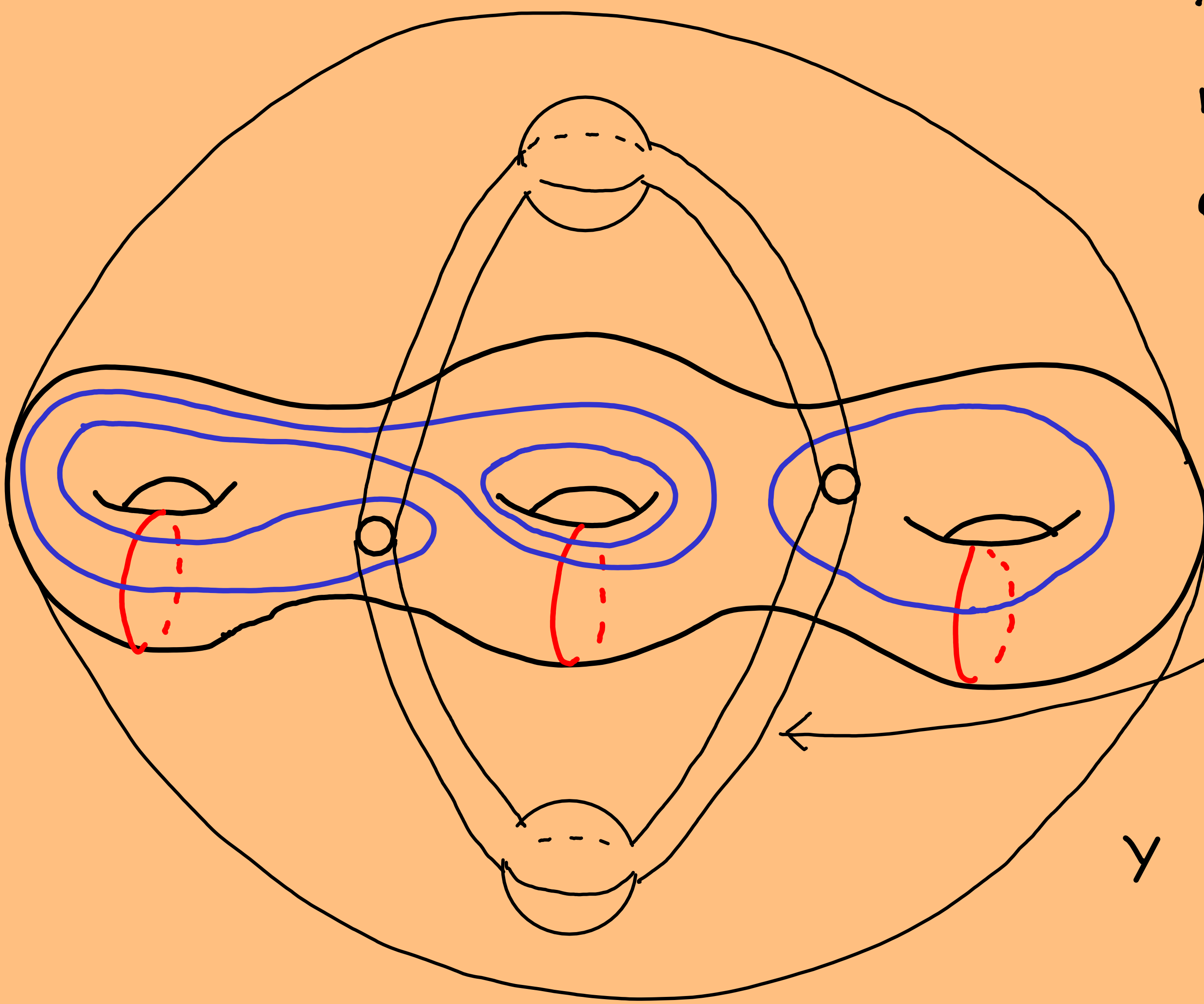


remove a neighborhood of z & w from Σ

neighborhood of K with meridional sutures

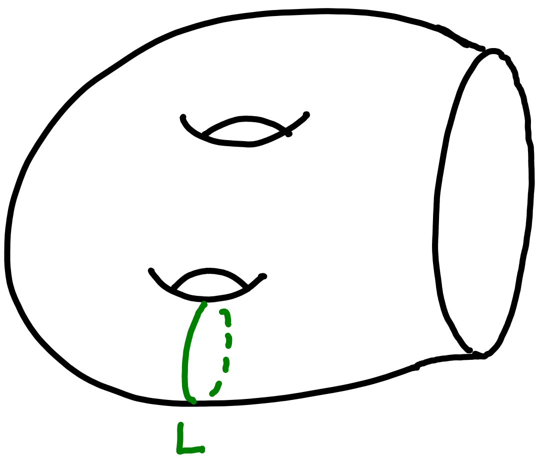
$\Rightarrow \widehat{HFK}(Y,K) \cong \widehat{HF}(Y - \mathring{N}_K, m \cup -m)$

Open Books & Legendrian Knots

$L \hookrightarrow (Y, \xi)$

