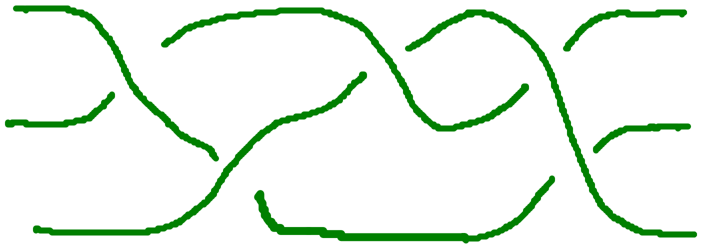
FLOER HOMOLOGY FOR TANGLES



BY

VERA VÉRTESI

(JOINT WITH:
INA PETKOVA)

MANY EQUIVALENT THEORIES

Donaldson invariants (1982) $\boxed{4}$

Instanton Floer homology $\boxed{3}$
(Floer, 1988)

Seiberg-Witten invariants (1994)
monopole Floer homology $\boxed{3}$
(Kronheimer-Mrowka, 2007)

Taubes 2008

Gromov-Taubes invariant $\boxed{4}$
(Taubes 199?)
embedded contact homology $\boxed{3}$
(Hutchings, 2010)

Kutluhan-Lee-Taubes
2010

Colin-Ghiggini-Honda
2011

Heegaard Floer homology $\boxed{3}$, $\boxed{4}$
(Ozsváth-Szabó, 2001)

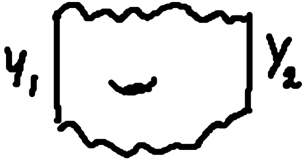
WHAT IS HEEGAARD FLOER THEORY ?

$\mathbb{F} = \mathbb{Z}$ or $\mathbb{Z}/(2)$

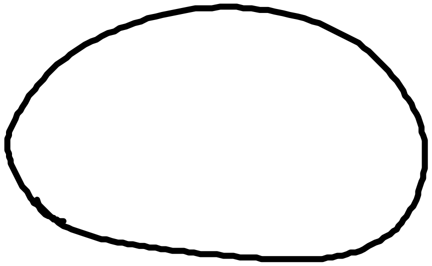
$$\begin{tikzcd}
Y \text{ 3-manifold} \arrow[r, rightsquigarrow] & \widehat{HF}(Y) & \text{graded } \mathbb{F}\text{-module} \\
& HF^-(Y) \arrow[d] & \mathbb{F}[U]\text{-module} \\
& HF^\infty(Y) \arrow[d] & \mathbb{F}[U,U^{-1}]\text{-module} \\
& HF^+(Y) \arrow[uu, bend left] & \mathbb{F}[U]\text{-module}
\end{tikzcd}$$

(determined by $H^*(Y)$)

W cobordism btwn 3-manifolds $\rightsquigarrow$ $\hat{F}_W^{+,-,\infty} : \widehat{HF}^{+,-,\infty}(Y_1) \longrightarrow \widehat{HF}^{+,-,\infty}(Y_2)$



X 4-manifold



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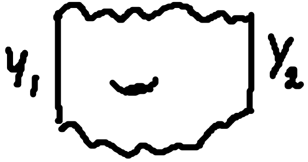

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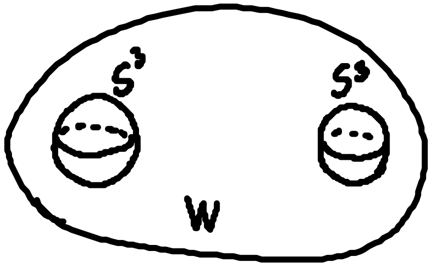

$$\begin{tikzcd} F_W : \widehat{HF}(S^3) \arrow[r] & \widehat{HF}(S^3) \\ \mathbb{Z} \arrow[u, phantom, sloped, "\cong"] \arrow[r, "\cdot n"] & \mathbb{Z} \arrow[u, phantom, sloped, "\cong"] \end{tikzcd}$$

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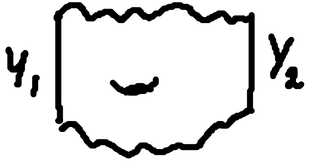

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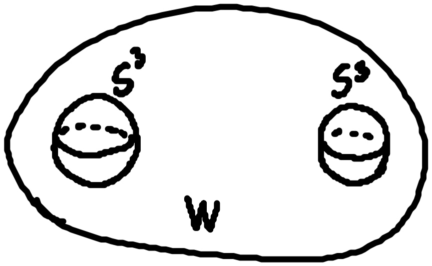

$$\begin{tikzcd} F_W : \widehat{HF}(S^3) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \widehat{HF}(S^3) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Z} \end{tikzcd}$$

! This does not work !

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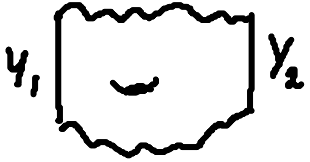

X 4-manifold $\rightsquigarrow$ a number for every spin^c-structure

$K \subset Y$ knot $\rightsquigarrow$ $\widehat{HFK}(Y,K)$ bigraded $\mathbb{F}$-module

$HFK^-(Y,K)$ $\mathbb{F}[U]$-module

WHY IS HEEGAARD FLOER THEORY USEFUL?

Geometric content

- Ozsváth - Szabó: Detects smooth structures on 4-manifolds
- Ozsváth - Szabó, Ni: Detects the genus of knots
  Thurston norm of 3-manifolds
- Ozsváth - Szabó, Ghiggini, Ni, Juhász, ...
  Detects fiberedness of knots and 3-manifolds
- Ozsváth - Szabó, ... Bounds the slice genus
  minimal class representatives of homology classes

Computability

- defined using a PDE but sometimes can be combinatorial:
- Manolescu - Ozsváth - Sarkar: $\widehat{HFK}^-$ for knots
- Sarkar - Wang, Ozsváth - Stipsicz - Szabó: $\widehat{HF}(Y)$, easier version of HF^-
- Manolescu - Ozsváth - Thurston: $\widehat{HF}^{\pm,\infty}$, 4-manifold invariant

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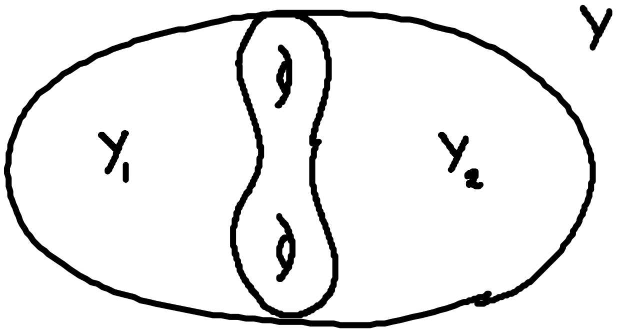
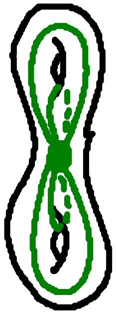
- Manolescu-Ozsváth-Thurston: $\widehat{HF}^{\pm,\infty}$, 4-manifold invariant
- still HARD to compute in practice

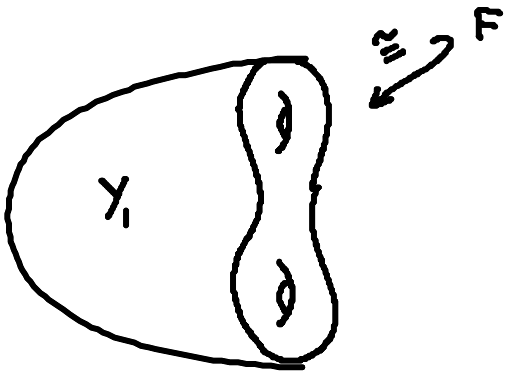
NEW APPROACH - BORDERED FLOER HOMOLOGY (Lipshitz - Ozsváth - Thurston)



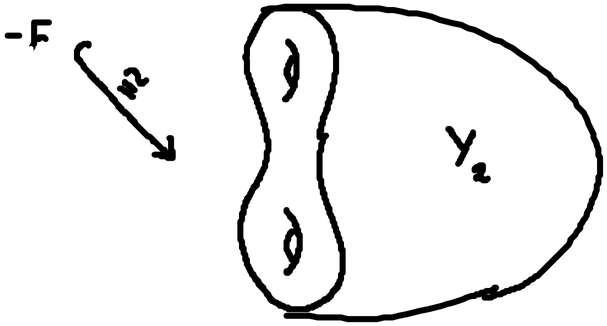


model surface w/ a given handle-decomposition

$\mathcal{F} \rightsquigarrow \mathcal{A}(\mathcal{F})$ - Differential Graded Algebra (DGA)



$\rightsquigarrow \widehat{CFA}_{\mathcal{A}(F)}(Y_1)$ - right $\mathcal{A}_\infty$-module over $\mathcal{A}(F)$



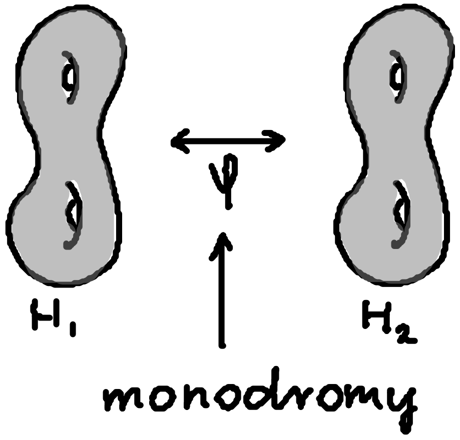
${}^{\mathcal{A}(F)}\widehat{CFD}$ - left DG-module over $\mathcal{A}(-F)$

$\rightsquigarrow$

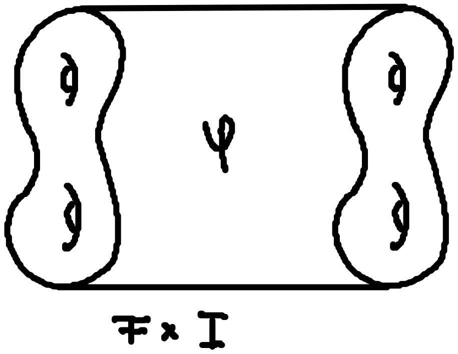
PAIRING THM: $\widehat{CFA}(Y_1) \tilde{\otimes} \widehat{CFD}(Y_2) \cong \widehat{CF}(Y_1 \underset{F}{\cup} Y_2)$

