On a Kronecker limit formula for real quadratic fields

Dedicated to Professor Y. Kawada for his 60th birthday

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Introduction.

0-1. The study of Kronecker limit formulas for real quadratic fields was initiated by E. Hecke [4] and has been developed by several authors. We refer to E. Hecke [6], [7], G. Herglotz [8], C. Meyer [10], C. L. Siegel [13] and D. Zagier [18].

Applying the limit formula, they studied analytic expressions for the value at $s=1$ of Hecke's L-series $L_F(s, \chi)$ associated with a character χ of the group of the narrow ideal classes of a real quadratic field $F \subset R$ modulo $\mathfrak{f}$ ($\mathfrak{f}$ is an integral ideal of F). For an integral principal ideal (μ) of F, $\chi((\mu))$ is given by one of the following three formulas:

(i) $\chi((\mu))=\chi_0(\mu)$,
(ii) $\chi((\mu))=\chi_0(\mu) \operatorname{sgn} N(\mu)$,
(iii) $\chi((\mu))=\chi_0(\mu) \operatorname{sgn}(\mu)$,
(iii)′ $\chi((\mu))=\chi_0(\mu) \operatorname{sgn}(\mu')$,

where χ_0 is a character of the group of invertible residue classes modulo $\mathfrak{f}$ of F and $N(\mu)$ is the norm of μ. Those authors studied $L_F(1, \chi)$ mainly for χ of type (i) or (ii) and obtained quite remarkable results. In this paper we present a formula which represents $L_F(1, \chi)$, for χ of type (iii), as a finite linear combination of special values of the logarithm of the *double gamma function* previously studied by E. W. Barnes.

0-2. For a pair $\omega=(\omega_1, \omega_2)$ of positive numbers, set

$$-\gamma_{21}(\omega)=\frac{1}{\omega_1^2}\sum_{n=1}^{\infty}\left\{\psi'\left(\frac{n\omega_2}{\omega_1}\right)-\frac{\omega_1}{n\omega_2}\right\}+\frac{\pi^2}{6\omega_1^2}-\frac{1}{\omega_1\omega_2}\log\omega_2+\frac{\gamma}{\omega_1\omega_2}$$

and

$$-\gamma_{22}(\omega)=\frac{1}{\omega_1}\sum_{n=1}^{\infty}\left\{\psi\left(\frac{n\omega_2}{\omega_1}\right)-\log\left(\frac{n\omega_2}{\omega_1}\right)+\frac{\omega_1}{2n\omega_2}\right\}+\frac{1}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\log\omega_1$$
$$-\frac{1}{2\omega_1}(\gamma-\log 2\pi)+\frac{\omega_1-\omega_2}{2\omega_1\omega_2}\log\left(\frac{\omega_2}{\omega_1}\right)-\frac{\gamma}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right),$$

where γ is the Euler constant and ψ is the logarithmic derivative of the gamma function.

Further, put

$$\Gamma_2(z,\omega)^{-1}=z\exp\left\{\gamma_{22}(\omega)z+\frac{z^2}{2}\gamma_{21}(\omega)\right\}$$

$$\times\Pi'\left(1+\frac{z}{n\omega_1+m\omega_2}\right)\exp\left\{-\frac{z}{n\omega_1+m\omega_2}+\frac{z^2}{2(n\omega_1+m\omega_2)^2}\right\}$$

where the product is over all pairs $(n, m)\neq 0$ of *non-negative* integers. The function $\Gamma_2(z,\omega)$ is the *double gamma function* which was introduced and studied in detail by E. W. Barnes in [1] and [2].

It satisfies the following difference equations:

$$\Gamma_2(z+\omega_1,\omega)=\sqrt{2\pi}\,\Gamma\left(\frac{z}{\omega_2}\right)^{-1}\Gamma_2(z,\omega)\exp\left\{\left(\frac{1}{2}-\frac{z}{\omega_2}\right)\log\omega_2\right\},$$

$$\Gamma_2(z+\omega_2,\omega)=\sqrt{2\pi}\,\Gamma\left(\frac{z}{\omega_1}\right)^{-1}\Gamma_2(z,\omega)\exp\left\{\left(\frac{1}{2}-\frac{z}{\omega_1}\right)\log\omega_1\right\}.$$

Let F be the real quadratic field with discriminant d and let χ be a primitive character of the group of ideal classes modulo $\mathfrak{f}$ of F *of type* (iii). Set

$$\xi(s,\chi)=\pi^{-s}(dN(\mathfrak{f}))^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_F(s,\chi).$$

It is known that $\xi(s,\chi)$ satisfies the following functional equation:

$$\xi(s,\chi)=w(\chi)\xi(1-s,\chi^{-1}),$$

where $w(\chi)$ is a certain complex number of modulus 1. Let $\varepsilon>1$ be the fundamental totally positive unit of F.

Choose integral ideals $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{h_0}$ so that they form a complete set of representatives of the *narrow* ideal classes of F. For each k $(1\leqq k\leqq h_0)$, set

$$R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})=\left\{z=x+\varepsilon y\in(\mathfrak{a}_k\mathfrak{f})^{-1};\ \begin{matrix}x,y\in\boldsymbol{Q},\ 0<x\leqq 1,\ 0\leqq y<1,\\ \mathfrak{a}_k\mathfrak{f}(z)\text{ is prime to }\mathfrak{f}\end{matrix}\right\}.$$

We note that the set $R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ is *finite.*

THEOREM. *Notations being as above, the value of $L_F(s,\chi)$ at $s=1$ is given by*

$$w(\chi)^{-1}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi) \tag{0-1}$$

$$=\sum_{k=1}^{h_0}\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi_k^{-1}(z)\log\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\}$$

(z' and ε' are conjugates of z and ε, respectively), where $\chi_k(z)=\chi(\mathfrak{a}_k\mathfrak{f}(z))$.

In particular, assume that χ corresponds to, in class field theory, a quadratic extension K of F with relative discriminant $\mathfrak{d}$ *in which exactly one of the two archimedean primes of F ramifies.* Let ε_0 be the fundamental unit of F and

assume that ε_0 is (up to ± 1) the m-th power $(m\geqq 1)$ of a primitive unit η_1 of K and that η_1 and η form a system of fundamental units of K. Denote by η^σ the conjugate of η with respect to F and assume that $\eta>|\eta^\sigma|>0$.

COROLLARY TO THE THEOREM. *Notations being as above,*

$$(\eta/|\eta^\sigma|)^{h_K}=\prod_{k=1}^{h_0}\prod_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\}^{mh_F\chi_k(z)}$$

where h_K and h_F are class numbers of K and F, respectively, and $\chi_k(z)=\chi(\mathfrak{a}_k\mathfrak{f}(z))$.

The formula (0-1) is further transformed into a linear combinations of logarithms of the function $F(z,(1,\varepsilon))=\Gamma_2(z,(1,\varepsilon))/\Gamma_2(1+\varepsilon-z,(1,\varepsilon))$.

0-3. As mentioned in the introduction to [4], the original motivation for Hecke to undertake the study of Kronecker limit formulas for real quadratic fields was to find transcendental functions suitable special values of which generate abelian extensions of real quadratic fields (cf. [14]). On the other hand, it is conjectured that, if χ is of type (iii), $L_F(1,\chi)$ would be a linear combination of the logarithms of units of certain abelian extensions of F (see [9] and [15]). With these situations in mind, it would be not too optimistic to suppose that the formula (0-1) suggests that double gamma functions may play a role in arithmetic of real quadratic fields. A brief summary of the present paper has been announced in [12].

0-4. The present article consists of three sections. The first section consists of six subsections. In 1 and 2 the difference equations for the double gamma function are proved. In 3 an integral representation for the logarithm of the double gamma function is established. Both are due to Barnes (though the proofs presented in this paper may be somewhat different from those of Barnes). In 4, we derive a formula which represents the first derivative at $s=0$ of the following Dirichlet series (0-2) in terms of the logarithms of the double gamma function.

$$\sum_{m,n=0}^{\infty}\left\{\prod_{i=1}^{2}L_i(m,n)^{-s}\right\}, \tag{0.2}$$

where L_1 and L_2 are inhomogeneous linear forms in m and n with positive coefficients. In 5, we obtain an asymptotic series for $\log\Gamma_2$ which is useful for the numerical computations. In 6, an infinite product expression for the function $\Gamma_2(z,\omega)/\Gamma_2(\omega_1+\omega_2-z,\omega)$ is proved. It is reminiscent of the infinite products for elliptic theta functions. The second section consists of four subsections. In 1, we show that any L-function of a real quadratic field is a finite linear combination of Dirichlet series of type (0.2) (cf. [11] and [18]). In 2, it is shown that if χ is primitive and of type (iii), $L_F(1,\chi)$ coincides, up to an elementary factor, with $\{(d/ds)L_F(s, \chi)\}_{s=0}$. Thus, the result of § 1, 4 yields our Theorem 1. In 3, the Corollary is derived. In 4, the formula (0.1) is transformed into a more suggestive form (see Corollary 2 to Theorem 1). In section 3, we discuss several numerical examples.

Notation. As usual, we denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. We denote by $\Gamma(z)$, $\zeta(s)$ and by $B_k(z)$ the gamma function, the Riemann zeta function and the k-th Bernoulli polynomial, respectively. We set $\psi(z)=\Gamma'(z)/\Gamma(z)$. The k-th Bernoulli number B_k is given by $B_k=B_k(0)$.

§ 1. The theory of double gamma functions was initiated and developed by E. W. Barnes in his memoires [1] and [2]. In this section we discuss certain properties of double gamma functions which are necessary for later applications. Propositions 1 and 2 are due to Barnes, while Propositions 3, 4 and 5 seem to be new.