L can be put homologically nontrivialy on the page of an OB

Thm (Lisca - Ozsváth - Stipsicz - Szabó) the isotopy class of L on S uniquely determines the Legendrian isotopy class of L.



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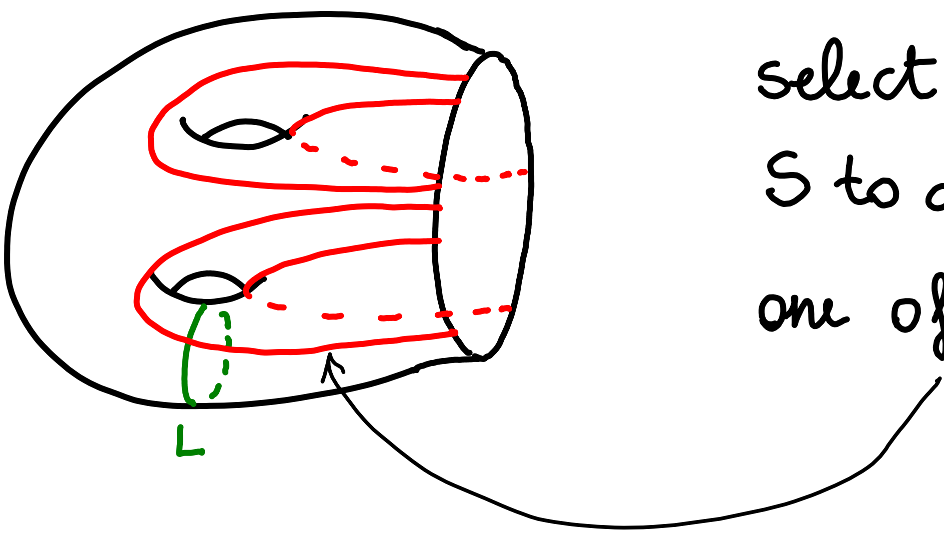

select the curves that cut up S to a disc such that exactly one of them intersects L