COMPUTATION - STRATEGY

- take a Heegaard decomposition of Y

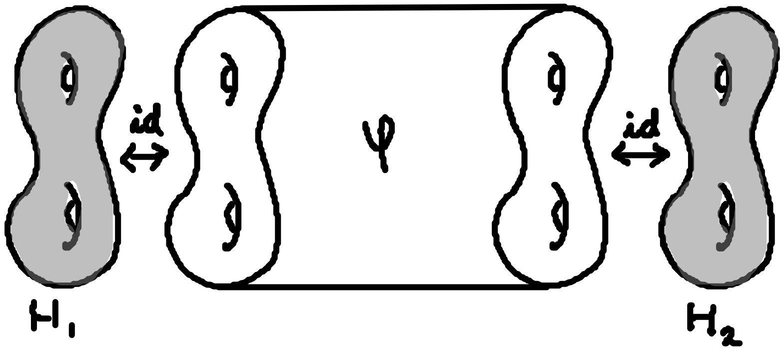


- define



$\mathfrak{s}(F)$ $\widehat{CFAD}_{\mathfrak{s}(F)}(\varphi)$

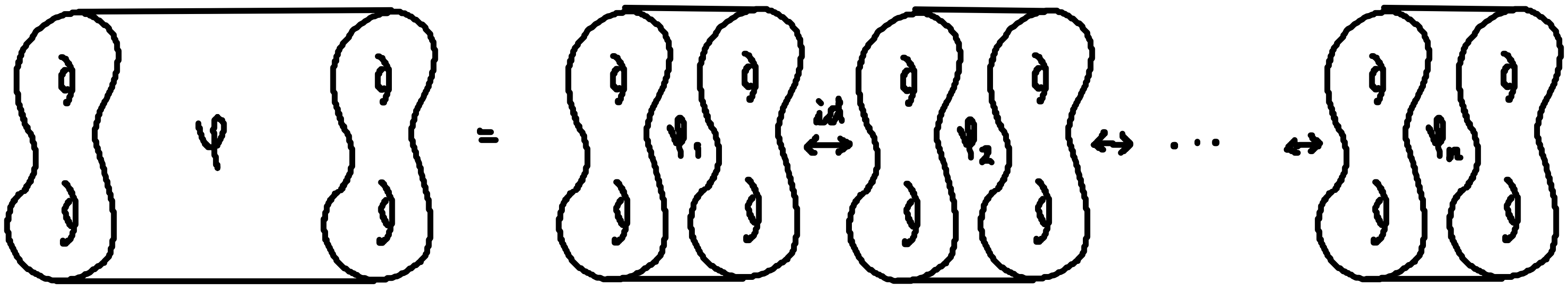
bimodule



$\rightsquigarrow$ $\widehat{CF}(Y) \cong \widehat{CFA}(H) \otimes \widehat{CFAD}(\varphi) \otimes \widehat{CFD}(H)$

simple simple

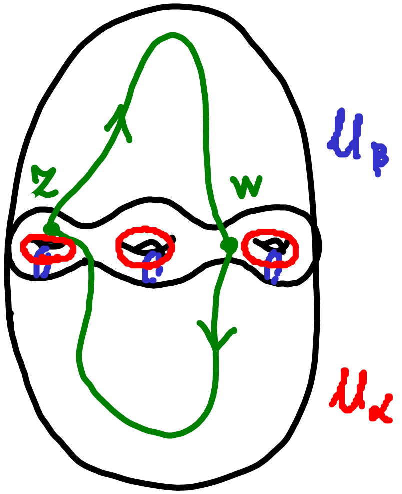
- "cut" φ into elementary pieces



$\widehat{CFAD}(\varphi) \cong \widehat{CFAD}(\varphi_1) \tilde{\otimes} \widehat{CFAD}(\varphi_2) \tilde{\otimes} \quad \tilde{\otimes} \widehat{CFAD}(\varphi_n)$

simple

KNOT FLOER HOMOLOGY

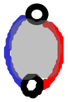


$K \hookrightarrow Y$, $(\Sigma, \underline{\alpha}, \underline{\beta}, \mathbb{O}, \mathbb{X})$ Heegaard diagram

w/ Σ - Heegaard surface

- $\underline{\alpha} = \{\alpha_1, \dots, \alpha_k\}$ set of curves bounding in U_α
- $\underline{\beta} = \{\beta_1, \dots, \beta_k\}$ set of curves bounding in U_β
- $\mathbb{O} = \{O_1, \dots, O_{g-k+1}\}$ + intersection of K & Σ
- $\mathbb{X} = \{X_1, \dots, X_{g-k+1}\}$ - intersection σ K & Σ

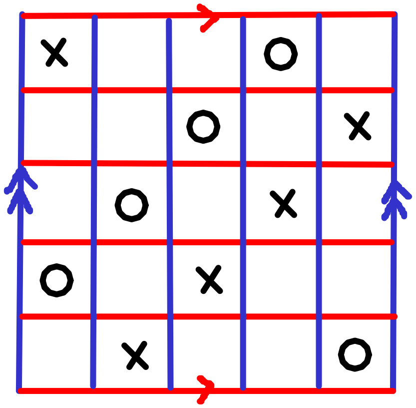
$\rightsquigarrow$ chain complex with generators $\vec{x} = (x_1, \dots, x_k)$ k-tuples of intersection points between $\underline{\alpha}$ - & $\underline{\beta}$ - curves with exactly 1 point on each over $\mathbb{Z}/(2)[U_1, \dots, U_n]$: CFK^-

boundary map : counts holomorphic curves in $\Sigma \times$ 

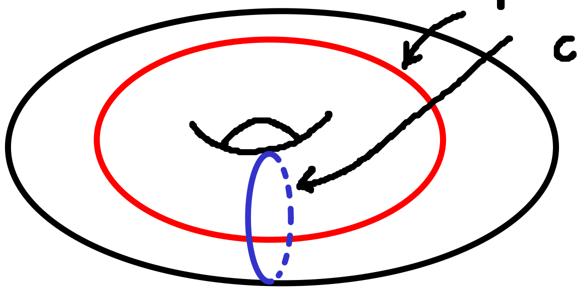
$$\partial_K \vec{x} = \sum_{\vec{y}} \sum_{\substack{\phi \\ \mu(\phi)=1 \\ \phi \cap \mathbb{X} = \phi}} U_1^{\phi \cap O_1} \cdots U_{g-k+1}^{\phi \cap O_{g-k+1}} \# \mathcal{M}(\phi)/_{\mathbb{R}} \cdot \vec{y}$$

Thm (OSz) This homology is an invariant of the isotopy class of the knot : $\widehat{HFK}(K)$ the knot Floer homology

EXAMPLE - COMBINATORIAL KNOT FLOER HOMOLOGY



On a torus:



parallel curves to these

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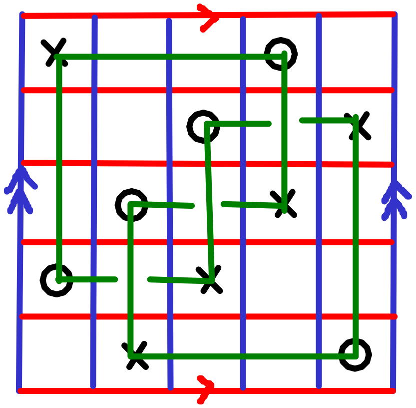

← right handed trefoil

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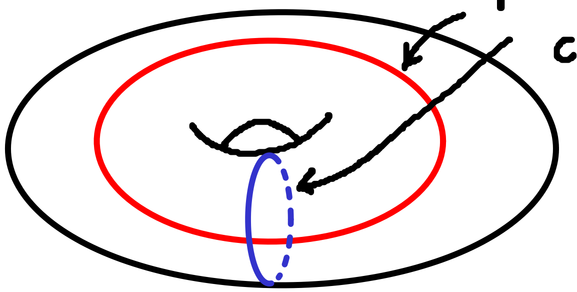

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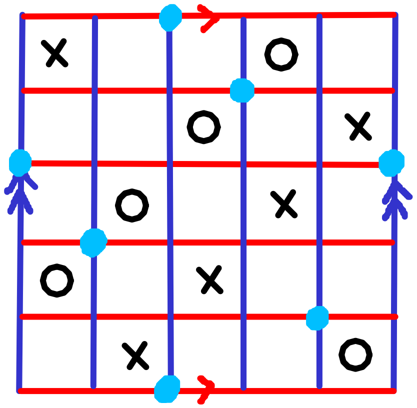

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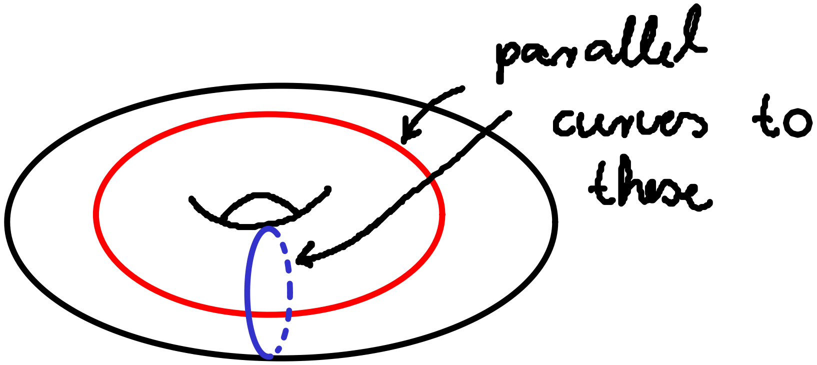

chain complex with generators $\vec{x}=(x_1,\dots,x_n)$ -tuples of intersection points between α- & β-curves with exactly 1 point on each over $\mathbb{Z}/(2)[U_1,\dots,U_n]$: CFK^- : ●

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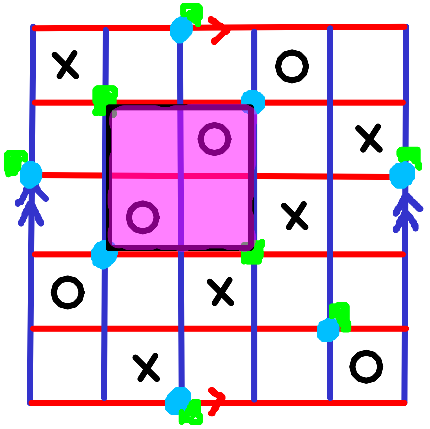

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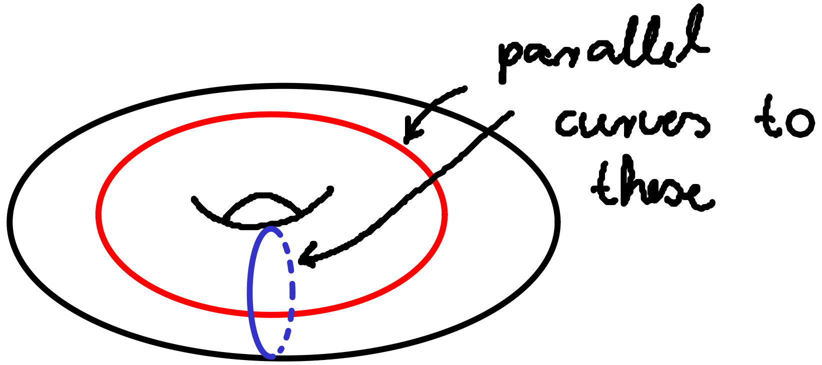

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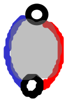
boundary map: counts holomorphic curves in $\Sigma \times$ 

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here the projection of all holomorphic curves are rectangles & $\#\mathcal{M}=1$

∂_K ● $= U_3U_4$ 🟩 $+$

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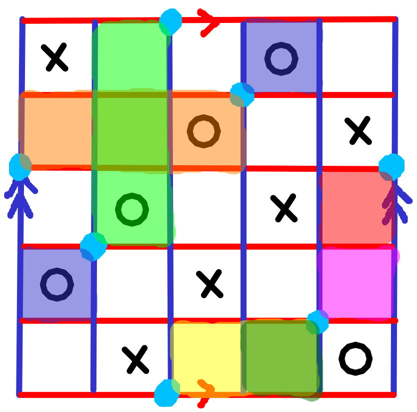

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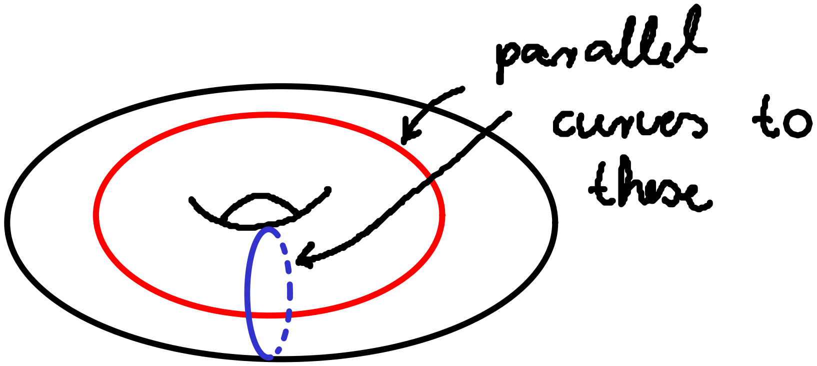

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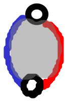
boundary map: counts holomorphic curves in $\Sigma \times$ 