1. Let $\omega=(\omega_1, \omega_2)$ be a pair of *positive* numbers. Set

(1.1)
$$-\gamma_{22}(\omega)=\frac{1}{\omega_1}\sum_{n=1}^{\infty}\Big\{\psi\Big(\frac{n\omega_2}{\omega_1}\Big)-\log\Big(\frac{n\omega_2}{\omega_1}\Big)+\frac{\omega_1}{2n\omega_2}\Big\}$$
$$+\frac{1}{2}\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big)\log\omega_1-\frac{1}{2\omega_1}(\gamma-\log 2\pi)$$
$$+\frac{\omega_1-\omega_2}{2\omega_1\omega_2}\log\Big(\frac{\omega_2}{\omega_1}\Big)-\frac{\gamma}{2}\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big),$$

where we denote by γ the Euler constant and by ψ the logarithmic derivative of the gamma function. Further, put

(1.2)
$$-\gamma_{21}(\omega)=\frac{1}{\omega_1^2}\sum_{n=1}^{\infty}\Big\{\psi'\Big(\frac{n\omega_2}{\omega_1}\Big)-\frac{\omega_1}{n\omega_2}\Big\}+\frac{\pi^2}{6\omega_1^2}-\frac{1}{\omega_1\omega_2}\log\omega_2+\frac{\gamma}{\omega_1\omega_2}.$$

LEMMA 1. *Notations being as above, both $\gamma_{21}(\omega)$ and $\gamma_{22}(\omega)$ are symmetric with respect to ω_1 and ω_2.*

PROOF. If $0<\operatorname{Re} s<1$,
$$\int_0^{\infty}\frac{t^{s-1}}{(n+t)^2}dt=\Gamma(s)\Gamma(2-s)n^{s-2}\qquad(n>0).$$
Since $\psi'(t)=t^{-2}+\sum_{n=1}^{\infty}(n+t)^{-2}$, the Mellin inversion formula implies that, if $t>0$,

(1.3)
$$\psi'(t)-t^{-2}=\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_0}\zeta(2-s)\Gamma(s)\Gamma(2-s)t^{-s}ds\qquad(0<\sigma_0<1)$$
$$=\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_1}\zeta(2-s)\Gamma(s)\Gamma(2-s)t^{-s}ds+t^{-1}\qquad(1<\sigma_1<2).$$

Thus, if one puts $f_1(t)=\sum_{n=1}^{\infty}\Big\{\psi'(nt)-\frac{1}{nt}\Big\}$, one has
$$f_1(t)=\frac{\pi^2}{6t^2}+\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_1}\zeta(2-s)\zeta(s)\Gamma(s)\Gamma(2-s)t^{-s}ds$$
$$=\frac{\pi^2}{6t^2}+\frac{1}{2\pi i}\int_{\operatorname{Re} s=2-\sigma_1}\zeta(2-s)\zeta(s)\Gamma(s)\Gamma(2-s)t^{-s}ds+t^{-1}\log t$$
$$=\frac{\pi^2}{6t^2}+t^{-1}\log t+\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_1}\zeta(2-s)\zeta(s)\Gamma(2-s)\Gamma(s)t^{s-2}ds$$
$$=\frac{\pi^2}{6t^2}+t^{-1}\log t+t^{-2}f_1\Big(\frac{1}{t}\Big)-\frac{\pi^2}{6}.$$
Since
$$-\gamma_{21}(\omega)=\frac{1}{\omega_1^2}f_1\Big(\frac{\omega_2}{\omega_1}\Big)-\frac{1}{\omega_1\omega_2}\log\omega_2+\frac{\gamma}{\omega_1\omega_2}+\frac{\pi^2}{6\omega_1^2},$$
the equality $\gamma_{21}(\omega_1, \omega_2)=\gamma_{21}(\omega_2, \omega_1)$ follows immediately from the equality $f_1(t)+\pi^2/6=\pi^2/6t^2+t^{-1}\log t+t^{-2}f_1(1/t)$. It follows from (1.3) that
$$\psi(t)+t^{-1}=-\gamma+\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_0}\zeta(2-s)\Gamma(s)\Gamma(1-s)t^{1-s}ds\qquad(0<\sigma_0<1)$$
$$=\log t+\frac{1}{2t}+\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_2}\zeta(2-s)\Gamma(s)\Gamma(1-s)t^{1-s}ds\quad(2<\sigma_2<3).$$
Now put $f_2(t)=\sum_{n=1}^{\infty}\{\psi(nt)-\log(nt)+1/2nt\}$. Then one has
$$f_2(t)=\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_2}\zeta(s-1)\zeta(2-s)\Gamma(s)\Gamma(1-s)t^{1-s}ds$$
$$=\frac{1}{2t}\log t+\frac{1}{t}\log\sqrt{2\pi}-\frac{\gamma}{2t}+\frac{1}{2}\log t-\log\sqrt{2\pi}+\frac{\gamma}{2}$$
$$+\frac{1}{2\pi i}\int_{\operatorname{Re} s=3-\sigma_2}\zeta(s-1)\zeta(2-s)\Gamma(s)\Gamma(1-s)t^{1-s}ds.$$
Thus,

(1.4)
$$f_2(t)+\log\sqrt{2\pi}-\frac{\gamma}{2}-\frac{1}{2}\log t$$
$$=\frac{1}{t}\Big\{f_2\Big(\frac{1}{t}\Big)-\frac{\gamma}{2}+\log\sqrt{2\pi}-\frac{1}{2}\log t^{-1}\Big\}.$$

Since
$$-\gamma_{22}(\omega)=\frac{1}{\omega_1}\Big\{f_2\Big(\frac{\omega_2}{\omega_1}\Big)+\log\sqrt{2\pi}-\frac{\gamma}{2}-\frac{1}{2}\log\Big(\frac{\omega_2}{\omega_1}\Big)\Big\}$$
$$+\frac{1}{2}\Big(\frac{1}{\omega_2}\log\omega_1+\frac{1}{\omega_1}\log\omega_2\Big)$$
$$+\frac{\omega_1-\omega_2}{2\omega_1\omega_2}\log\Big(\frac{\omega_2}{\omega_1}\Big)-\frac{\gamma}{2}\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big),$$

the equality $\gamma_{22}(\omega_1, \omega_2)=\gamma_{22}(\omega_2, \omega_1)$ follows from (1.4).

2. For a pair $\omega=(\omega_1, \omega_2)$ of positive numbers, set

(1.5) $$\Gamma_2^{-1}(z, \omega)=z\exp\left\{\gamma_{22}(\omega)z+\frac{z^2}{2}\gamma_{21}(\omega)\right\}$$
$$\times\prod{}'\left(1+\frac{z}{m\omega_1+n\omega_2}\right)\exp\left\{-\frac{z}{m\omega_1+n\omega_2}+\frac{z^2}{2(m\omega_1+n\omega_2)^2}\right\},$$

where the product is taken over all pairs (m, n) of non-negative integers which are not simultaneously equal to zero ($\gamma_{22}(\omega)$ and $\gamma_{21}(\omega)$ are given by (1.1) and (1.2) respectively).

PROPOSITION 1. *As a function of z, $\Gamma_2(z, \omega)^{-1}$ is an entire function of order 2 which is symmetric with respect to ω_1 and ω_2. Moreover it satisfies the following difference equations.*

$$\Gamma_2(z+\omega_1, \omega)^{-1}=\frac{1}{\sqrt{2\pi}}\Gamma_2^{-1}(z, \omega)\Gamma\left(\frac{z}{\omega_2}\right)\exp\left\{\left(\frac{z}{\omega_2}-\frac{1}{2}\right)\log\omega_2\right\},$$
$$\Gamma_2(z+\omega_2, \omega)^{-1}=\frac{1}{\sqrt{2\pi}}\Gamma_2^{-1}(z, \omega)\Gamma\left(\frac{z}{\omega_1}\right)\exp\left\{\left(\frac{z}{\omega_1}-\frac{1}{2}\right)\log\omega_1\right\}.$$

PROOF. The first two assertions are immediate consequences of (1.5) and Lemma 1. To prove the difference equations, set

$$f(\omega, z)=z\prod{}'\left(1+\frac{z}{m\omega_1+n\omega_2}\right)\exp\left\{-\frac{z}{m\omega_1+n\omega_2}+\frac{z^2}{2(m\omega_1+n\omega_2)^2}\right\}.$$

Then,

$$\log f(\omega, z)=\log z+\sum_{m=1}^{\infty}\left\{\log(z+m\omega_1)-\log(m\omega_1)-\frac{z}{m\omega_1}+\frac{z^2}{2m^2\omega_1^2}\right\}$$
$$+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left\{\log(z+m\omega_1+n\omega_2)-\log(m\omega_1+n\omega_2)-\frac{z}{m\omega_1+n\omega_2}+\frac{z^2}{2(m\omega_1+n\omega_2)^2}\right\}.$$

Recalling that $\log\Gamma(z)=-\log z-\gamma z-\sum_{n=1}^{\infty}\{\log(n+z)-\log n-(z/n)\}$ and that $\psi(z)=-(1/z)-\gamma-\sum_{n=1}^{\infty}\{1/(n+z)-(1/n)\}$, we have,

$$\log z+\sum_{m=1}^{\infty}\left\{\log(z+m\omega_1)-\log(m\omega_1)-\frac{z}{m\omega_1}+\frac{z^2}{2m^2\omega_1^2}\right\}$$
$$=-\log\Gamma\left(\frac{z}{\omega_1}\right)+\log\omega_1-\frac{\gamma z}{\omega_1}+\frac{\pi^2}{12\omega_1^2}z^2$$

and

$$\sum_{m=0}^{\infty}\left\{\log(z+m\omega_1+n\omega_2)-\log(m\omega_1+n\omega_2)-\frac{z}{m\omega_1+n\omega_2}+\frac{z^2}{2(m\omega_1+n\omega_2)^2}\right\}$$
$$=\log\Gamma\left(\frac{n\omega_2}{\omega_1}\right)-\log\Gamma\left(\frac{z+n\omega_2}{\omega_1}\right)+\frac{z}{\omega_1}\psi\left(\frac{n\omega_2}{\omega_1}\right)+\frac{z^2}{2\omega_1^2}\psi'\left(\frac{n\omega_2}{\omega_1}\right)$$
$$(n=1, 2, 3, \cdots).$$