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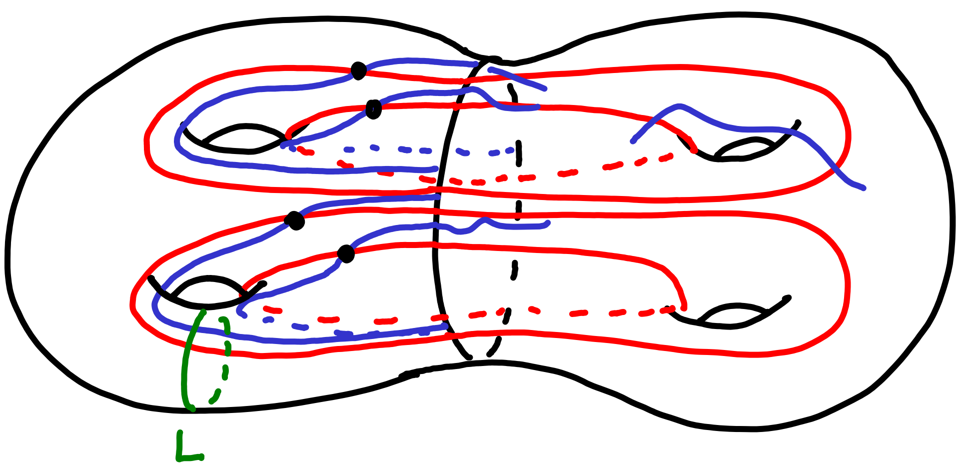
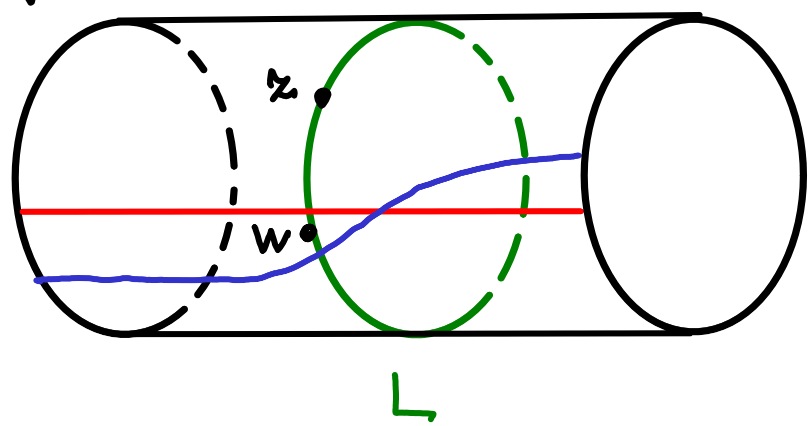


in the neighborhood of L:



there is a unique element of $\widehat{CFK}(-Y, K)$ on the LHS,

$\hat{\partial}_K = 0 \Rightarrow$ this defines an element of $\widehat{HFK}(-Y, K)$

Thm (LOSS) This element $\hat{\mathcal{L}}(L)$ is an invariant of L

PROPERTIES OF $\hat{\mathfrak{L}}$

$L \hookrightarrow (Y, \xi)$ Legendrian knot

Prop (LOSS) $c(\xi) = 0 \Rightarrow \hat{\mathfrak{L}}(L) = 0$

Def L loose if its complement is OT

L is exceptional or non-loose otherwise

Cor L loose $\Rightarrow \hat{\mathfrak{L}}(L) = 0$

Thm (Stipsicz–V, Vela-Vick)

complement of L contains Giroux torsion $\Rightarrow \hat{\mathfrak{L}}(L) = 0$

no gluing result for $\widehat{HFK}$ or $\hat{\mathfrak{L}}$

Thm (LOSS)

$$\hat{\mathfrak{L}}(L^-) = \hat{\mathfrak{L}}(L)$$

$$\hat{\mathfrak{L}}(L^+) = 0$$

Transverse Invariant

Remember Legendrian approximations of transverse knots were determined up to negative stabilization

$\hat{\mathfrak{L}}(L^-) = \hat{\mathfrak{L}}(L)$

$\Rightarrow$ **Def** $\hat{\vartheta}(T) = \hat{\mathfrak{L}}(L)$, where L is a Legendrian approximation of T – transverse invariant

the results for Legendrian knots remain true

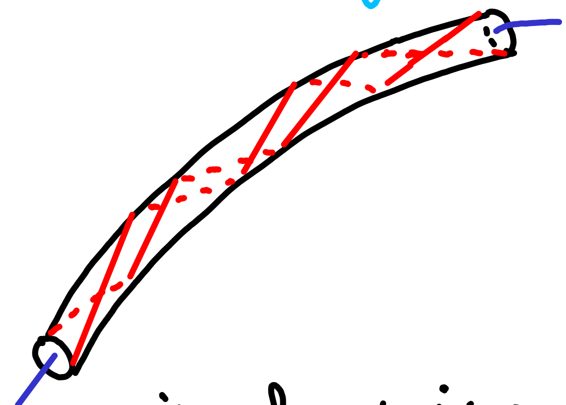
in particular: $\hat{\vartheta}(T) = 0$ if its complement contains OT disc or Giroux torsion