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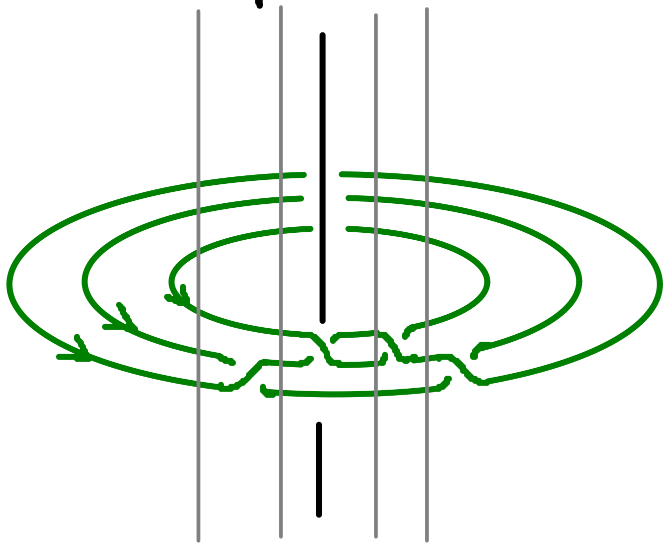
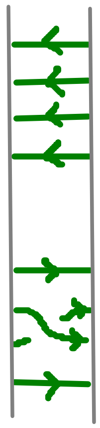
BRAIDS

- every knot can be put in braid position

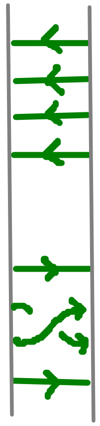


- elementary pieces :

σ_i



σ_i^{-1}

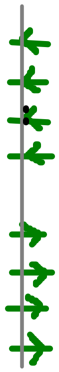


relations : $\cdot\ \sigma_i \sigma_i^{-1} = \mathbb{1}$

$\cdot\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

$\cdot\ \sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geqslant 2$

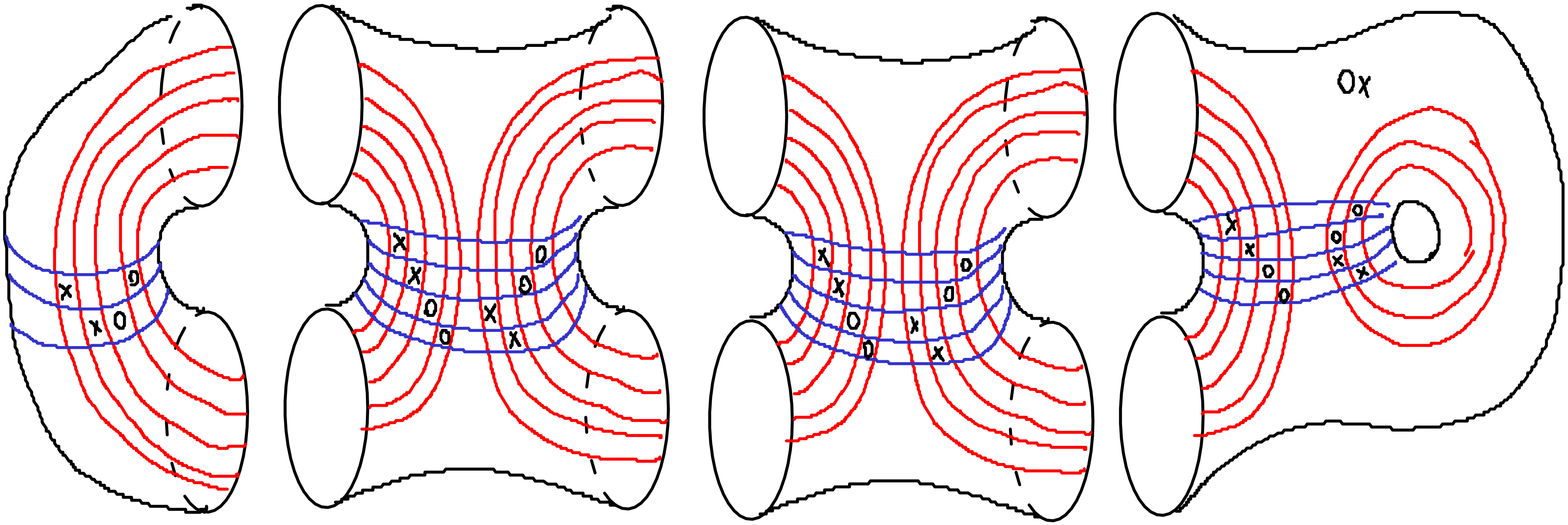
Goal:

- define differential graded algebra $\mathcal{A}_n$ corresponding to 
- define differential graded bimodules

$$\mathcal{M}_i = {}_{\mathcal{A}_n}\widehat{\mathrm{CFTAD}}(\sigma_i)^{\mathcal{A}_n} \ \& \ \overline{\mathcal{M}}_i = {}_{\mathcal{A}_n}\widehat{\mathrm{CFTAD}}(\sigma_i^{-1})^{\mathcal{A}_n}$$

such that :
- $\mathcal{M}_i \tilde{\otimes} \overline{\mathcal{M}}_i \simeq [\mathbb{1}]$
- $\mathcal{M}_i \tilde{\otimes} \mathcal{M}_{i+1} \otimes \mathcal{M}_i \simeq \mathcal{M}_{i+1} \tilde{\otimes} \mathcal{M}_i \otimes \mathcal{M}_{i+1}$
- $\mathcal{M}_i \tilde{\otimes} \mathcal{M}_j \simeq \mathcal{M}_j \tilde{\otimes} \mathcal{M}_i$ if $|i-j| \geqslant 2$

EXAMPLE - BRAIDS



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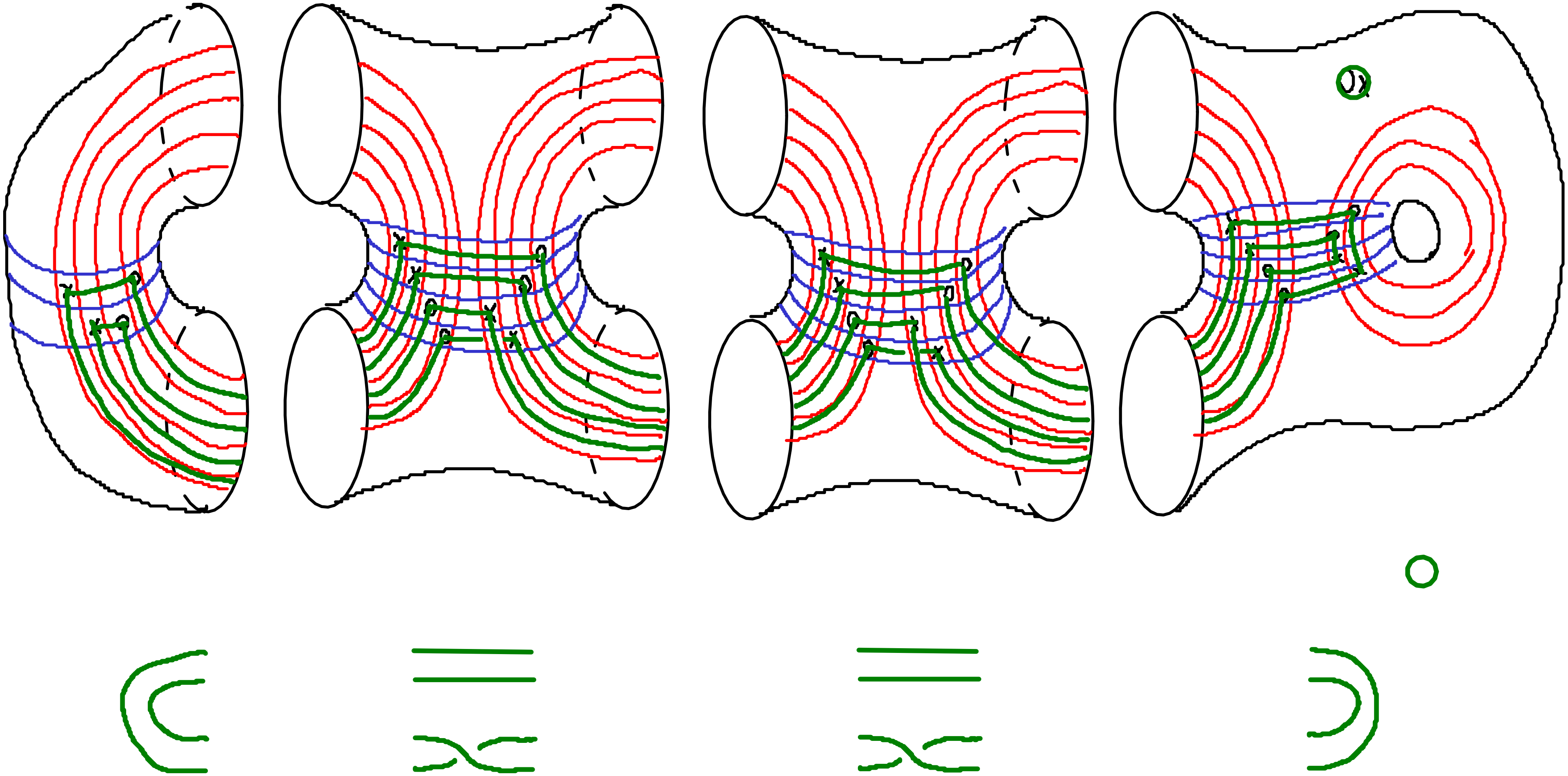
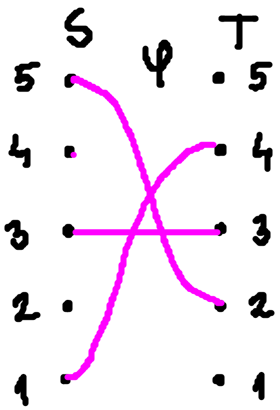


right handed trefoil

THE ALGEBRA $\mathcal{A}_n$

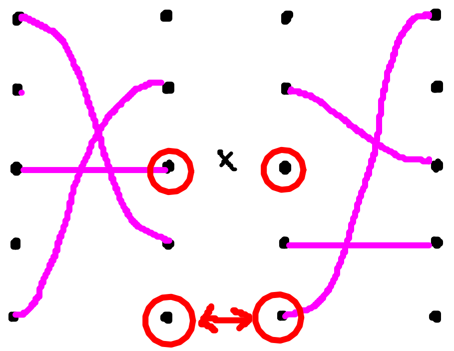


generators: (S, T, φ) where $S, T \subseteq \{1, 2, \dots, 2n+1\}$

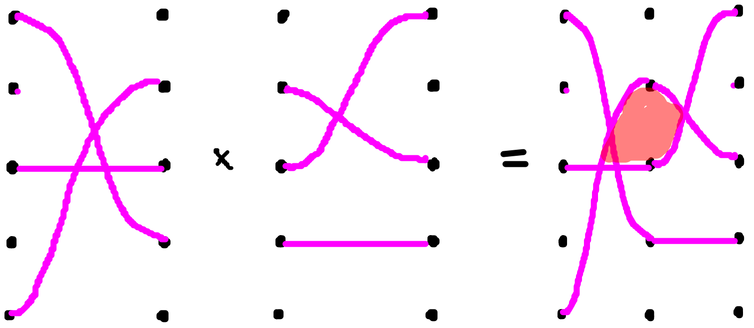
$$\varphi : S \xrightarrow{1:1} T$$

over $\mathbb{Z}/(2)[U_1, \dots, U_n]$ (pretend $U_i = 0$)

multiplication: concatenation (if possible)