Hence,

$$\log f(\omega, z)=\log\omega_1-\log\Gamma\left(\frac{z}{\omega_1}\right)-\frac{\gamma z}{\omega_1}+\frac{\pi^2}{12\omega_1^2}z^2$$
$$+\sum_{n=1}^{\infty}\left\{\log\Gamma\left(\frac{n\omega_2}{\omega_1}\right)-\log\Gamma\left(\frac{z+n\omega_2}{\omega_1}\right)+\frac{z}{\omega_1}\log\left(\frac{n\omega_2}{\omega_1}\right)\right.$$
$$\left.-\frac{z}{2n\omega_2}+\frac{z^2}{2n\omega_1\omega_2}\right\}+\frac{z}{\omega_1}\sum_{n=1}^{\infty}\left\{\psi\left(\frac{n\omega_2}{\omega_1}\right)-\log\left(\frac{n\omega_2}{\omega_1}\right)+\frac{\omega_1}{2n\omega_2}\right\}$$
$$+\frac{z^2}{2\omega_1^2}\sum_{n=1}^{\infty}\left\{\psi'\left(\frac{n\omega_2}{\omega_1}\right)-\frac{\omega_1}{n\omega_2}\right\}.$$

Hence,

$$\log\Gamma_2^{-1}(z, \omega)=\frac{z^2}{2\omega_1\omega_2}\log\omega_2-\frac{\gamma z^2}{2\omega_1\omega_2}-\frac{z}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\log\omega_1$$
$$-\frac{z}{2\omega_1}(\gamma+\log 2\pi)-\frac{(\omega_1-\omega_2)z}{2\omega_1\omega_2}\log\left(\frac{\omega_2}{\omega_1}\right)+\frac{\gamma z}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)$$
$$+\log\omega_1-\log\Gamma\left(\frac{z}{\omega_1}\right)+\sum_{n=1}^{\infty}\left\{\log\Gamma\left(\frac{n\omega_2}{\omega_1}\right)-\log\Gamma\left(\frac{z+n\omega_2}{\omega_1}\right)\right.$$
$$\left.+\frac{z}{\omega_1}\log\left(\frac{n\omega_2}{\omega_1}\right)-\frac{z}{2n\omega_2}+\frac{z^2}{2n\omega_1\omega_2}\right\}.$$

Thus,

$$\log\Gamma_2(z+\omega_1, \omega)^{-1}-\log\Gamma_2(z, \omega)^{-1}$$
$$=\frac{z}{\omega_2}\log\omega_2+\frac{1}{2}\log\omega_2-\log\omega_1-\frac{1}{2}\log 2\pi-\frac{\gamma z}{\omega_2}-\log\left(\frac{z}{\omega_1}\right)$$
$$+\sum_{n=1}^{\infty}\left\{-\log\left(\frac{z+n\omega_2}{\omega_1}\right)+\log n+\log\left(\frac{\omega_2}{\omega_1}\right)+\frac{z}{n\omega_2}\right\}$$
$$=\log\Gamma\left(\frac{z}{\omega_2}\right)+\left(\frac{z}{\omega_2}-\frac{1}{2}\right)\log\omega_2-\frac{1}{2}\log 2\pi.$$

The first difference equation has been verified. Since $\Gamma_2(z, \omega)$ is symmetric with respect to ω_1 and ω_2, the second difference equation is a consequence of the first.

3. In this subsection, an integral representation for the logarithm of the double gamma function is derived. We begin with some notational preparations.

For a positive number $\lambda<1$, denote by $I(\lambda, \infty)$ (resp. $I(\lambda, 1)$) the integral path in the complex plane consisting of the oriented linear segment $(+\infty, \lambda)$ (resp. $(1, \lambda)$), the counterclockwise circle of radius λ around the origin and the oriented linear segment $(\lambda, +\infty)$ (resp. $(\lambda, 1)$). The outline of the integral path $I(\lambda, \infty)$ (resp. $I(\lambda, 1)$) is indicated in Figure 1 (resp. Figure 2).

Figure 1.

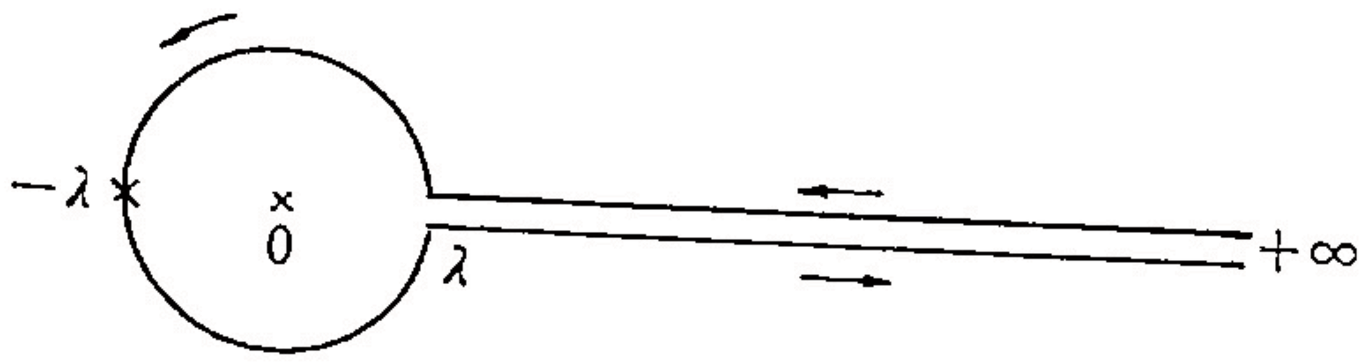
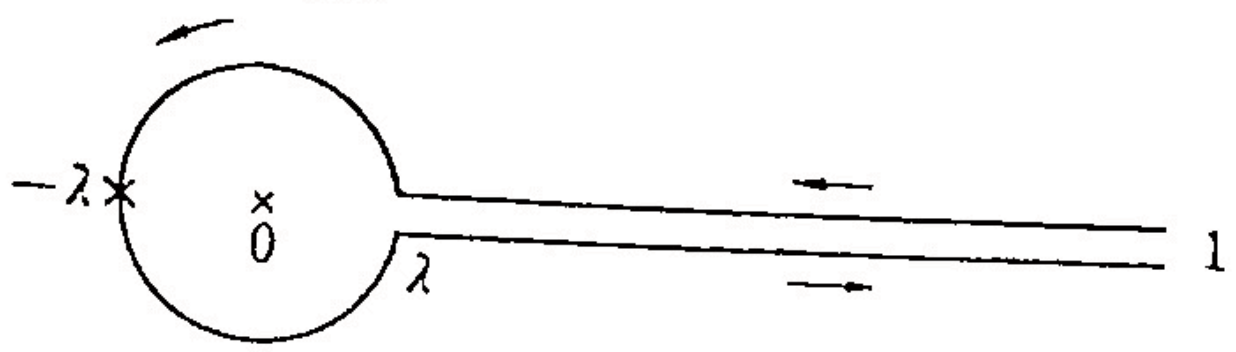


Figure 2.



Let $\zeta(s, z)=\sum_{n=0}^{\infty}(n+z)^{-s}$ ($\mathrm{Re}\, s>1$, $\mathrm{Re}\, z>0$) be the Riemann-Hurwitz zeta function. Then the following formulas for $\zeta(s, z)$ are well-known (see e.g. Chapter 13 of [16]).

(1.6) $$\zeta(s, z)=\frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i}\int_{I(\lambda,\infty)}\frac{\exp(-zt)}{1-\exp(-t)}t^{s-1}dt \quad (0<\lambda<2\pi).$$

(1.7) $$\frac{d}{ds}\zeta(s, z)\Big|_{s=0}=\log\Big\{\frac{\Gamma(z)}{\sqrt{2\pi}}\Big\}.$$

It follows from (1.6) and (1.7) that

(1.8) $$\log\Big\{\frac{\Gamma(z)}{\sqrt{2\pi}}\Big\}=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{\exp(-zt)}{1-\exp(-t)}\frac{\log t}{t}dt+(\gamma-\pi i)\Big(\frac{1}{2}-z\Big).$$

PROPOSITION 2. *Let* $\omega=(\omega_1, \omega_2)$ *be a pair of positive numbers. There exists a positive constant* $\rho_2(\omega)$ *which does not depend upon* z *such that*

(1.9) $$\log\Big\{\frac{\Gamma_2(z, \omega)}{\rho_2(\omega)}\Big\}=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{\exp(-zt)}{\{1-\exp(-\omega_1 t)\}\{1-\exp(-\omega_2 t)\}}\frac{\log t}{t}dt$$
$$+\frac{(\gamma-\pi i)}{2\omega_1\omega_2}\Big\{B_2\Big(\frac{z}{\omega_1}\Big)\omega_1^2+2B_1\Big(\frac{z}{\omega_1}\Big)B_1\omega_1\omega_2+B_2\omega_2^2\Big\}$$
$$\Big(\mathrm{Re}\, z>0,\ 0<\lambda<\frac{2\pi}{\omega_1}, \frac{2\pi}{\omega_2}\Big),$$

where $\log t$ *is understood to be real valued on the upper linear segment* $(+\infty, \lambda)$ *of* $I(\lambda, \infty)$.

PROOF. Denote by $J(z, \omega)$ the right side of (1.9). Then it follows from the equality (1.8) and Proposition 1 that

$$J(z+\omega_1, \omega)-J(z)=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{\exp(-zt)}{\exp(-t\omega_2)-1}\frac{\log t}{t}dt+\frac{\gamma-\pi i}{2\omega_1\omega_2}(2z\omega_1-\omega_1\omega_2)$$
$$=-\log\Gamma\Big(\frac{z}{\omega_2}\Big)+\log\sqrt{2\pi}+\Big(\frac{1}{2}-\frac{z}{\omega_2}\Big)\log(\omega_2)$$
$$=\log\Gamma_2(z+\omega_1, \omega)-\log\Gamma_2(z, \omega).$$

Hence,
$$J(z+\omega_1, \omega)-\log\Gamma_2(z+\omega_1, \omega)=J(z, \omega)-\log\Gamma_2(z, \omega).$$
Since J and $\log\Gamma_2$ are symmetric with respect to ω_1 and ω_2,
$$J(z+\omega_2, \omega)-\log\Gamma_2(z+\omega_2, \omega)=J(z, \omega)-\log\Gamma_2(z, \omega).$$
Thus the function $J(z, \omega)-\log\Gamma_2(z, \omega)$, which is holomorphic in the right half plane, is continued analytically to an entire function with periods ω_1 and ω_2. Thus, if ω_1 and ω_2 are linearly independent over $\boldsymbol{Q}$,
$$\frac{d}{dz}\{J(z, \omega)-\log\Gamma_2(z, \omega)\}=0.$$
Since the left side of the above equality depends continuously upon ω_1 and ω_2,
$$\frac{d}{dz}\{J(z, \omega)-\log\Gamma_2(z, \omega)\}=0, \quad \text{for any } \omega.$$
Thus, $\log\rho_2(\omega)=\log\Gamma_2(z, \omega)-J(z, \omega)$ does not depend upon z.