Thm (Vela-Vick) $\hat{\vartheta}(T) \neq 0$ for a binding of an OB of ξ

Cor the binding of an OB for ξ pierces all OT discs & Giroux torsions

STANDARD NEIGHBORHOOD OF A LEGENDRIAN KNOT

Fact A Legendrian knot has a standard neighborhood with convex boundary having a 2-component dividing curve each of which represents the Thurston - Bennequin framing



$L \hookrightarrow (Y, \xi)$ Legendrian knot N_L its standard neighborhood

$\rightsquigarrow (Y - N_L, \xi|_{Y-N_L})$ is a contact 3-manifold with boundary

$c(L) := \hat{c}(\xi|_{Y-N_L}) \in \widehat{HF}(-(Y-N_L), \Gamma_{\partial N_L})$

- $c(L)$ is an invariant of L

Connection Between the Invariants ?

thus we have two invariants : $\widehat{\mathcal{L}}(L)$, $c(L)$

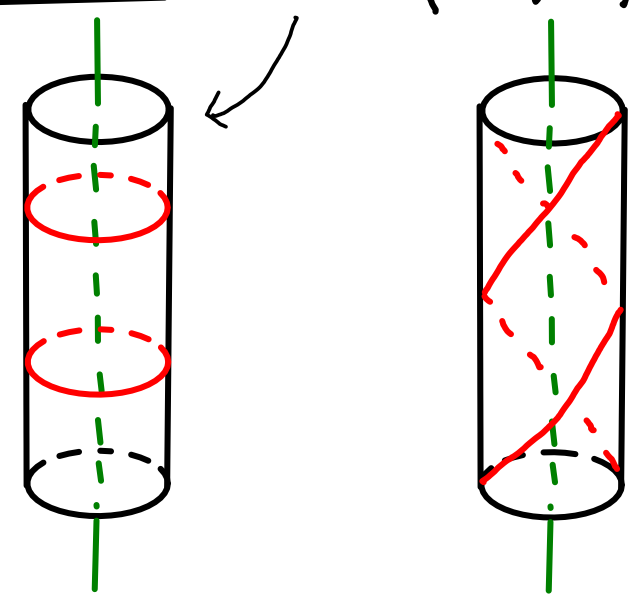
$\widehat{\mathcal{L}}(L) \in \widehat{HFK}(-Y,K)$ $\qquad$ $c(L) \in \widehat{HF}(-Y-N_L, \Gamma_{\partial N_L})$

- they live in different spaces
- for an unoriented knot $\widehat{\mathcal{L}}$ has two versions :

$$\widehat{\mathcal{L}}(L) \quad \& \quad \widehat{\mathcal{L}}(\overline{L})$$

but $c(L)$ does not see the orientation

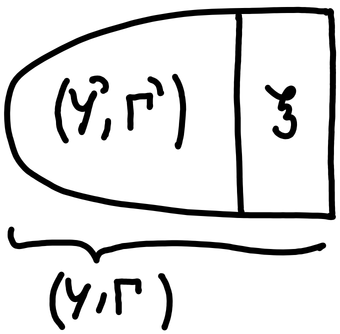
Remember $\widehat{HFK}(-Y,K) \cong \widehat{HF}(-Y, -m \cup m)$



Thurston-Bennequin framing

Connection Between the Invariants

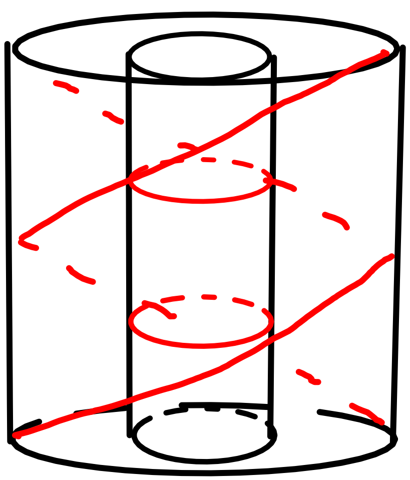
Remember (HKM) $\exists\ \varphi_\xi : \widehat{HF}(-Y',\Gamma') \longrightarrow \widehat{HF}(-Y,\Gamma)$



moreover if (Y',Γ') comes from (Y',ξ') then

$$\varphi_\xi : c(\xi') \longmapsto c(\xi' \cup \xi)$$

thus a map $\widehat{HF}(Y-N_L,\Gamma_{\partial N_L}) \longrightarrow \widehat{HF}(Y-N_L, m\cup -m)$ can be defined using a contact structure on



there are two tight such contact structures : basic slices η_+ & η_-

Thm (Stipsicz - V)