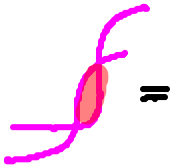


$= 0$



$= 0$

& **relation:**



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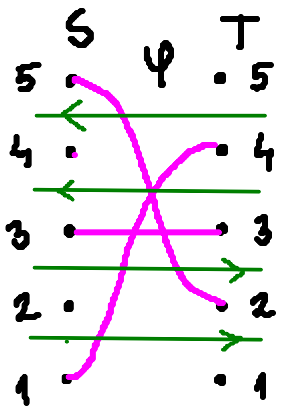
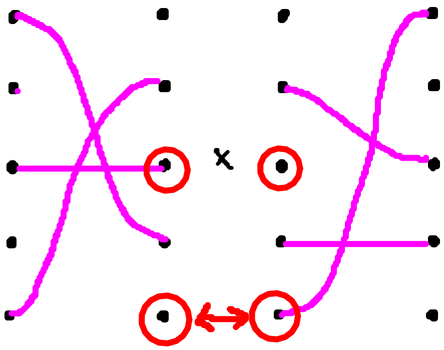


generators: (S, T, φ) where $S, T \subseteq \{1, 2, \dots, 2n+1\}$

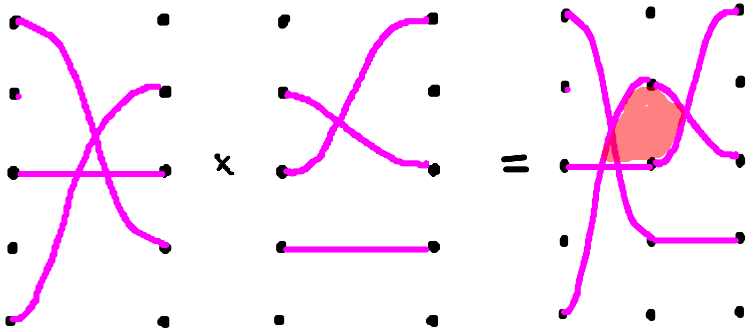
$$\varphi : S \xrightarrow{1:1} T$$

over $\mathbb{Z}/(2)[U_1, \dots, U_n]$ (pretend $U_i = 0$)

multiplication: concatenation (if possible)

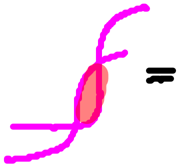


$= 0$

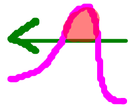


$= 0$

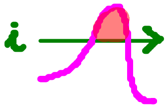
& **relation:**



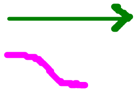
$= 0$

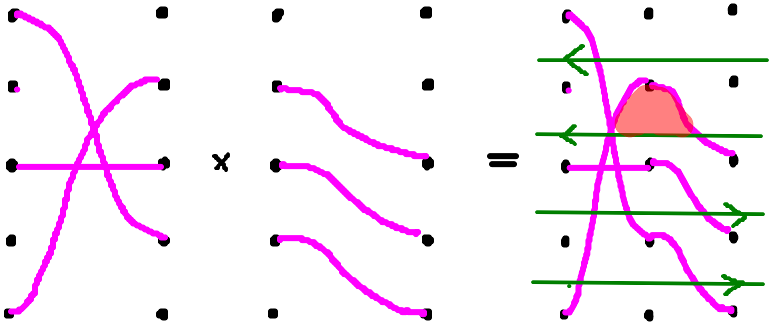


$= 0$

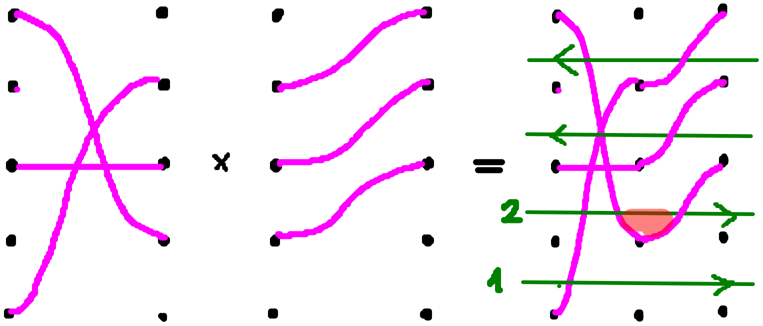


$= U_i$

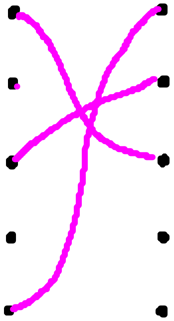




$= 0$

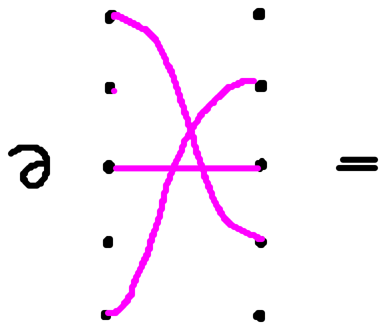


$= U_2$



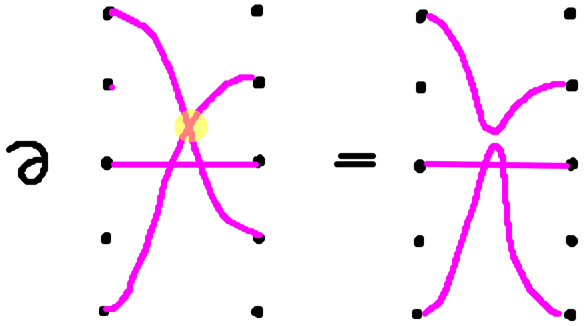
THE ALGEBRA $\mathcal{A}_n$

differential: resolving intersections



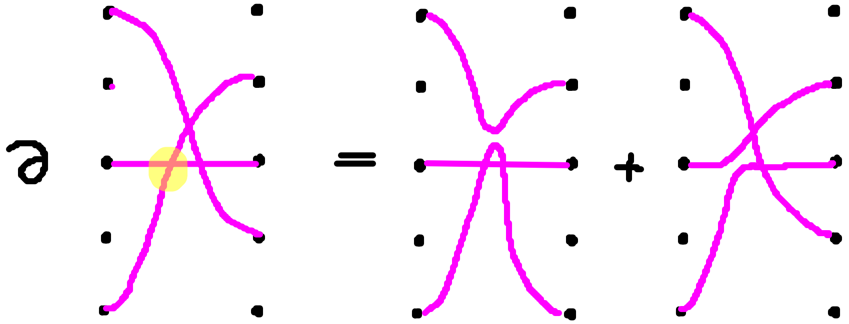
THE ALGEBRA $\mathcal{A}_n$

differential: resolving intersections



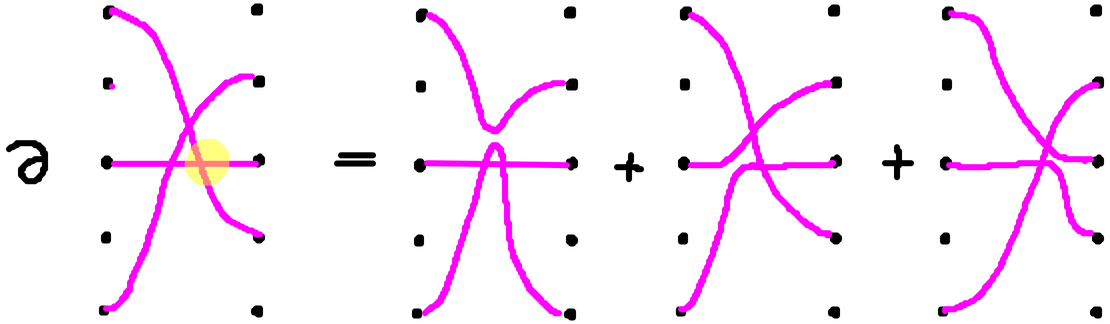
THE ALGEBRA $\mathcal{A}_n$

differential: resolving intersections

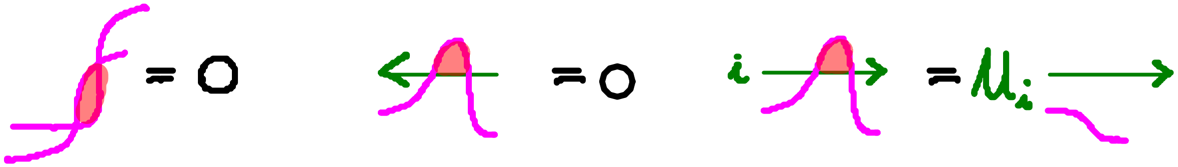


THE ALGEBRA $\mathcal{A}_n$

differential: resolving intersections

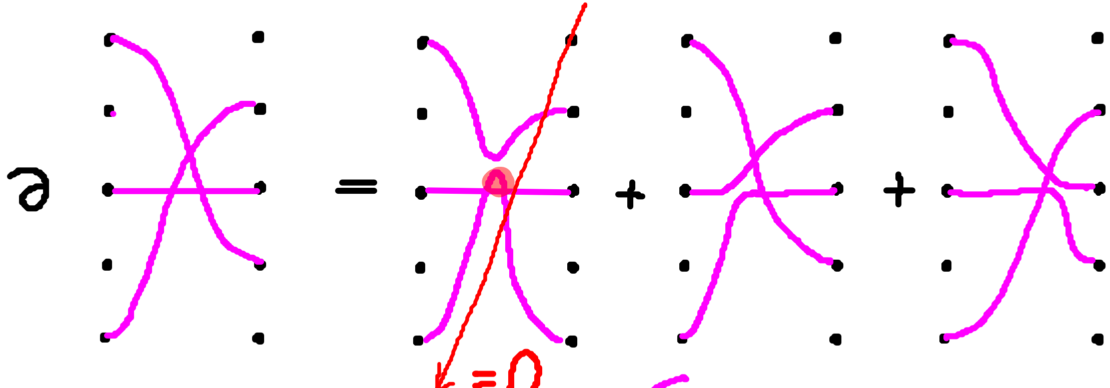


& the **relation:**

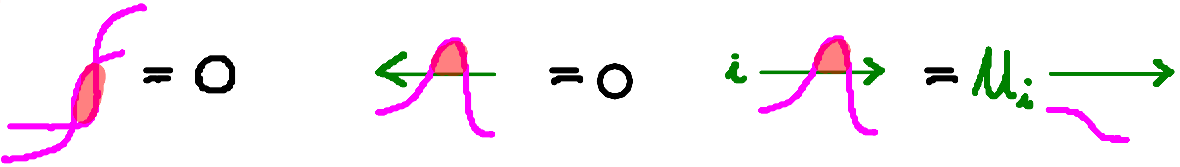


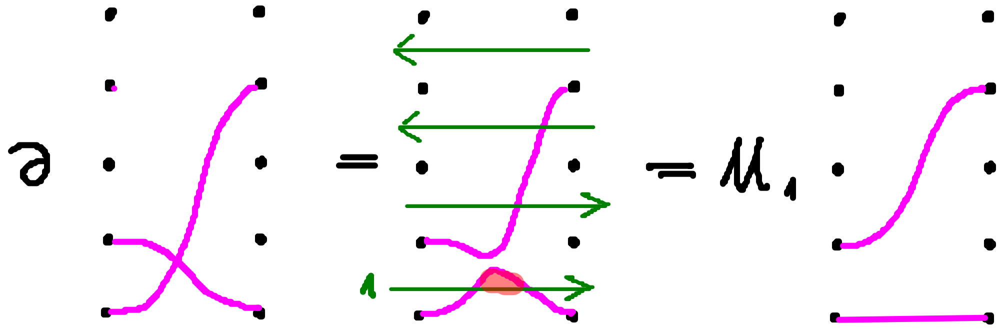
THE ALGEBRA A_n

differential: resolving intersections

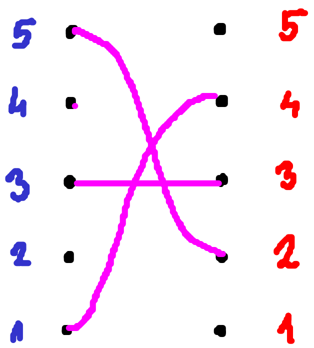


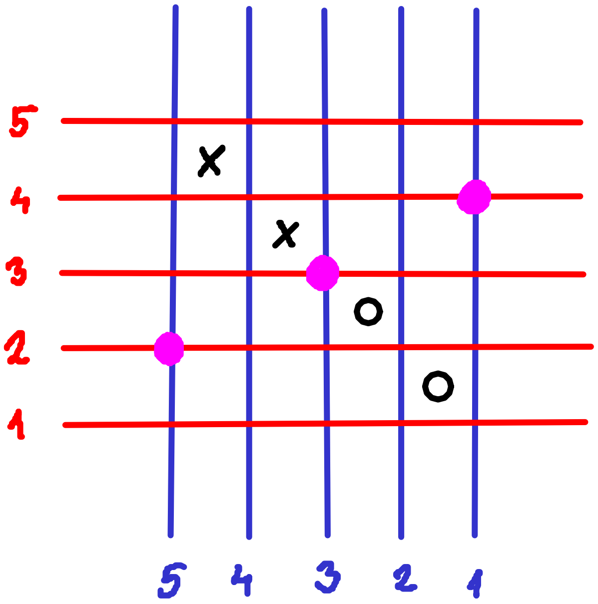
& the **relation**:





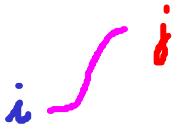
ANOTHER DESCRIPTION OF $\mathcal{A}_n$

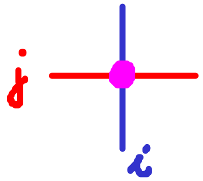




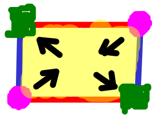
generators ⟷ tuples of intersection points

at most 1 on each curve





differential ⟷ empty rectangles

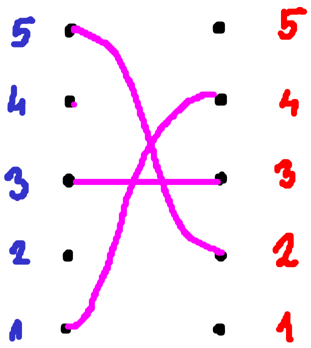


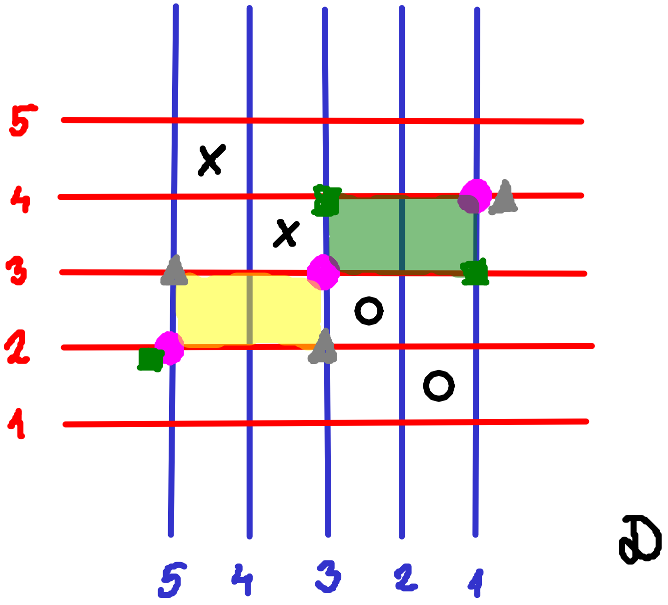
(no X,

no ●

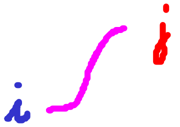
keep track of O_i by U_i)

ANOTHER DESCRIPTION OF $\mathcal{A}_n$

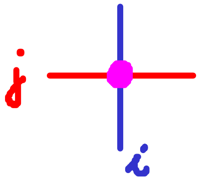




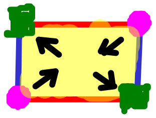
generators ⟷ tuples of intersection points



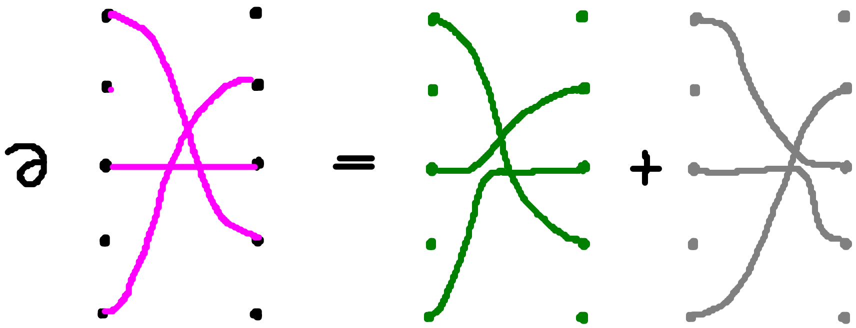
at most 1 on each curve



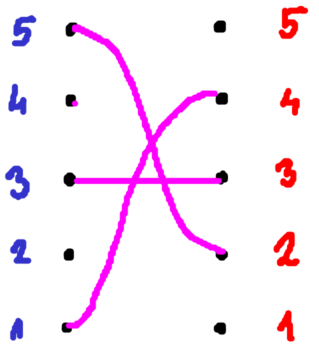
differential ⟷ empty rectangles

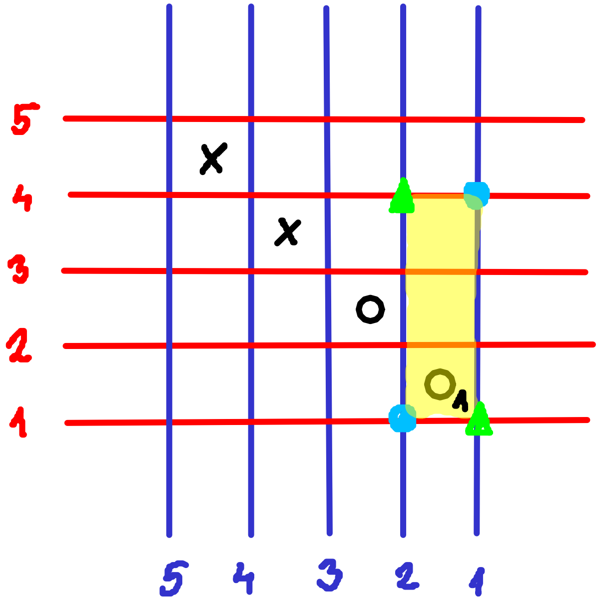


(no X,
no ●
keep track of O_i by U_i)



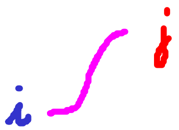
ANOTHER DESCRIPTION OF $\mathcal{A}_n$

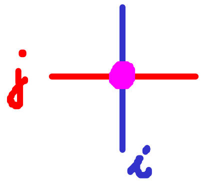




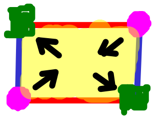
generators ⟷ tuples of intersection points

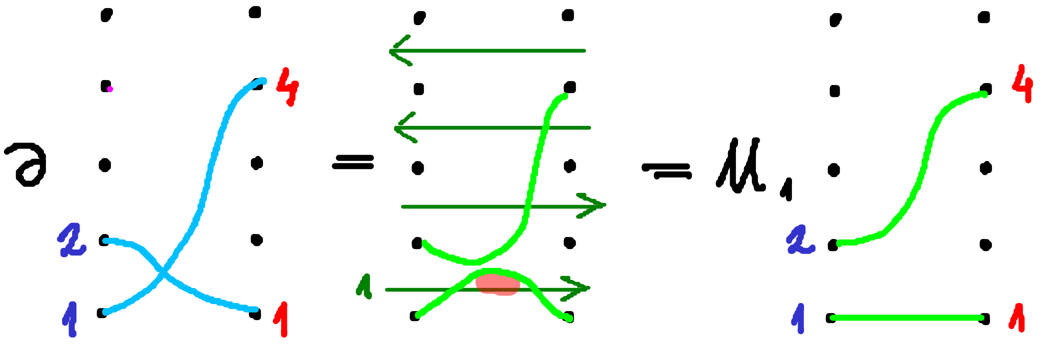
at most 1 on each curve





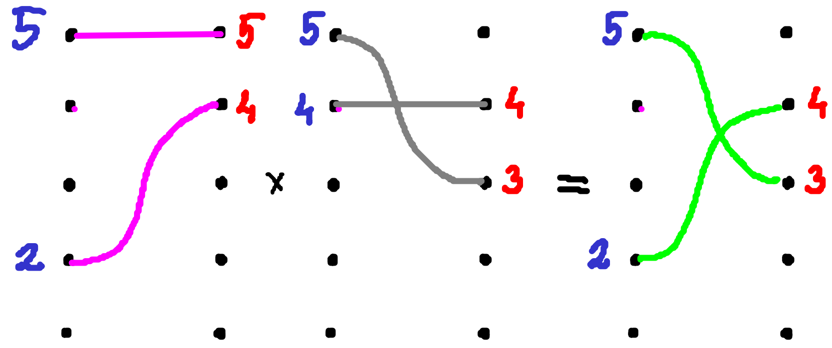
differential ⟷ empty rectangles



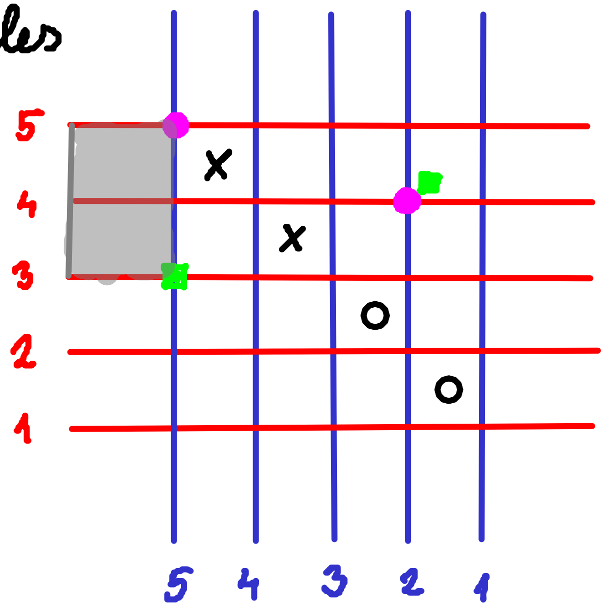


ANOTHER DESCRIPTION OF $\mathcal{A}_n$

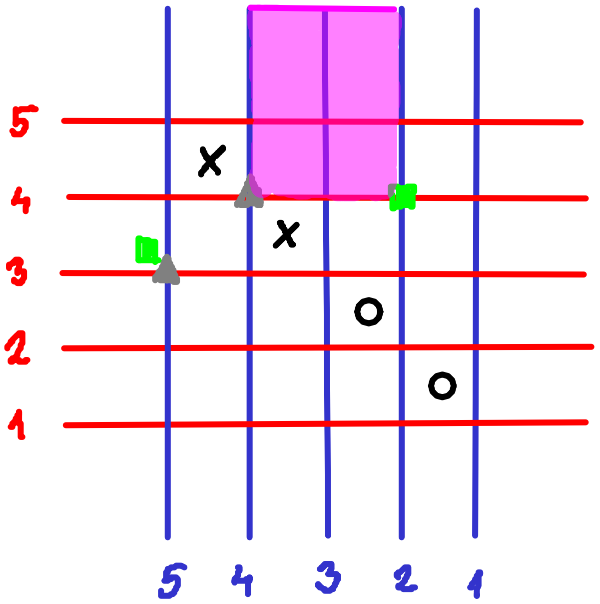
multiplication ⟷ half rectangles



- right action:

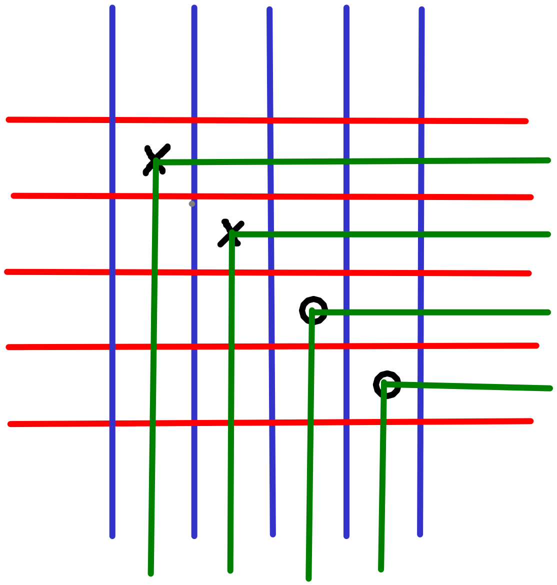


- left action



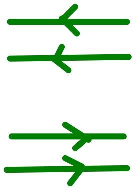
Thus the diagram gives $\mathcal{A}_n$ as a bimodule : ${}_{\mathcal{A}_n}\mathcal{A}_{n\,\mathcal{A}_n}$