COROLLARY TO PROPOSITION 2. *For any positive number* t,
$$\log\Gamma_2(tz, t\omega)-\log\Gamma_2(z, \omega)=\Big\{\frac{1}{2}\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big)z-\frac{z^2}{2\omega_1\omega_2}-1\Big\}\log t,$$
$$\log\rho_2(t\omega)-\log\rho_2(\omega)=\Big(\frac{\omega_2}{12\omega_1}+\frac{\omega_1}{12\omega_2}-\frac{3}{4}\Big)\log t.$$

PROOF. The first equality follows immediately from the definition of the double gamma function. It follows from (1.9) that
$$\log\Big\{\frac{\Gamma_2(tz, t\omega)}{\rho_2(t\omega)}\Big\}-\log\Big\{\frac{\Gamma_2(z, \omega)}{\rho_2(\omega)}\Big\}$$
$$=\frac{(-\log t)}{2\pi i}\int_{I(\lambda,\infty)}\frac{\exp(-zu)}{\{1-\exp(-\omega_1 u)\}\{1-\exp(-\omega_2 u)\}}\frac{du}{u}.$$
A simple computation shows that

(1.10) $$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{\exp(-zu)}{\{1-\exp(-\omega_1 u)\}\{1-\exp(-\omega_2 u)\}}\frac{du}{u}$$
$$=\frac{1}{2\omega_1\omega_2}\Big\{B_2\Big(\frac{z}{\omega_1}\Big)\omega_1^2+2B_1\Big(\frac{z}{\omega_1}\Big)B_1\omega_1\omega_2+B_2\omega_2^2\Big\}$$
$$\Big(\mathrm{Re}\, z>0,\ 0<\lambda<\frac{2\pi}{\omega_1}, \frac{2\pi}{\omega_2}\Big).$$

The second equality is now an easy consequence of the first equality.

4. Let $a=(a_1, a_2)$ be a pair of *positive* numbers and let $x=(x_1, x_2)$ be a pair of *non-negative* numbers *which do not vanish simultaneously*. Set $\zeta(s, a, x)=\sum_{m,n=0}^{\infty}\prod_{k=1}^{2}\{x_1+m+(x_2+n)a_k\}^{-s}$. The Dirichlet series $\zeta(s, a, x)$ converges absolutely if $\operatorname{Re} s>1$ and is continued analytically to a meromorphic function in the whole complex plane (see Proposition 1. of [11]).

PROPOSITION 3. *Notations being as above,*

$$\frac{d}{ds}\zeta(s, a, x)\Big|_{s=0}$$
$$=\log\Big\{\frac{\Gamma_2(x_1+x_2a_1, (1, a_1))\Gamma_2(x_1+x_2a_2, (1, a_2))}{\rho_2((1, a_1))\qquad \rho_2((1, a_2))}\Big\}+\frac{a_1-a_2}{4a_1a_2}\log\Big(\frac{a_2}{a_1}\Big)B_2(x_1),$$

(for the notation $\rho_2((1, a))$, see Proposition 2).

PROOF. Let us recall certain computations in the proof of Proposition 1. of [11]. Since

$$\Gamma(s)^2\prod_{k=1}^{2}\{x_1+m+(x_2+n)a_k\}^{-s}$$
$$=\int_0^\infty\int_0^\infty(t_1t_2)^{s-1}\exp\{-(t_1+t_2)(x_1+m)-(x_2+n)(a_1t_1+a_2t_2)\}dt_1dt_2$$
$$(\operatorname{Re} s>0),$$

we have, for $\operatorname{Re} s>1$,

$$\Gamma(s)^2\zeta(s, a, x)=\int_0^\infty\int_0^\infty(t_1t_2)^{s-1}g(t_1, t_2)dt_1dt_2,$$

where

$$g(t_1, t_2)=\frac{\exp\{(1-x_1)(t_1+t_2)+(1-x_2)(a_1t_1+a_2t_2)\}}{\{1-\exp(t_1+t_2)\}\{1-\exp(a_1t_1+a_2t_2)\}}.$$

Set

$$D_1=\{(t_1, t_2)\in \boldsymbol{R}_+^2,\quad t_1\geqq t_2\}$$

and

$$D_2=\{(t_1, t_2)\in \boldsymbol{R}_+^2,\quad t_1\leqq t_2\}.$$

In D_1 (resp. D_2), let us make the following change of variables: $t_1=t$, $t_2=tu$ (resp. $t_1=tu$, $t_2=t$). Then we have

$$\Gamma(s)^2\zeta(s, a, x)=\int_0^\infty t^{2s}\frac{dt}{t}\int_0^1 u^s\{g(t, tu)+g(tu, t)\}\frac{du}{u}.$$

For a sufficiently small positive number λ, we have

(1.11) $$\zeta(s, a, x)=\frac{1}{(2\pi i)^2}\frac{\exp(-2\pi is)}{1+\exp(2\pi is)}\Gamma(1-s)^2$$
$$\times\int_{I(\lambda,\infty)}t^{2s}\frac{dt}{t}\int_{I(\lambda,1)}u^s\frac{du}{u}\{g(t, tu)+g(tu, t)\}$$

(the integral path $I(\lambda, \infty)$ (resp. $I(\lambda, 1)$) is as indicated in Fig. 1 (resp. Fig. 2)). Applying residue calculus, we have

$$\frac{d}{ds}\zeta(s, a, x)\Big|_{s=0}=(-3\pi i+2\gamma)\zeta(0, a, x)$$
$$+\frac{1}{2\pi i}\Big\{\int_{I(\lambda,\infty)}\frac{\exp t\{1+a_1-(x_1+x_2a_1)\}}{\{\exp(t)-1\}\{\exp(a_1t)-1\}}\frac{\log t}{t}dt$$
$$+\int_{I(\lambda,\infty)}\frac{\exp t\{1+a_2-(x_1+x_2a_2)\}}{\{\exp(t)-1\}\{\exp(a_2t)-1\}}\frac{\log t}{t}dt\Big\}$$
$$+\frac{1}{4\pi i}\int_{I(\lambda,1)}\Big\{\frac{1}{2}\frac{1+u}{a_1+a_2u}B_2(x_1)+B_1(x_1)B_1(x_2)+\frac{1}{2}\frac{a_1+a_2u}{1+u}B_2(x_2)\Big\}\frac{\log u}{u}du$$
$$+\frac{1}{4\pi i}\int_{I(\lambda,1)}\Big\{\frac{1}{2}\frac{1+u}{a_1u+a_2}B_2(x_1)+B_1(x_1)B_1(x_2)+\frac{1}{2}\frac{a_1u+a_2}{1+u}B_2(x_2)\Big\}\frac{\log u}{u}du,$$

where $\log t$ (resp. $\log u$) is understood to be real valued on the upper linear segment $(+\infty, \lambda)$ (resp. $(1, \lambda)$) of $I(\lambda, \infty)$ (resp. $I(\lambda, 1)$). It follows from (1.10) and (1.11) that

$$\zeta(0, a, x)=\frac{1}{2}\sum_{k=1}^{2}\Big(\frac{1}{2a_k}B_2(x_1+x_2a_k)+B_1B_1(x_1+x_2a_k)+\frac{1}{2}B_2a_k\Big)$$
$$=\frac{1}{4}\Big(\frac{1}{a_1}+\frac{1}{a_2}\Big)B_2(x_1)+B_1(x_1)B_1(x_2)+\frac{1}{4}(a_1+a_2)B_2(x_2).$$

Hence, Proposition 2 implies that

$$\frac{d}{ds}\zeta(s, a, x)\Big|_{s=0}=-\pi i\zeta(0, a, x)+\log\Big\{\frac{\Gamma_2(x_1+x_2a_1, (1, a_1))\Gamma_2(x_1+x_2a_2, (1, a_2))}{\rho_2((1, a_1))\qquad \rho_2((1, a_2))}\Big\}$$
$$+\zeta(0, a, x)\frac{1}{2\pi i}\int_{I(\lambda,1)}\frac{\log u}{u}du+\int_0^1\Big(\frac{1+u}{a_1+a_2u}+\frac{1+u}{a_1u+a_2}-\frac{1}{a_1}-\frac{1}{a_2}\Big)\frac{du}{u}\frac{1}{4}B_2(x_1).$$

On the other hand, simple computations show that

$$\frac{1}{2\pi i}\int_{I(\lambda,1)}\frac{\log u}{u}du=\pi i$$

and

$$\int_0^1\Big(\frac{1+u}{a_1u+a_2}-\frac{1}{a_2}\Big)\frac{du}{u}=\frac{a_2-a_1}{a_1a_2}\log\Big(\frac{a_1+a_2}{a_2}\Big)$$

Thus, the proposition follows.

5. In this subsection, we derive an asymptotic series for $\log\Gamma_2(z, \omega)$ which is useful for the numerical computation of the double gamma function. We use notations in 3 without further comment. For $z>0$, set

(1.12) $$\operatorname{LG}(z)=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{\exp(-zt)}{1-\exp(-t)}\frac{\log t}{t^2}dt+\frac{\gamma-\pi i}{2}B_2(z),\quad (0<\lambda<2\pi),$$

where $\log t$ is real valued on the upper linear segment $(+\infty, \lambda)$ of $I(\lambda, \infty)$.