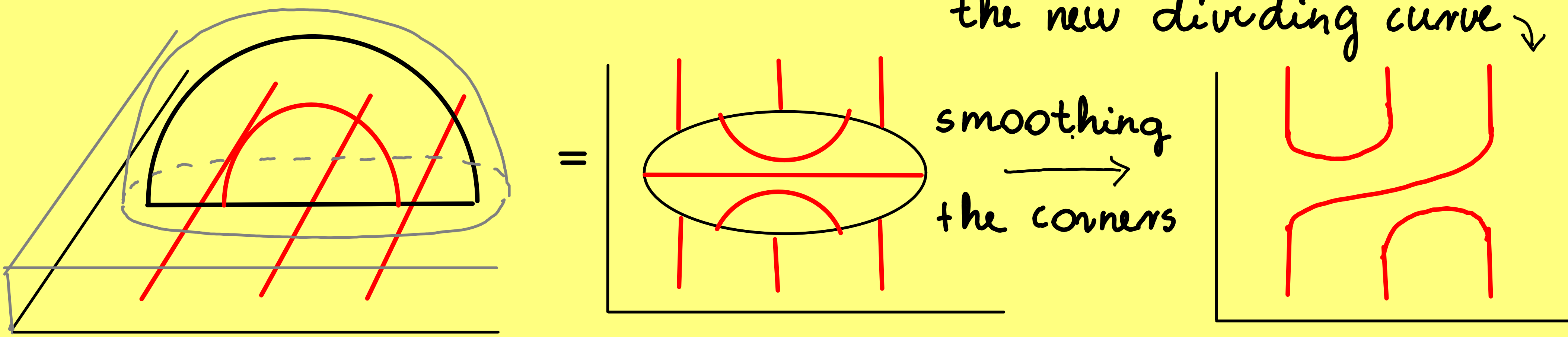
$$\widehat{HF}(-(Y-N_L),\Gamma_{\partial N_L}) \underset{-}{\overset{+}{\rightrightarrows}} \widehat{HF}(-(Y-N_L), m\cup -m) \cong \widehat{HFK}(-Y,K)$$

$$c(\xi|_{Y-N_L}) \longmapsto \hat{\mathfrak{L}}(L)$$
$$c(\xi|_{Y-N_L}) \longmapsto \hat{\mathfrak{L}}(\overline{L})$$

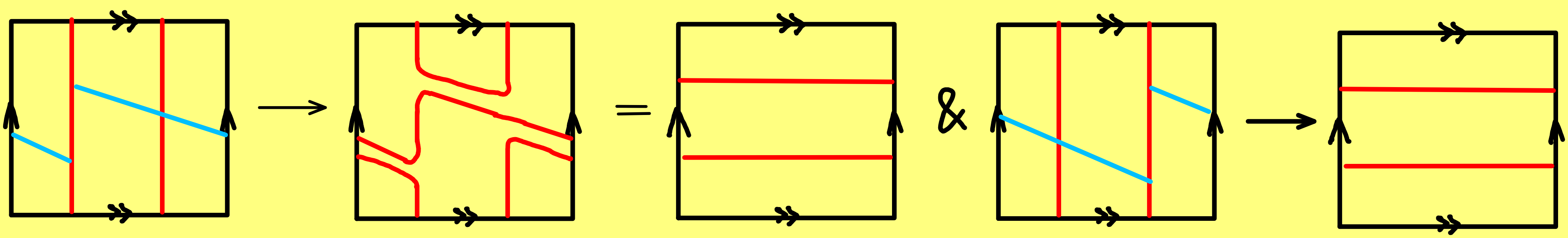
Convex Surface Theory

Def attaching a "half overtwisted disc" to a convex surface is a bypass attachment:

the new dividing curve



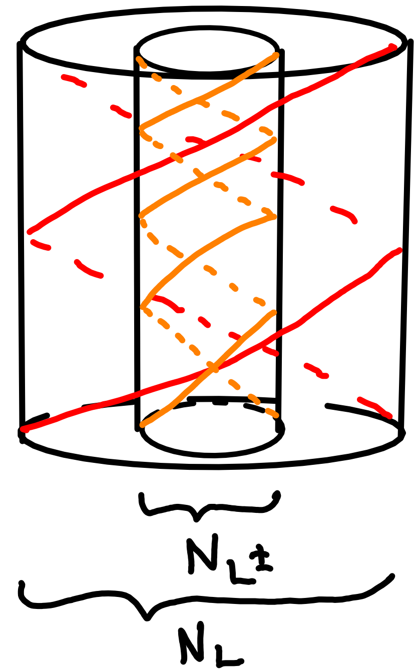
e.g. attaching a bypass to the torus:



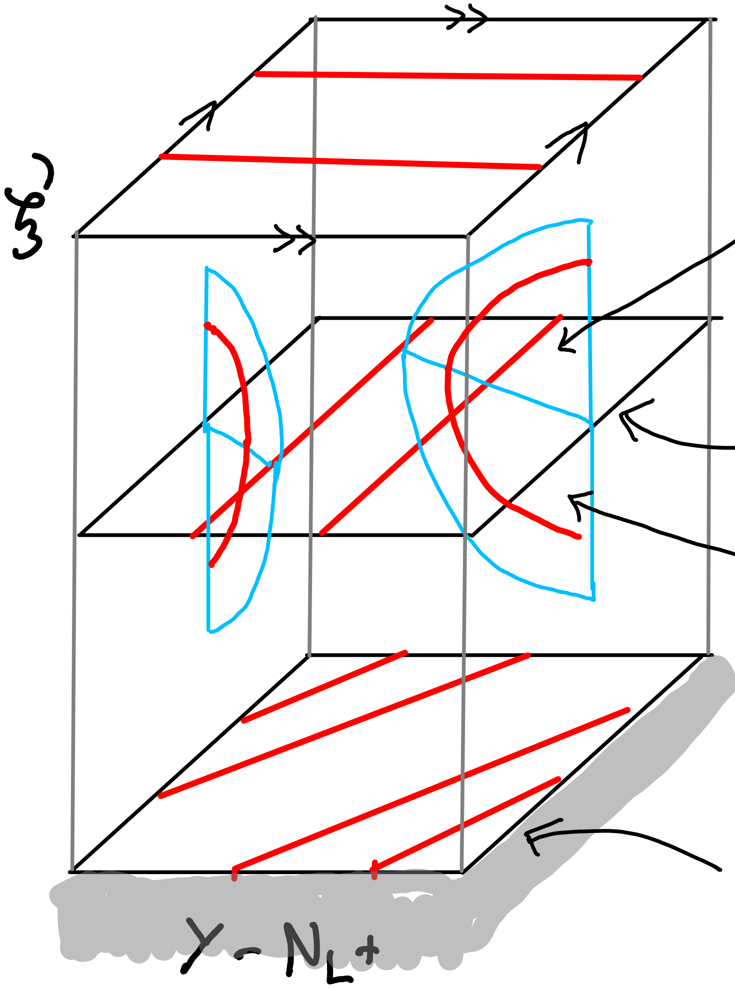
these are the two basic slices η_+ & η_-