BACK TO BRAIDS

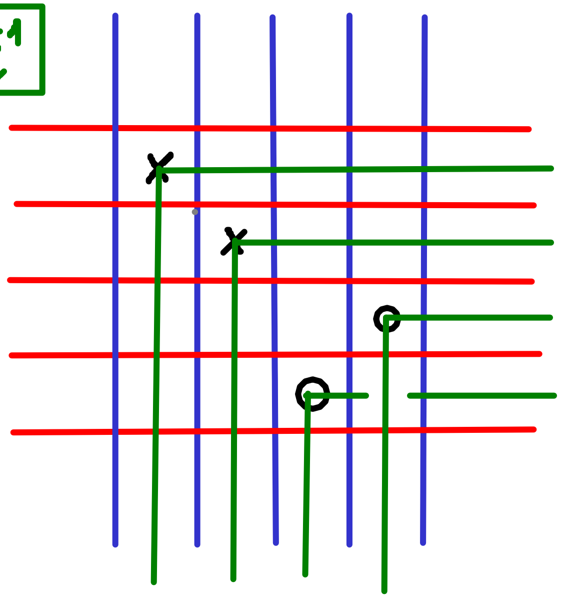


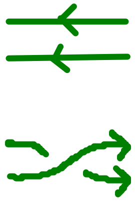
$\boxed{\mathbb{1}}$

trivial braid

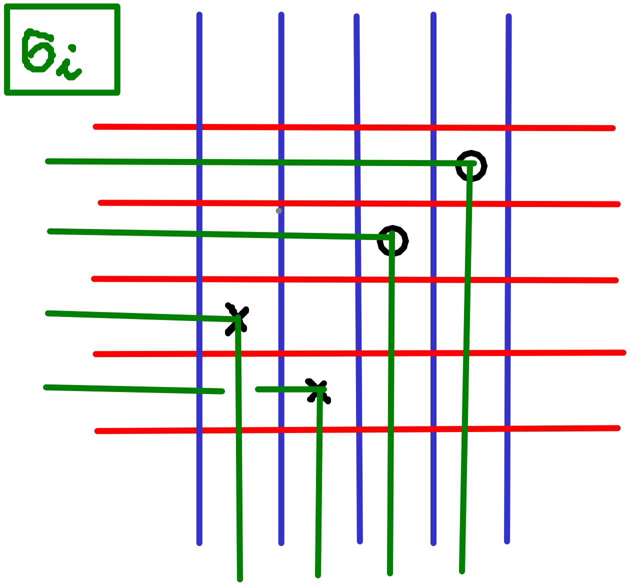


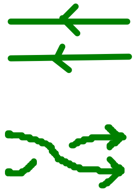
$\boxed{\sigma_i^{-1}}$





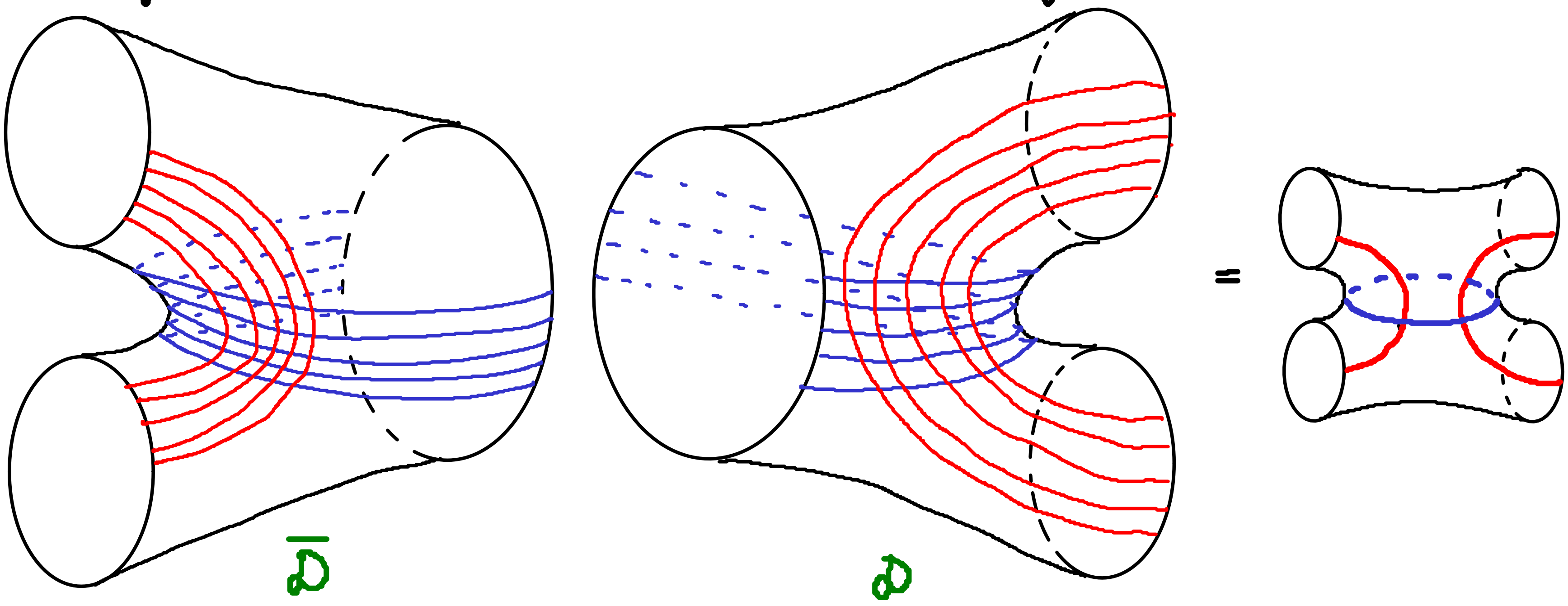
$\boxed{\sigma_i}$





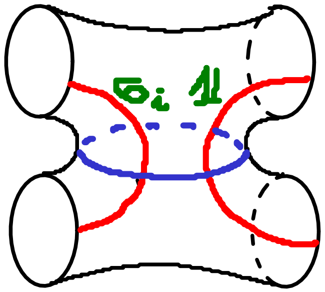
BUILDING BLOCKS

each of the above can be put on the diagram

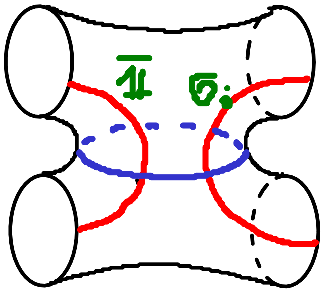


diagram

$\mathcal{H}(\sigma_i)$



$\mathcal{H}(\sigma_i^{-1})$



bimodule

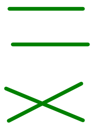
${}_{\mathcal{A}_n}\mathcal{M}_i{}_{\mathcal{A}_n}$

${}_{\mathcal{A}_n}\overline{\mathcal{M}}_i{}_{\mathcal{A}_n}$

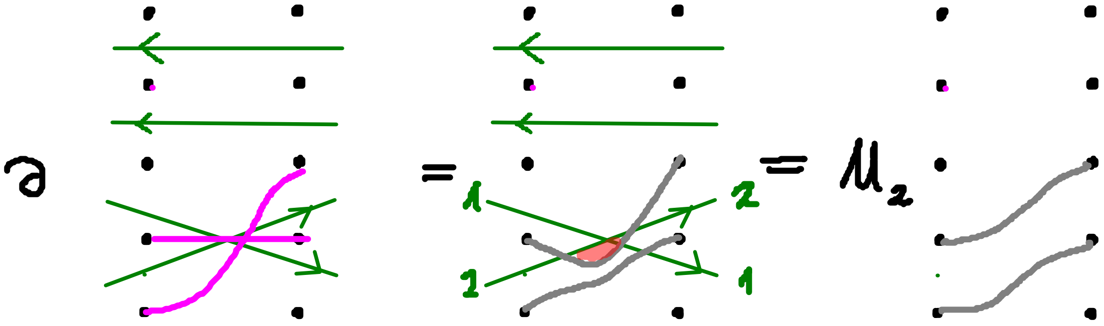
MODULES CORRESPONDING TO σ_i & σ_i^{-1}

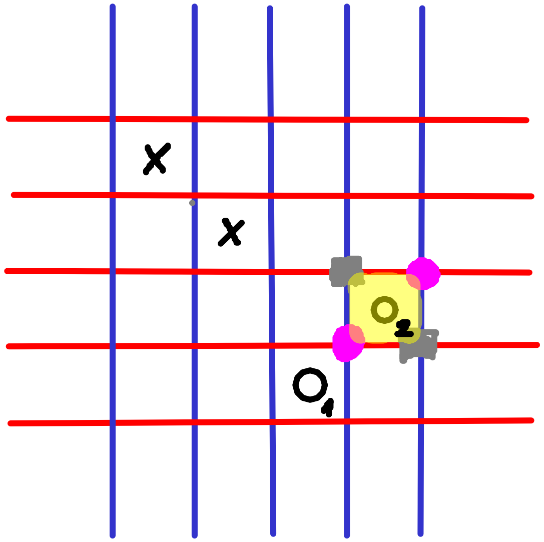
generators : same

differential : same except the relations are defined by the new



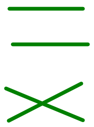
curves

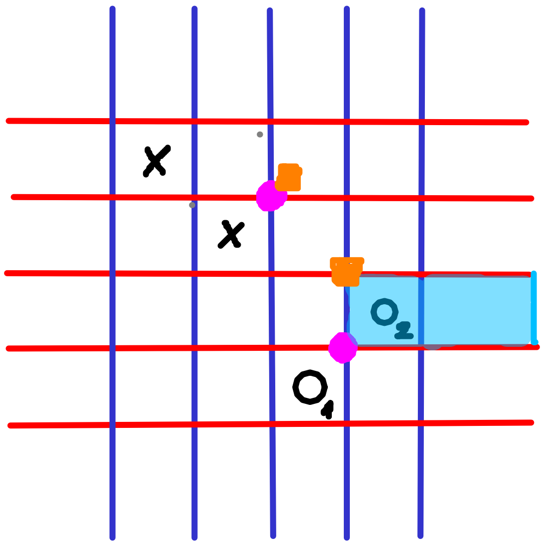




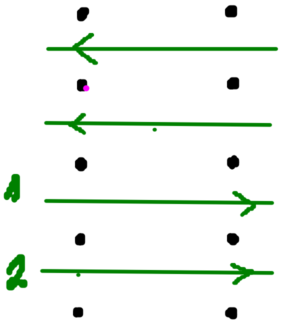
MODULES CORRESPONDING TO σ_i & σ_i^{-1}

generators : same

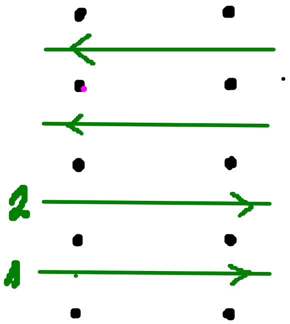
differential : same except the relations are defined by the new  curves