LEMMA 2. (i) $\frac{d}{dz}\mathrm{LG}(z)=-\log\left\{\frac{\Gamma(z)}{\sqrt{2\pi}}\right\}$,

(ii) $\mathrm{LG}(z)=\frac{1}{2}B_2(z)-\left[\frac{d}{ds}\zeta(s,z)\right]_{s=-1}$,

(iii) $\mathrm{LG}(z+1)-\mathrm{LG}(z)=-z\log z+z$,

(iv) $\mathrm{LG}(z)=-\frac{z^2}{2}\log z+\frac{3}{4}z^2-B_1(z\log z-z)-\frac{B_2}{2}\log z$

$$+\sum_{k=1}^{n-1}\frac{1}{z^{2k}}\frac{B_{2k+2}}{2k(2k+1)(2k+2)}+R_n(z),$$

where the remainder term $R_n(z)$ satisfies the inequality

$$|R_n(z)|<\frac{1}{z^{2n}}\frac{|B_{2n+2}|}{(2n)(2n+1)(2n+2)}\qquad(z>0).$$

PROOF. The first (resp. second) assertion follows easily from (1.12) and (1.8) (resp. (1.6)). Since $\zeta(s,z+1)=\zeta(s,z)-z^{-s}$, the third equality is an immediate consequence of the second one. Set

$$S_n(z)=\log\frac{\Gamma(z)}{\sqrt{2\pi}}-\left\{\left(z-\frac{1}{2}\right)\log z-z\right\}-\sum_{k=1}^{n}\frac{B_{2k}}{(2k-1)(2k)}z^{1-2k}.$$

Then it is well-known (see e.g. Chapter 12 of [16]) that

$$(1.13)\qquad |S_n(z)|<\frac{|B_{2n+2}|}{(2n+2)(2n+1)}z^{-1-2n}\qquad\text{for } z>0.$$

In view of (i), there exists a constant c (which is independent of z) such that

$$(1.14)\qquad \mathrm{LG}(z)=-\frac{z^2}{2}\log z+\frac{3}{4}z^2-B_1(z\log z-z)-\frac{B_2}{2}\log z$$
$$+\sum_{k=1}^{n-1}\frac{B_{2k+2}}{(2k)(2k+1)(2k+2)}z^{-2k}+R_n(z)+c,$$

where we put $R_n(z)=\int_\infty^z S_n(t)dt$. The inequality (1.13) implies that

$$|R_n(z)|<\frac{|B_{2n+2}|}{2n(2n+1)(2n+2)}z^{-2n}.$$

On the other hand, it follows easily from (1.12) that

$$\mathrm{LG}(z)=\int_0^\infty\left\{\frac{1}{1-\exp(-t)}-\frac{1}{t}+B_1-\frac{B_2}{2}t\right\}\frac{\exp(-zt)}{t^2}dt$$
$$+\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left(\frac{1}{t}-B_1+\frac{B_2}{2}t\right)\frac{\exp(-zt)}{t^2}\log t\,dt+\frac{\gamma-\pi i}{2}B_2(z).$$

A straightforward computation shows that

$$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left(\frac{1}{t}-B_1+\frac{B_2}{2}t\right)\frac{\exp(-zt)}{t^2}\log t\,dt$$
$$=-\frac{z^2}{2}\log z+\frac{3}{4}z^2-B_1(z\log z-z)-\frac{B_2}{2}\log z+\frac{\pi i-\gamma}{2}B_2(z).$$

Hence, we see that

$$\lim_{z\to+\infty}\left\{\mathrm{LG}(z)+\frac{z^2}{2}\log z-\frac{3}{4}z^2+B_1(z\log z-z)+\frac{B_2}{2}\log z\right\}=0.$$

Thus, the constant c in the equality (1.14) is equal to zero.

Let $\omega=(\omega_1,\omega_2)$ be a pair of positive numbers and put $\tau=\omega_2/\omega_1$.

PROPOSITION 4. *Notations being as above, if $z>0$,*

$$(1.15)\qquad \log\frac{\Gamma_2(z,\omega)}{\rho_2(\omega)}=-\frac{\log\omega_1}{2\omega_1\omega_2}\left\{\omega_1^2B_2\left(\frac{z}{\omega_1}\right)+2\omega_1\omega_2B_1\left(\frac{z}{\omega_1}\right)B_1+B_2\omega_2^2\right\}$$
$$+\frac{1}{\tau}\mathrm{LG}\left(\frac{z}{\omega_1}\right)-B_1\log\left\{\frac{\Gamma(z/\omega_1)}{\sqrt{2\pi}}\right\}-\frac{B_2\tau}{2}\psi\left(\frac{z}{\omega_1}\right)$$
$$+\sum_{k=2}^{n}\frac{B_{2k}\tau^{2k-1}}{(2k-1)(2k)}\zeta\left(2k-1,\frac{z}{\omega_1}\right)+R_n(z)\quad(n=2,3,\cdots)$$

where $R_n(z)=-\frac{\tau^{2n+1}}{2n+1}\int_0^\infty B_{2n+1}(\{u\})\zeta\left(2n+1,\frac{z}{\omega_1}+\tau u\right)du$, ($\{u\}=u-[u]$, ($[u]$ is the integer part of u)). Furthermore, the remaining term R_n satisfies the following inequality:

$$|R_n(z)|\leq\frac{\tau^{2n+1}}{(2n+1)(2n+2)}\zeta\left(2n+1,\frac{z}{\omega_1}\right)|B_{2n+2}|.$$

PROOF. Set $w=z/\omega_1$. Proposition 2 and equalities (1.8), (1.10) and (1.12) imply that

$$(1.16)\qquad \log\left\{\frac{\Gamma_2(z,\omega)}{\rho_2(\omega)}\right\}=\frac{(\gamma-\pi i-\log\omega_1)}{2\omega_1\omega_2}\{\omega_1^2B_2(w)+2\omega_1\omega_2B_1B_1(w)+B_2\omega_2^2\}$$
$$+\int_0^\infty\frac{\exp(-wt)}{1-\exp(-t)}\left\{\frac{\exp(\tau t)}{\exp(\tau t)-1}-\frac{1}{\tau t}-B_1-\frac{B_2}{2}\tau t\right\}\frac{dt}{t}$$
$$+\frac{1}{\tau}\mathrm{LG}(w)-B_1\log\left\{\frac{\Gamma(w)}{\sqrt{2\pi}}\right\}-\frac{B_2\tau}{2}\psi(w)$$
$$+\frac{\pi i-\gamma}{2}\left\{\frac{1}{\tau}B_2(w)+2B_1B_1(w)+B_2\tau\right\}.$$

On the other hand, differentiating three times both sides of the equality (1.9) with respect to z, we have

$$\frac{d^3}{dz^3}\log\{\Gamma_2(z,\omega)\}=-\int_0^\infty\frac{\exp(-zt)}{\{1-\exp(-\omega_1t)\}\{1-\exp(-\omega_2t)\}}t^2dt$$

$$=-2\sum_{m,n=0}^{\infty}(z+m\omega_1+n\omega_2)^{-s}$$

$$=-2\omega_2^{-3}\sum_{n=0}^{\infty}\zeta(3,\ w'+n\tau'),$$

where we put $\tau'=\frac{\omega_1}{\omega_2}$ and $w'=\frac{z}{\omega_2}$. A straightforward computation shows that

$$\zeta(s,\ z)=\frac{z^{1-s}}{s-1}+\frac{1}{2z^s}+\frac{sB_2}{2z^{1+s}}-\frac{s(s+1)(s+2)}{6}\int_0^\infty\frac{B_3(\{u\})}{(u+z)^{s+3}}du \quad (\mathrm{Re}\,s>-2,\ z>0),$$

where we put $\{u\}=u-[u]$. Taking into account (i) of Lemma 2, we have

(1.17) $$\frac{d^3}{dz^3}\log\{\Gamma_2(z,\ \omega)\}$$

$$=-\frac{1}{\omega_1^2\omega_2}\zeta(2,\ w)-\frac{1}{\omega_1^3}\zeta(3,\ w)-\frac{3B_2\omega_2}{\omega_1^4}\zeta(4,\ w)+\frac{20\omega_2^3}{\omega_1^6}\int_0^\infty B_3(\{u\})\zeta(6,\ w+\tau u)du$$

$$=\frac{d^3}{dz^3}\Big\{\frac{1}{\tau}\mathrm{LG}(w)+\frac{1}{2}\log\Gamma(w)-\frac{B_2}{2}\tau\psi(w)-\frac{\tau^3}{3}\int_0^\infty B_3(\{u\})\zeta(3,\ w+\tau u)du\Big\}.$$

From (1.16) and (1.17) we infer that

(1.18) $$\log\Big\{\frac{\Gamma_2(z,\ \omega)}{\rho_2(\omega)}\Big\}=-\frac{\log\omega_1}{2}\Big\{\frac{1}{\tau}B_2(w)+2B_1B_1(w)+B_2\tau\Big\}+\frac{1}{\tau}\mathrm{LG}(w)$$

$$-B_1\log\Big\{\frac{\Gamma(w)}{\sqrt{2\pi}}\Big\}-\frac{B_2\tau}{2}\psi(w)-\frac{\tau^3}{3}\int_0^\infty B_3(\{u\})\zeta(3,\ w+\tau u)du.$$

In fact, (1.17) implies that the difference between the left side and the right side of (1.18) is a quadratic polynomial in z. On the other hand (1.16) implies that the difference approaches 0 when $z\to+\infty$. Hence the equality (1.18) follows. Applying the integration by part, we have

$$-\frac{\tau^3}{3}\int_0^\infty B_3\{u\}\zeta(3,\ w+\tau u)du$$

$$=\sum_{k=2}^{n}\frac{B_{2k}\tau^{2k-1}}{(2k-1)(2k)}\zeta(2k-1,\ w)-\frac{\tau^{2n+1}}{2n+1}\int_0^\infty B_{2n+1}(\{u\})\zeta(2n+1,\ w+\tau u)du.$$

Thus, we obtain (1.15). Set $R_n(z)=-\frac{\tau^{2n+1}}{2n+1}\int_0^\infty B_{2n+1}(\{u\})\zeta(2n+1,\ w+\tau u)du$. Then

(1.19) $$R_n(z)=\frac{\tau^{2n+1}B_{2n+2}}{(2n+1)(2n+2)}\zeta(2n+1,\ w)+R_{n+1}(z).$$

On the other hand,

$$R_n(z)=\frac{-\tau^{2n+2}}{2n+2}\int_0^\infty[B_{2n+2}(\{u\})-B_{2n+2}]\zeta(2n+2,\ w+\tau u)du.$$

Since $(-1)^n[B_{2n+2}-B_{2n+2}(\{u\})]\geqq 0$, $(-1)^nR_n(z)>0$. Hence $R_n(z)$ and $R_{n+1}(z)$ have

opposite signs. Thus, the equality (1.19) implies that

$$|R_n(z)|<\frac{\tau^{2n+1}|B_{2n+2}|}{(2n+1)(2n+2)}\zeta(2n+1,\ w) \quad \text{(cf. 69. of [17])}.$$

REMARK. If $\tau=\omega_2/\omega_1<1$ and if $w=z/\omega_1$ is a large positive number, the asymptotic series (1.15) is quite useful for the numerical computation of $\log\Gamma_2(z,\ \omega)$.