Reproving Stabilization Properties of $\hat{\mathfrak{L}}$

N_L & $N_{L^\pm}$ differ by a bypass attachment



the bypass for L^+:



half OT disc coming from the basic slice

∂N_L

half OT disc comming from the stabilization

∂N_{L^+}

ξ' OT

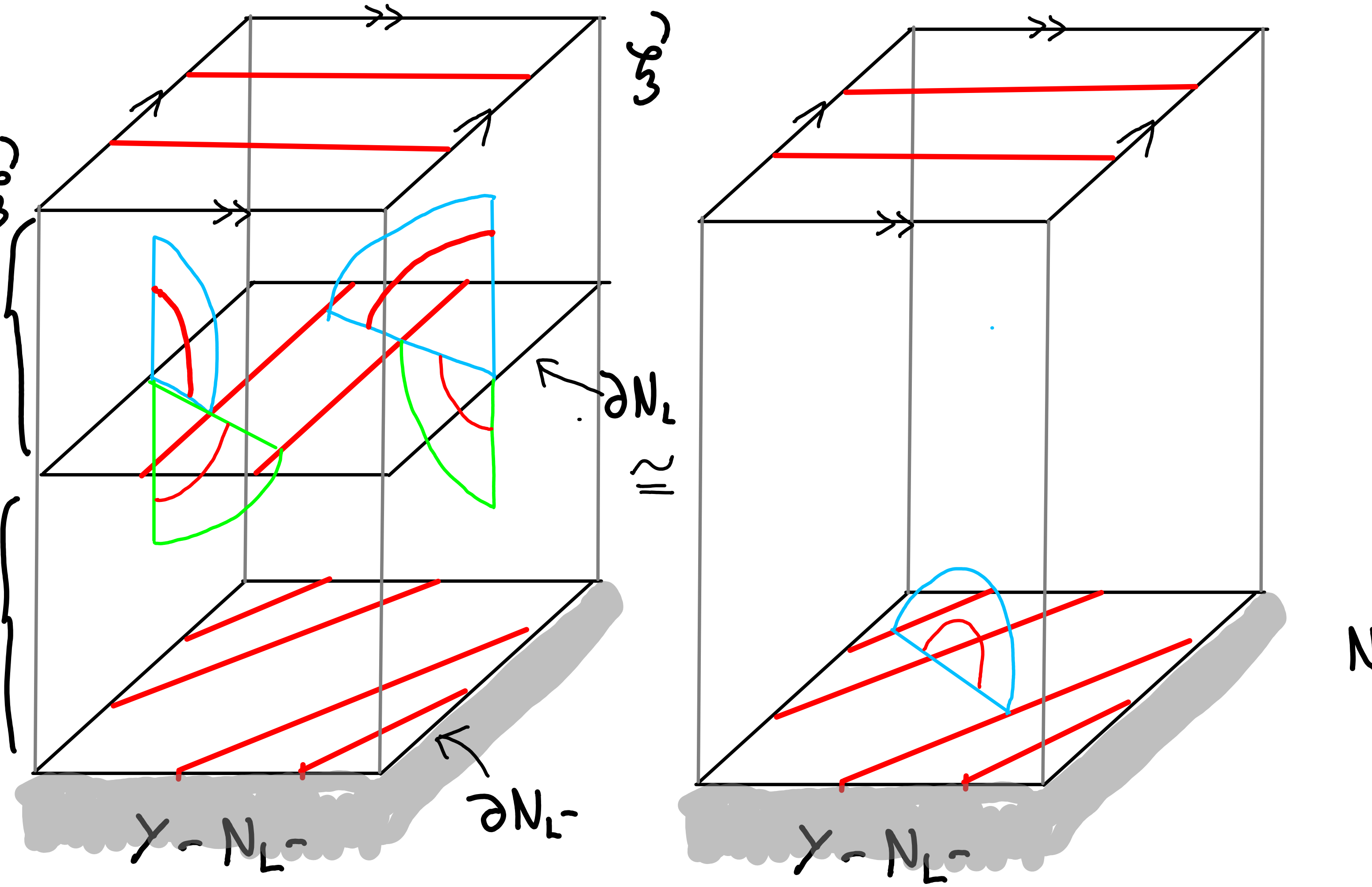
$$\hat{\mathfrak{L}}(L^+) = c(\xi') = 0$$

Reproving Stabilization Properties of $\hat{\mathcal{L}}$

the bypass for L^-:

basic slice η_+

negative stabilisation



$$\Rightarrow \hat{\mathcal{L}}(L^-) = \hat{\mathcal{L}}(L)$$

Thanks for your attention!