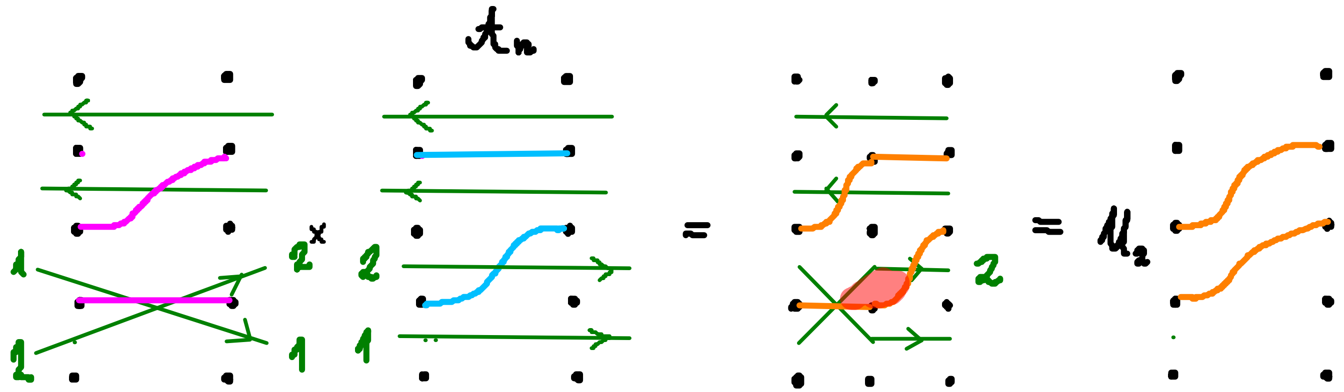
multiplication by



from right &

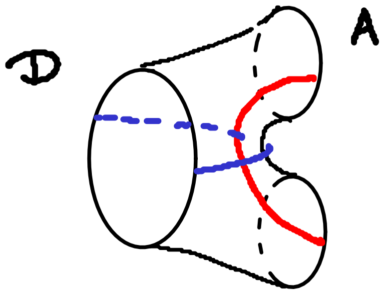


from the left



TYPE DA-STRUCTURE

In order to understand gluing we need to work with DA-structures

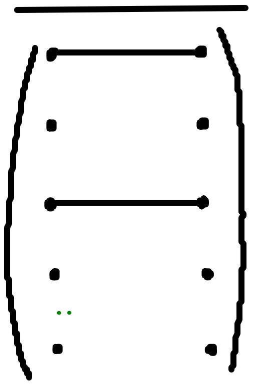


$^{\overline{\mathcal{A}}_n} N_{\mathcal{A}_n}$

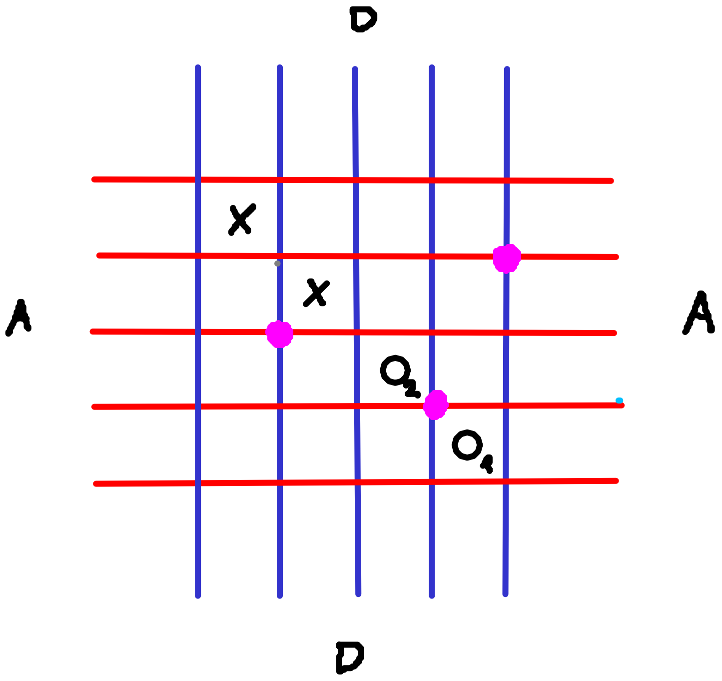
- where the left $\mathcal{A}_n$-action is the usual
- $\overline{\mathcal{A}}_n$ only acts by its idempotents:

$$I \otimes x = \begin{cases} x & I = I_D(x) \\ 0 & \text{otherwise} \end{cases}$$

$I_D(\bullet) =$



(the non-occupied strands)



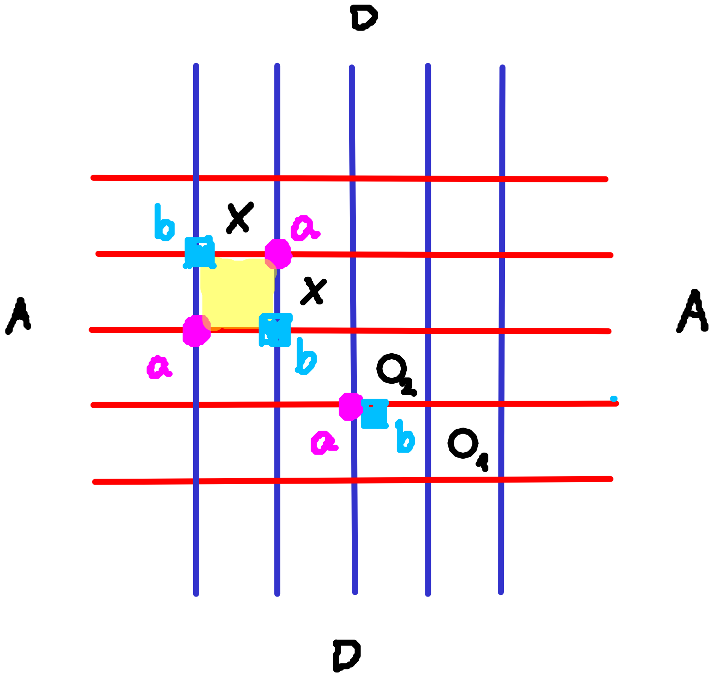
- and another map: $\boxed{\delta^1 : N \longrightarrow \overline{\mathcal{A}}_n \otimes N}$ defined by rectangles and partial rectangles

(no X,
no ●
keep track of O_i by U_i)

TYPE DA-STRUCTURE — δ^1

$$\delta^1 : \mathcal{N} \longrightarrow \overline{\mathcal{A}}_n \otimes \mathcal{N}$$

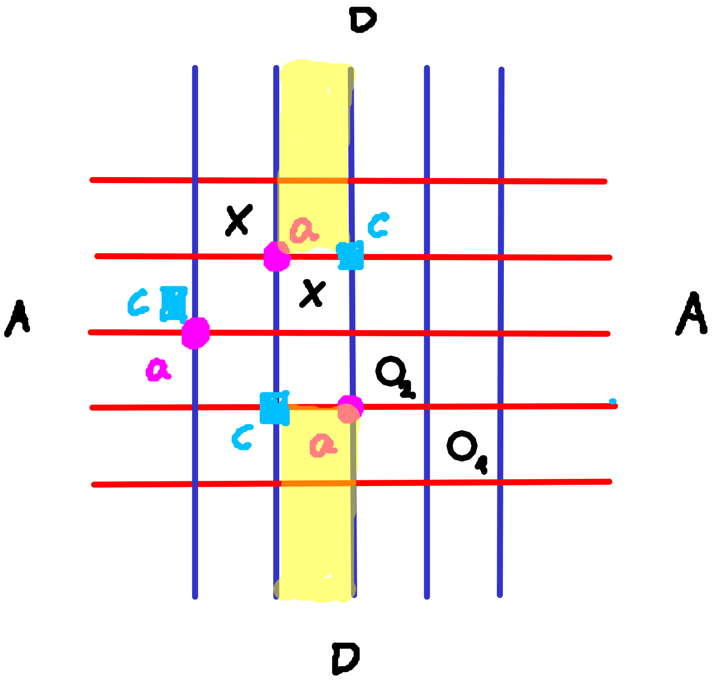
$$\delta^1(a) = I_D(a) \otimes b$$



TYPE DA-STRUCTURE — δ^1

$$\delta^1 : \mathcal{N} \longrightarrow \overline{\mathcal{A}_n} \otimes \mathcal{N}$$

$$\delta^1(a) = I_D(a) \otimes b + I_D(a) \otimes c \; +$$

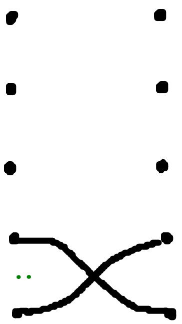


TYPE DA-STRUCTURE — δ^1

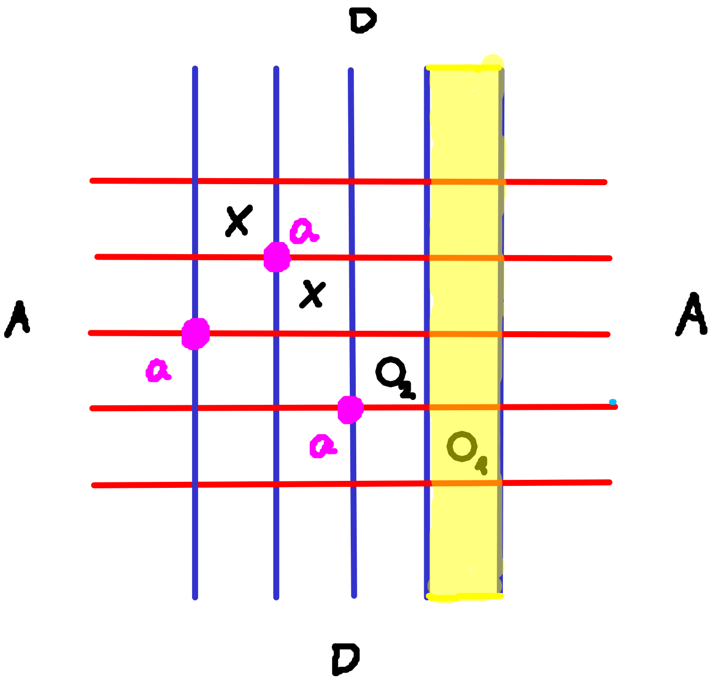
$$\delta^1 : N \longrightarrow \overline{\mathcal{A}_n} \otimes N$$

$$\delta^1(a) = I_D(a) \otimes b + I_D(a) \otimes c +$$

$+ \mathcal{M}_1$



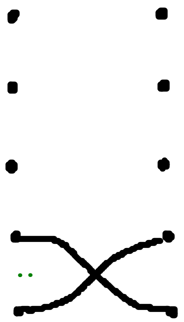
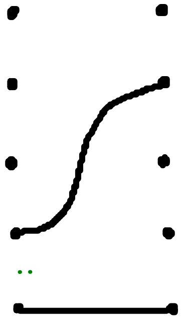
$\otimes \; a$

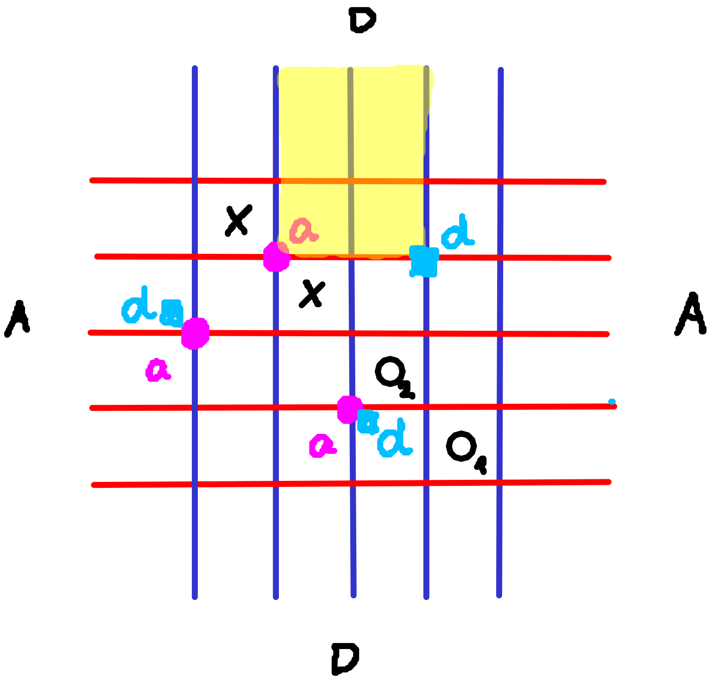


TYPE DA-STRUCTURE — δ^1

$$\delta^1 : \mathcal{N} \longrightarrow \overline{\mathcal{A}}_n \otimes \mathcal{N}$$

$$\delta^1(a) = I_D(a) \otimes b + I_D(a) \otimes c +$$

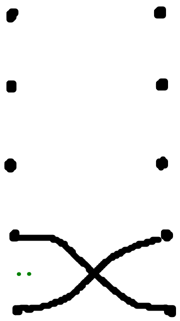
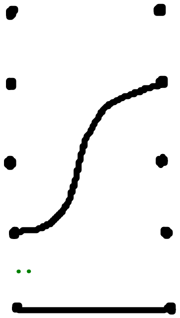
$+\mathcal{M}_1$  $\otimes\, a \;+$  $\otimes\, d$