6. So far, we have assumed that the parameters ω_1 and ω_2 of the double gamma function $\Gamma_2(z,\ \omega)$ are both positive. However, analytic expressitons defining $\Gamma_2(z,\ \omega)$, $\gamma_{21}(\omega)$ and $\gamma_{22}(\omega)$ are all meaningful if ω_2/ω_1 is not a negative real number. In this paragraph *we assume that ω_1 is positive and ω_2 is a complex number which is not non-positive real.* Then it is easy to see that Lemma 1 and Proposition 1 are both true if $\log z$ is understood to be a holomorphic function on $C-(-\infty,0]$ which is real valued on the positive real axis. Set $q=\exp(2\pi i(\omega_2/\omega_1))$ and $q'=\exp(-2\pi i(\omega_1/\omega_2))$.

PROPOSITION 5. *Notations being as above, assume that $\omega_1>0$ and $\mathrm{Im}\,\omega_2>0$. Then*

$$\frac{\Gamma_2(\omega_1+\omega_2-z,\ \omega)}{\Gamma_2(z,\ \omega)}$$

$$=\sqrt{i}\exp\frac{\pi i}{12}\Big(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\Big)\frac{\prod_{n=0}^{\infty}\Big(1-q^n\exp\frac{2\pi i}{\omega_1}z\Big)}{\prod_{n=1}^{\infty}\Big(1-q'^n\exp\frac{2\pi i}{\omega_2}z\Big)}\exp\frac{\pi i}{2}\Big\{\frac{z^2}{\omega_1\omega_2}-\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big)z\Big\}.$$

PROOF. Set $F(z,\ \omega)=\Gamma_2(\omega_1+\omega_2-z,\ \omega)/\Gamma_2(z,\ \omega)$. It follows easily from Proposition 1 that

$$\frac{F(z,\ \omega)}{F(z+\omega_1,\ \omega)}=2\sin\frac{\pi z}{\omega_2} \quad \text{and} \quad \frac{F(z,\ \omega)}{F(z+\omega_2,\ \omega)}=2\sin\frac{\pi z}{\omega_1}.$$

The zero's (resp. poles) of the function $F(z,\ \omega)$ are all simple and located at $z=-(n\omega_1+m\omega_2)$ $(n,\ m=0,1,2,\cdots)$ (resp. $z=n\omega_1+m\omega_2$ $(n,\ m=1,2,3,\cdots)$). Set $f_1(z,\ \omega)=\prod_{n=0}^{\infty}(1-q^n\exp(2\pi i/\omega_1)z)$ and $f_2(z,\ \omega)=\prod_{n=1}^{\infty}(1-q'^n\exp(2\pi i/\omega_2)z)$. Then f_1 and f_2 are entire functions of z which satisfy the following difference equations:

$$f_1(z+\omega_1,\ \omega)=f_1(z,\ \omega),\qquad f_1(z+\omega_2,\ \omega)\Big(1-\exp\frac{2\pi i}{\omega_1}z\Big)=f_1(z,\ \omega),$$

$$f_2(z+\omega_1,\ z)=\Big(1-\exp\frac{2\pi i}{\omega_2}z\Big)f_2(z,\ \omega),\qquad f_2(z+\omega_2,\ \omega)=f_2(z,\ \omega).$$

The zero's of $f_1(z,\ \omega)$ (resp. $f_2(z,\ \omega)$) are all simple and are located at $z=n\omega_1-m\omega_2$ $(n\in\boldsymbol{Z},\ m=0,1,2,\cdots)$, (resp. $z=n\omega_1+m\omega_2$ $(n=1,2,\cdots,\ m\in\boldsymbol{Z})$). Set

$$\tilde{F}(z,\omega)=\frac{f_1(z,\omega)}{f_2(z,\omega)}\exp\frac{\pi i}{2}\left\{\frac{z^2}{\omega_1\omega_2}-\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)z\right\}.$$

It is now easily seen that $F(z,\omega)/\tilde{F}(z,\omega)$ is an entire function of z with periods ω_1 and ω_2. Hence, $F(z,\omega)/\tilde{F}(z,\omega)$ does not depend upon z. On the other hand, $\lim_{z\to 0}F(z,\omega)/z=\Gamma_2(\omega_1+\omega_2,\omega)=\frac{2\pi}{\sqrt{\omega_1\omega_2}}$ and $\lim_{z\to 0}\tilde{F}(z,\omega)/z=-\frac{2\pi i}{\omega_1}\frac{\prod_{n=1}^{\infty}(1-q^n)}{\prod_{n=1}^{\infty}(1-q'^n)}$. It follows from the transformation formula for the Dedekind η-function that

$$\exp\left(\frac{\pi i}{12}\frac{\omega_2}{\omega_1}\right)\prod_{n=1}^{\infty}(1-q^n)=\sqrt{\frac{i\omega_1}{\omega_2}}\exp\left(-\frac{\pi i}{12}\frac{\omega_1}{\omega_2}\right)\prod_{n=1}^{\infty}(1-q'^n).$$

Thus, we have

$$F(z,\omega)\frac{\sqrt{\omega_1\omega_2}}{2\pi}=\tilde{F}(z,\omega)\left(-\frac{\omega_1}{2\pi i}\right)\sqrt{\frac{\omega_2}{i\omega_1}}\exp\left\{\frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\right)\right\}.$$

Proposition 5 has been proved.

§ 2.

1. Let F be a real quadratic field. We fix an embedding of F into $\boldsymbol{R}$. Denote by h_0 the number of *narrow* ideal classes of F and choose integral ideals $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{h_0}$ so that they form a complete set of representatives of narrow ideal classes of F.

Denote by E (resp. E_+) the group of units (resp. totally positive units) of F. Let $\varepsilon>1$ be the generator of the infinite cyclic group E_+. For each $x\in F$, we denote by x' the conjugate of x. Let $\mathfrak{f}$ be an integral ideal of F. We denote by $R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ $(1\leqq k\leqq h_0)$ the set consisting of all elements z of $(\mathfrak{a}_k\mathfrak{f})^{-1}$ which satisfy the following conditions (2.1) and (2.2).

(2.1) The number z is of the form $z=x+\varepsilon y$, where x and y are rational numbers which satisfy the inequalities $0<x\leqq 1$ and $0\leqq y<1$.

(2.2) The integral ideal $\mathfrak{a}_k\mathfrak{f}(z)$ is prime to $\mathfrak{f}$.

It is easy to see that $R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ is a *finite* subset of F. Let $I(\mathfrak{f})$ be the group of fractional ideals of F which are prime to $\mathfrak{f}$. Further, let $P(\mathfrak{f})$ be the subgroup of $I(\mathfrak{f})$ consisting of all the principal ideals generated by totally positive μ with the congruence condition $\mu\equiv 1$ (mod. $\mathfrak{f}$). Then the group $I(\mathfrak{f})/P(\mathfrak{f})$ is the group of narrow ideal classes modulo $\mathfrak{f}$ of F. Let χ be a character of the group $I(\mathfrak{f})/P(\mathfrak{f})$. Denote by $L_F(s,\chi)$ the Hecke L-function of F associated with the character χ:

$L_F(s,\chi)=\sum\chi(\mathfrak{g})N(\mathfrak{g})^{-s}$, where the summation with respect to $\mathfrak{g}$ is over all the integral ideals of F. If $\mathfrak{g}$ is not prime to $\mathfrak{f}$, we put $\chi(\mathfrak{g})=0$.

LEMMA 3. *Notations being as above,*

$$L_F(s,\chi)=\sum_{k=1}^{h_0}\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi(\mathfrak{a}_k\mathfrak{f}(z))N(\mathfrak{a}_k\mathfrak{f})^{-s}\zeta(s,(\varepsilon,\varepsilon'),(x_1,x_2))^{(1)}\quad (z=x_1+\varepsilon x_2).$$

PROOF. It follows from the definition of $L_F(s,\chi)$ that

$$L_F(s,\chi)=\sum_{k=1}^{h_0}\sum_{\mathfrak{g}\approx\mathfrak{a}_k\mathfrak{f}}\chi(\mathfrak{g})N(\mathfrak{g})^{-s},$$

where the summation with respect to $\mathfrak{g}\approx\mathfrak{a}_k\mathfrak{f}$ is over all the integral ideals of F which are in the same narrow ideal class as $\mathfrak{a}_k\mathfrak{f}$. On the other hand

$$\sum_{\mathfrak{g}\approx\mathfrak{a}_k\mathfrak{f}}\chi(\mathfrak{g})N(\mathfrak{g})^{-s}=N(\mathfrak{a}_k\mathfrak{f})^{-s}\sum\chi(\mathfrak{a}_k\mathfrak{f}(\mu))\,|N(\mu)|^{-s},\tag{2.3}$$

where the summation with respect to μ is over all totally positive numbers in $(\mathfrak{a}_k\mathfrak{f})^{-1}$ which are not associated with each other under the action of E_+. Denote by $\tilde{R}(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ the set consisting of all z of $(\mathfrak{a}_k\mathfrak{f})^{-1}$ which satisfy the condition (2.1). Then it is easy to see that the mapping from $E_+\times\tilde{R}(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})\times Z_+^2$ into F given by $(u,z,(m,n))\mapsto u(z+m+n\varepsilon)$ is a bijection from the former set onto $F_+\cap(\mathfrak{a}_k\mathfrak{f})^{-1}$ (we denote by Z_+ the set of all non-negative integers and by F_+ the set of all totally positive numbers of F). Moreover, since χ is a character modulo $\mathfrak{f}$,

$$\chi(\mathfrak{a}_k\mathfrak{f}(z+m+n\varepsilon))=\chi(\mathfrak{a}_k\mathfrak{f}(z))\qquad\text{for any } m, n\in Z_+.$$

If the integral ideal $\mathfrak{a}_k\mathfrak{f}(z)$ is not prime to $\mathfrak{f}$, $\chi(\mathfrak{a}_k\mathfrak{f}(z))=0$. Hence the right side of (2.3) is equal to

$$N(\mathfrak{a}_k\mathfrak{f})^{-s}\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi(\mathfrak{a}_k\mathfrak{f}(z))\zeta(s,(\varepsilon,\varepsilon'),(x_1,x_2))\qquad(z=x_1+x_2\varepsilon).$$

Now, the lemma follows.

2. Now we assume that χ is a primitive character of the narrow ideal class group modulo $\mathfrak{f}$ of F which is given by the following formula (2.4), if (μ) is a principal integral ideal of F:

$$\chi((\mu))=\operatorname{sgn}(\mu)\chi_0(\mu),\tag{2.4}$$

where χ_0 is a character of the group of invertible residue classes modulo $\mathfrak{f}$. It is known that $L_F(s,\chi)$ is an entire function of s. Furthermore, if one puts

$$\xi(s,\chi)=\pi^{-s}(dN(\mathfrak{f}))^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_F(s,\chi),\tag{2.5}$$

where d is the discriminant of F, one knows that ξ satisfies the following func-

(1) For the definition of $\zeta(s,(\varepsilon,\varepsilon'),(x_1,x_2))$, see Proposition 3.

tional equation:

$$\xi(1-s;\chi)=w(\chi)\xi(s,\chi^{-1}), \tag{2.6}$$

where $w(\chi)$ is a complex number of modulus 1 which depends only on χ (see (45) of Hecke [5]).

THEOREM 1. *Notations being as above, let χ be a primitive character of the group of narrow ideal classes modulo $\mathfrak{f}$ of the form (2.4), then*

$$w(\chi)^{-1}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi)$$
$$=\sum_{k=1}^{h_0}\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi^{-1}(\mathfrak{a}_k\mathfrak{f}(z))\log\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\}.$$

PROOF. We note that the holomorphy of $L_F(s,\chi)$ at $s=1$ and the functional equation (2.6) implies that $L_F(0,\chi)=0$. Evaluating the both sides of the functional equation (2.6) at $s=0$, we have

$$\pi^{-1}\sqrt{dN(\mathfrak{f})}\,\Gamma\Big(\frac{1}{2}\Big)L_F(1,\chi)=2w(\chi)\Gamma\Big(\frac{1}{2}\Big)\Big\{\frac{d}{ds}L_F(s,\chi^{-1})\Big\}_{s=0}.$$

Hence,

$$\overline{w(\chi)}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi)=\Big\{\frac{d}{ds}L_F(s,\chi^{-1})\Big\}_{s=0}.$$

By Lemma 2,

$$L_F(s,\chi^{-1})=\sum_{k=1}^{h_0}\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi^{-1}(\mathfrak{a}_k\mathfrak{f}(z))N(\mathfrak{a}_k\mathfrak{f})^{-s}\zeta(s,(\varepsilon,\varepsilon'),(x_1,x_2)). \tag{2.7}$$

We note that for any character μ of the group of *narrow* ideal classes of F, $\chi^{-1}\mu$ is always a primitive character of the group of narrow ideal classes modulo $\mathfrak{f}$ of F of the form (2.4). Since $L_F(0,\chi^{-1}\mu)=0$ for any such μ, we have

$$\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi^{-1}(\mathfrak{a}_k\mathfrak{f}(z))\zeta(0,(\varepsilon,\varepsilon'),(x_1,x_2))=0 \qquad \text{for } k=1,2,\cdots,h_0.$$

Thus, we obtain

$$\overline{w(\chi)}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi) \tag{2.8}$$
$$=\sum_{k=1}^{h_0}\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi^{-1}(\mathfrak{a}_k\mathfrak{f}(z))\Big\{\frac{d}{ds}\zeta(s,(\varepsilon,\varepsilon'),(x_1,x_2))\Big\}_{s=0}\quad(z=x_1+\varepsilon x_2).$$

By Proposition 3,

$$\Big\{\frac{d}{ds}\zeta(s,(\varepsilon,\varepsilon'),(x_1,x_2))\Big\}_{s=0}=\log\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\} \tag{2.9}$$
$$-\log\{\rho_2((1,\varepsilon))\rho_2((1,\varepsilon'))\}$$
$$+\frac{\varepsilon-\varepsilon'}{4}B_2(x_1)\log(\varepsilon')^2.$$

We note that if $z=x_1+\varepsilon x_2\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ and if $0<x_1, x_2<1$, then $1+\varepsilon-z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ and $\chi^{-1}(\mathfrak{a}_k\mathfrak{f}(z))=-\chi^{-1}(\mathfrak{a}_k\mathfrak{f}(1+\varepsilon-z))$. Furthermore if $z=x_1\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ $(0<x_1<1)$, then $1-z$, $1+\varepsilon z$ and $1+\varepsilon(1-z)\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$ and $\chi(\mathfrak{a}_k\mathfrak{f}(z))=-\chi(\mathfrak{a}_k\mathfrak{f}(1-z))$ $=\chi(\mathfrak{a}_k\mathfrak{f}(1+\varepsilon z))=-\chi(\mathfrak{a}_k\mathfrak{f}(1+\varepsilon(1-z)))$.

Taking these situations into account, it is now easy to infer, from equalities (2.8) and (2.9) that

$$w(\chi)^{-1}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi)$$
$$=\sum_{k=1}^{h_0}\sum_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})}\chi(\mathfrak{a}_k\mathfrak{f}(z))^{-1}\log\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\}.$$

3. Let K be a quadratic extension of F with relative discriminant $\mathfrak{d}$ and *assume that exactly one of the two archimedean primes of F ramifies in K.* Further, let χ be the character of the narrow ideal class group modulo $\mathfrak{d}$ which corresponds to the extension K of F in class field theory. Then χ is primitive and of the form (2.4). Denote by $\zeta_K(s)$ (resp. $\zeta_F(s)$) the Dedekind zeta-function for K (resp. F). Then

$$\zeta_K(s)=\zeta_F(s)L_F(s,\chi). \tag{2.10}$$

As is well-known, the functional equations for $\zeta_K(s)$ and $\zeta_F(s)$, together with the above equality, imply that the factor $w(\chi)$ in (2.6) is, in the present case, equal to 1. As is well-known, the residues of $\zeta_K(s)$ and $\zeta_F(s)$ at $s=1$ is given by the following formula:

$$\operatorname*{Res}_{s=1}\zeta_K(s)=\frac{2^3\pi R_K h_K}{2\sqrt{d^2N(\mathfrak{d})}}\quad\text{and}\quad\operatorname*{Res}_{s=1}\zeta_F(s)=\frac{2^2R_Fh_F}{2\sqrt{d}},$$

where R_K (resp. R_F) and h_K (resp. h_F) are the regulator and the class number of K (resp. F).

Hence, it follows from (2.10) that

$$L_F(1,\chi)=\frac{2\pi}{\sqrt{dN(\mathfrak{d})}}\frac{R_K}{R_F}\frac{h_K}{h_F}.$$

For each $y\in K$, we denote by y^σ the conjugate of y with respect to F. We fix an embedding of K into the real number field $\boldsymbol{R}$ which extends the given embedding of F into $\boldsymbol{R}$. Let $\varepsilon_0>1$ be the fundamental unit of F. A unit u in K is said to be *primitive* if u is not a power of any unit of K other than u or u^{-1}. Assume that $\pm\varepsilon_0=\eta_1^m$ $(m>0)$ be the m-th power of a primitive unit η_1 of K. Take a unit η of K which satisfies the following conditions (2.11) and (2.12).

(2.11) The group of units of K is generated by η_1, η and ±1.

$$\eta>|\eta^\sigma|>0. \tag{2.12}$$

We note that, although η_1 and η are not uniquely determined, the positive integer m and $|\eta/\eta^\sigma|$ are uniquely determined. Then we have

$$\frac{R_K}{R_F}=\left|\frac{\log|\eta_1|\log|\eta^\sigma|-\log|\eta_1^\sigma|\log|\eta|}{\log\varepsilon_0}\right|=\frac{1}{m}\log\left(\frac{\eta}{|\eta^\sigma|}\right).$$

Thus,

$$L_F(1,\chi)=\frac{2\pi}{\sqrt{dN(\mathfrak{d})}}\frac{h_K}{mh_F}\log\left(\frac{\eta}{|\eta^\sigma|}\right). \tag{2.13}$$

Comparing (2.13) with the formula of Theorem 1, we obtain the following:

COROLLARY 1 TO THEOREM 1. *Notations being as above,*

$$\left(\frac{\eta}{|\eta^\sigma|}\right)^{h_K}=\prod_{k=1}^{h_0}\prod_{z\in R(\varepsilon,(\mathfrak{a}_k\mathfrak{d})^{-1})}\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\}^{mh_F\chi_k(z)},$$

where we put $\chi_k(z)=\chi(\mathfrak{a}_k\mathfrak{d}(z))$.

4. In this subsection we derive a modified version of Theorem 1 which seems to be more suggestive and to be more convienient for the numerical computation of $L_F(1,\chi)$. In the following we use natations in § 2.3 without further comment. Let $\mathfrak{f}$ be an integral ideal of F. *We assume that there is a character* χ *of the group of narrow ideal classes modulo* $\mathfrak{f}$ *of* F *of type* (2.4). Choose integral ideals $\mathfrak{a}_1,\cdots,\mathfrak{a}_{h_0}$ of F so that they form a complete set of representatives for narrow ideal classes of F. Set $R_k=R(\varepsilon,(\mathfrak{a}_k\mathfrak{f})^{-1})$, $(1\leqq k\leqq h_0)$. Let E_+ be the group of all the totally positive units of F. For each $u\in E_+$ and for each $z\in R_k$ there exists a unique $\overline{uz}\in R_k$ such that $\overline{uz}-uz\in\mathbf{Z}+\mathbf{Z}\varepsilon$. It is easy to see that E_+ acts on R_k via the mapping: $(u,z)\mapsto\overline{uz}$. Each E_+-orbit in R_k is said to be a *cycle*. The value of χ at the integral ideal $\mathfrak{a}_k\mathfrak{f}(z)$ $(z\in R_k)$ depends only upon the cycle which contains z. For each cycle C of R_k, set

$$\chi_k(C)=\chi(\mathfrak{a}_k\mathfrak{f}(z))\qquad(z\in C).$$

For a $z=x+\varepsilon y$ of R_k $(x,y\in\mathbf{Q})$, put

$$\overline{-z}=\begin{cases}1+\varepsilon-z & \text{if } 0<x<1 \text{ and } 0<y<1,\\ 1-z & \text{if } 0<x<1 \text{ and } y=0,\\ 1+\varepsilon(1-y) & \text{if } x=1 \text{ and } 0<y<1.\end{cases}$$