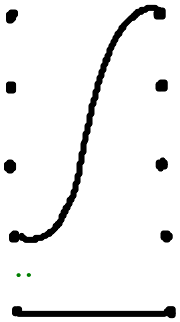


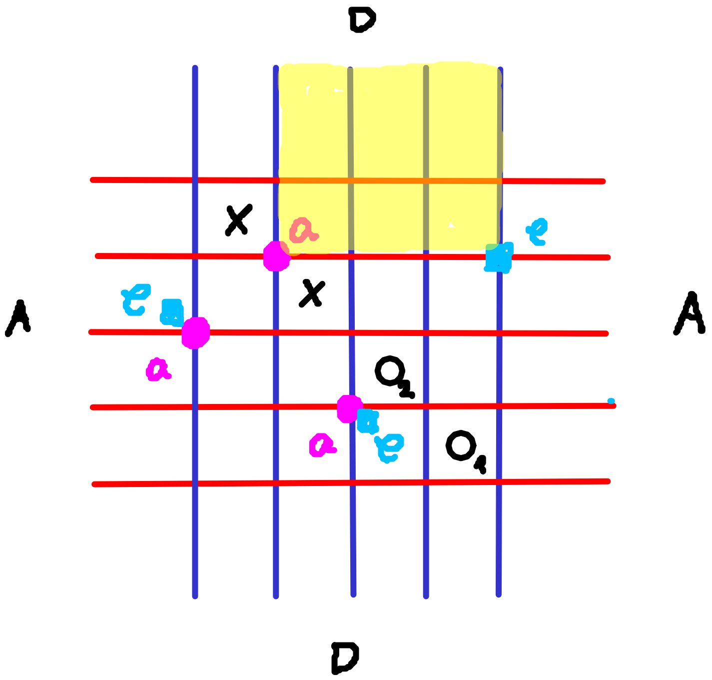
TYPE DA-STRUCTURE — δ^1

$$\delta^1: \mathcal{N} \longrightarrow \overline{\mathcal{A}}_n \otimes \mathcal{N}$$

$$\delta^1(a) = I_D(a) \otimes b + I_D(a) \otimes c +$$

$+ \mathcal{M}_1$  $\otimes a +$  $\otimes d$

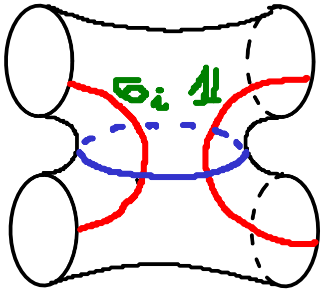
$+$  $\otimes e$



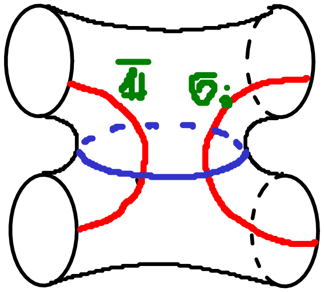
DA-TYPE BIMODULE FOR A BRAID

Remember:

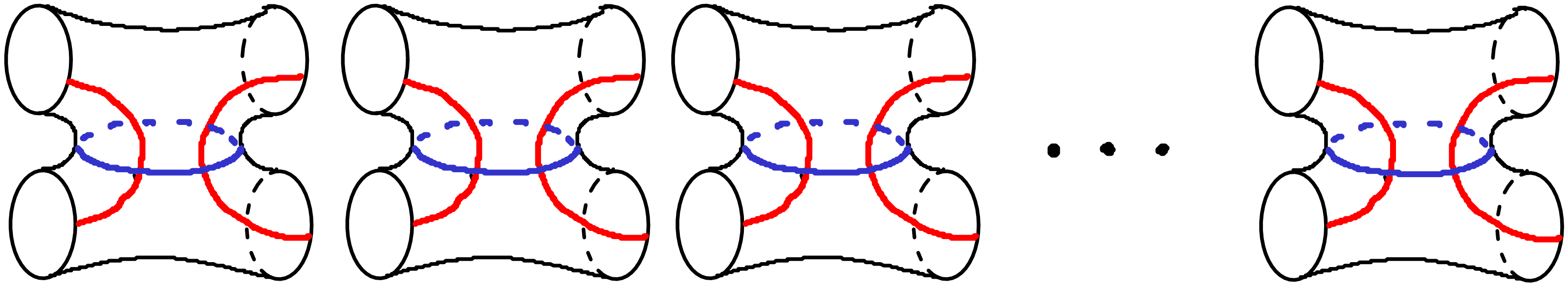
$\mathcal{H}(\sigma_i)$



$\mathcal{H}(\sigma_i^{-1})$



$$\boxed{w = \sigma_1 \sigma_2^{-1} \sigma_4 \cdots \sigma_2 \in \mathcal{B}_n}$$



$= \mathcal{H}(w)$

$\mathcal{H}(\sigma_1)$ $\mathcal{H}(\sigma_2^{-1})$ $\mathcal{H}(\sigma_4)$ $\mathcal{H}(\sigma_2)$

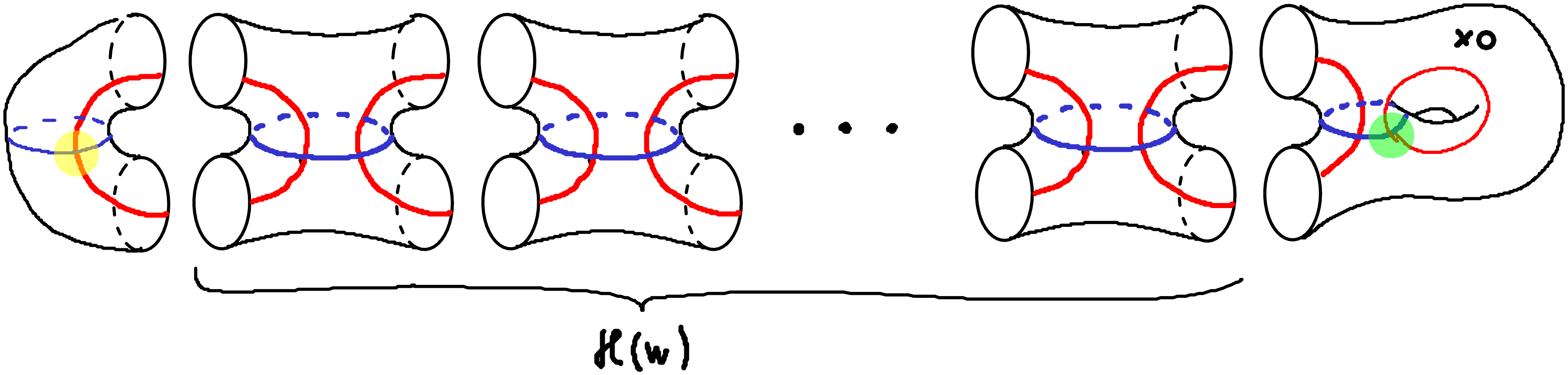
$${}^{\mathcal{A}}\mathcal{M}_{1\,\overline{\mathcal{A}}} \;\tilde{\otimes}\; {}^{\mathcal{A}}\overline{\mathcal{M}}_{2\,\overline{\mathcal{A}}} \;\tilde{\otimes}\; {}^{\mathcal{A}}\mathcal{M}_{4\,\overline{\mathcal{A}}} \;\tilde{\otimes}\; \cdots \;\tilde{\otimes}\; {}^{\mathcal{A}}\mathcal{M}_{2\,\overline{\mathcal{A}}} = \mathrm{CFTAD}$$

such that :
- $\mathcal{M}_i \,\tilde{\otimes}\, \overline{\mathcal{M}}_i \simeq [\mathbb{1}]$
- $\mathcal{M}_i \,\tilde{\otimes}\, \mathcal{M}_{i+1} \otimes \mathcal{M}_i \simeq \mathcal{M}_{i+1} \,\tilde{\otimes}\, \mathcal{M}_i \otimes \mathcal{M}_{i+1}$
- $\mathcal{M}_i \,\tilde{\otimes}\, \mathcal{M}_j \simeq \mathcal{M}_j \,\tilde{\otimes}\, \mathcal{M}_i$ if $|i-j| \geqslant 2$

these are in progress!

RELATION TO KNOT FLOER HOMOLOGY

$w \in B_n$ a braid word



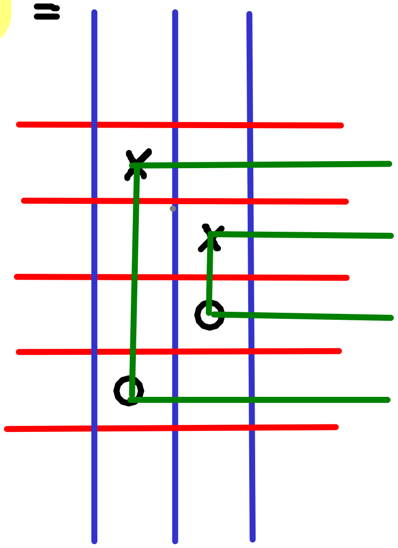
— = 2n+1 parallel α-curves

— = n+1 "

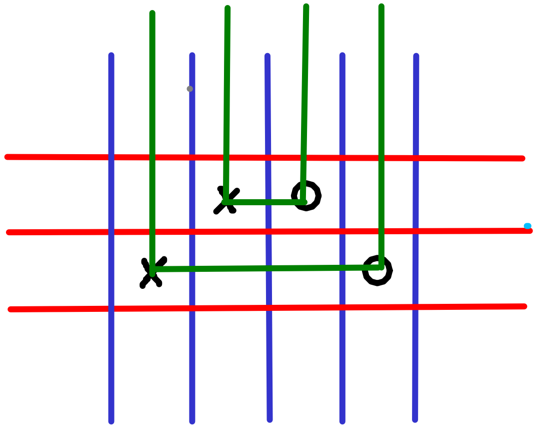
— = 2n+1 parallel β-curves

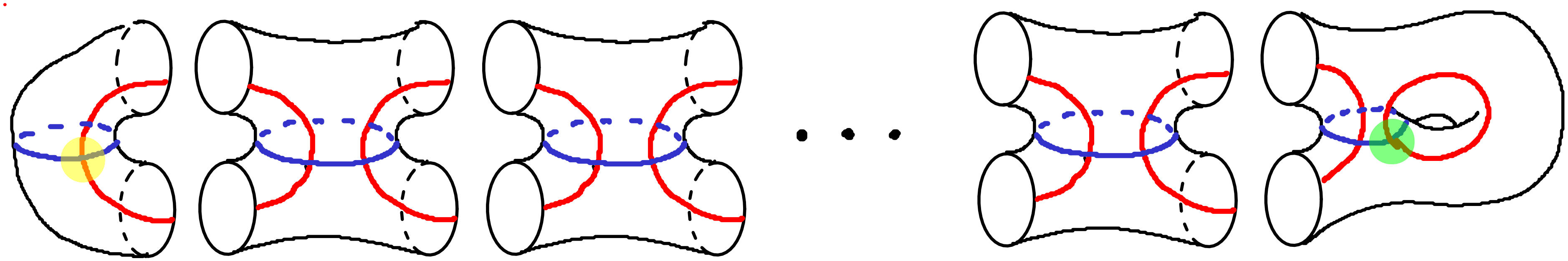
— = n+1 "

🟡 =



🟢 =





THANKS FOR YOUR ATTENTION!