The mapping: $z\mapsto\overline{-z}$ induces a permutation of order 2 on R_k. For any cycle C of R_k, set

$$-C=\{\overline{-z}\,;\,z\in C\}.$$

It is easy to see that $-C$ is also a cycle of R_k and that

$$\chi_k(-C)=-\chi_k(C). \tag{2.14}$$

Thus, the mapping: $C\mapsto -C$ induces a fixed point free permutation of order 2 on the set of all the cycles in R_k. Hence, we can choose suitable cycles $C_1^{(k)}$, $C_2^{(k)},\cdots,C_{l_k}^{(k)}$ of R_k such that

$$R_k=C_1^{(k)}\cup -C_1^{(k)}\cup\cdots\cup C_{l_k}^{(k)}\cup -C_{l_k}^{(k)}\qquad\text{(disjoint union)}. \tag{2.15}$$

The cycle $-C$ is said to be the *opposite cycle* of C. For each $z=x+y\varepsilon\in R_k$ $(x,y\in\mathbf{Q})$, set

$$\tilde{z}=\begin{cases}y+x\varepsilon & \text{if } 0<x<1, & 0<y<1,\\ 1+x\varepsilon & \text{if } 0<x<1, & y=0,\\ y & \text{if } x=1, & 0<y<1.\end{cases}$$

For each cycle C of R_k, put

$$\tilde{C}=\{\tilde{z}\,;\,z\in C\}.$$

We call $\tilde{C}$ the *conjugate* of C. The conjugate of C is a cycle in the set $R(\varepsilon,(\mathfrak{a}_k'\mathfrak{f}')^{-1})$, where $\mathfrak{a}_k'$ and $\mathfrak{f}'$ are conjugates of $\mathfrak{a}_k$ and $\mathfrak{f}$. Thus, the conjugate of C is not necessarily a subset of R_k. However, if $\tilde{C}$ has a non-empty intersection with R_k, $\tilde{C}$ is a cycle in R_k. Further, as in the proof of Proposition 5, set

$$F(z,\omega)=\frac{\Gamma_2(z,\omega)}{\Gamma_2(\omega_1+\omega_2-z,\omega)}\qquad(\omega=(\omega_1,\omega_2)).$$

Then we have the following:

COROLLARY 2 TO THEOREM 1. *Notations being as above,*

$$\overline{w(\chi)}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi)=\sum_{k=1}^{h_0}\sum\chi_k^{-1}(C)\log\{\prod_{z\in C}F(z,(1,\varepsilon))\prod_{\tilde{z}\in\tilde{C}}F(\tilde{z},(1,\varepsilon))\}$$

$$=\sum_{k=1}^{h_0}\sum\chi_k^{-1}(C)\log\{\prod_{z\in C}F(z',(1,\varepsilon'))\prod_{\tilde{z}\in\tilde{C}}F(\tilde{z}',(1,\varepsilon'))\},$$

where the summation with respect to C *is over all the cycles among* $C_1^{(k)},\cdots,C_{l_k}^{(k)}$ *such that* $\tilde{C}\neq -C$.

PROOF. The equalities (2.14) and (2.15) imply that

$$\sum_{z\in R_k}\chi^{-1}(\mathfrak{a}_k\mathfrak{f}(z))\log\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\}$$
$$=\sum_{i=1}^{l_k}\chi_k^{-1}(C_i^{(k)})\log\left\{\frac{\prod_{z\in C_i^{(k)}}\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))}{\prod_{z\in(-C_i^{(k)})}\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))}\right\}. \tag{2.16}$$

Let $C=\{z_1,\cdots,z_n\}$ be a cycle in R_k. Set $z_i=x_i+y_i\varepsilon$ $(x_i,y_i\in\mathbf{Q},\ 0<x_i\leqq 1,\ 0\leqq y_i<1)$. We may assume, without loss of generality, that

$$z_i=\begin{cases} x_i & (0<x_i<1) & \text{if } 1\leqq i\leqq m,\\ 1+\varepsilon x_{i-m} & & \text{if } m<i\leqq 2m,\\ x_i+\varepsilon y_i & (0<x_i<1,\ 0<y_i<1) & \text{if } 2m<i\leqq n.\end{cases}$$

Then we have,

$$-C=\{1-x_1, \cdots, 1-x_m, 1+\varepsilon(1-x_1), \cdots, 1+\varepsilon(1-x_m), 1+\varepsilon-z_{2m+1}, \cdots, 1+\varepsilon-z_n\}$$

and

$$\tilde{C}=\{1+\varepsilon x_1, \cdots, 1+\varepsilon x_m, x_1, \cdots, x_m, \varepsilon z'_{2m+1}, \cdots, \varepsilon z'_n\}.$$

We note that if $m>0$ (namely if the cycle C contains a rational number), the conjugate $\tilde{C}$ of C coincides with C. It follows from the difference equations of the double gamma function that

$$\frac{\Gamma_2(x, (1, \varepsilon))\Gamma_2(1+\varepsilon x, (1, \varepsilon))}{\Gamma_2(1-x, (1, \varepsilon))\Gamma_2(1+\varepsilon(1-x), (1, \varepsilon))}$$

$$=F(x, (1, \varepsilon))F(1+\varepsilon x, (1, \varepsilon))\exp\left\{\left(\frac{1}{2}-x\right)\log\varepsilon\right\}.$$

Since $\varepsilon\varepsilon'=1$,

$$\frac{\Gamma_2(x, (1, \varepsilon))\Gamma_2(1+\varepsilon x, (1, \varepsilon))\Gamma_2(x, (1, \varepsilon'))\Gamma_2(1+\varepsilon' x, (1, \varepsilon'))}{\Gamma_2(1-x, (1, \varepsilon))\Gamma_2(1+\varepsilon(1-x), (1, \varepsilon))\Gamma_2(1-x, (1, \varepsilon'))\Gamma_2(1+\varepsilon'(1-x), (1, \varepsilon'))}$$

$$=F(x, (1, \varepsilon))F(1+\varepsilon x, (1, \varepsilon))F(x, (1, \varepsilon'))F(1+\varepsilon' x, (1, \varepsilon')).$$

Thus,

$$\frac{\prod_{z\in C}\Gamma_2(z, (1, \varepsilon))\Gamma_2(z', (1, \varepsilon'))}{\prod_{z\in(-C)}\Gamma_2(z, (1, \varepsilon))\Gamma_2(z', (1, \varepsilon'))}=\prod_{z\in C}F(z, (1, \varepsilon))F(z', (1, \varepsilon')).$$

On the other hand it follows from Corollary to Proposition 2 that

$$F(z', (1, \varepsilon'))=F(\varepsilon z', (1, \varepsilon)).$$

Furthermore, the difference equations satisfied by $F(z, (1, \varepsilon))$ (see the proof of Proposition 5) imply that

$$F(\varepsilon x, (1, \varepsilon))F(\varepsilon+x, (1, \varepsilon))=F(x, (1, \varepsilon))F(1+\varepsilon x, (1, \varepsilon)). \tag{2.17}$$

Thus, we have

$$\prod_{z\in C}F(z, (1, \varepsilon))F(z', (1, \varepsilon'))=\prod_{z\in C}F(z, (1, \varepsilon))\prod_{z\in\tilde{C}}F(\tilde{z}, (1, \varepsilon))$$

$$=\prod_{z\in C}F(z', (1, \varepsilon'))\prod_{z\in\tilde{C}}F(\tilde{z}', (1, \varepsilon')).$$

Hence, the right side of (2.16) is equal to

$$\sum_{i=1}^{l_k}\chi_k^{-1}(C_i^{(k)})\log\{\prod_{z\in C_i^{(k)}}F(z, (1, \varepsilon))\prod_{z\in\tilde{C}_i^{(k)}}F(\tilde{z}, (1, \varepsilon))\} \tag{2.18}$$

$$=\sum_{i=1}^{l_k}\chi_k^{-1}(C_i^{(k)})\log\{\prod_{z\in C_i^{(k)}}F(z', (1, \varepsilon'))\prod_{z\in\tilde{C}_i^{(k)}}F(\tilde{z}', (1, \varepsilon'))\}.$$

It follows easily from the definition of F and the equality (2.17) that

$$\prod_{z\in C}F(z, (1, \varepsilon))=\{\prod_{z\in(-C)}F(z, (1, \varepsilon))\}^{-1} \quad \text{for any cycle } C \text{ in } R_k.$$

Hence, in (2.18), we may restrict the summation with respect to $C_i^{(k)}$ to those cycles C for which $\tilde{C}\neq -C$. The proof of Corollary 2 to Theorem 1 is now complete.

§ 3. We discuss several numerical examples. We use notations introduced in § 2.1~§ 2.4 without further comment.

1. Set $F=Q(\sqrt{5})$. The fundamental unit of F is $\varepsilon_0=(1+\sqrt{5})/2$ and the fundamental totally positive unit is $\varepsilon=\varepsilon_0^2=(3+\sqrt{5})/2$. The class number of F is 1. Put $\mathfrak{f}=(4)$. It is easy to see that the group $(\mathfrak{o}(F)/\mathfrak{f})^*$ of invertible residue classes modulo $\mathfrak{f}$ of F is isomorphic to the direct product of the group of order 2 generated by (-1) and the group of order 6 generated by $\bar{\varepsilon}_0$ ($\bar{\varepsilon}_0$ is the residue class modulo $\mathfrak{f}$ generated by ε_0). Hence, the group of narrow ideal classes modulo $\mathfrak{f}$ of F is isomorphic to $Z/(2)\times Z/(2)$. Denote by $\xi_{0,1}$ and $\xi_{0,2}$ characters of the group $(\mathfrak{o}(F)/\mathfrak{f})^*$ given by the following formula: For

$$\mu\equiv(-1)^a\varepsilon_0^b \mod \mathfrak{f}\ (a, b\in Z),$$

$$\xi_{0,1}(\mu)=(-1)^{a+b}, \qquad \xi_{0,2}(\mu)=(-1)^a.$$

We see there are exactly 2 characters of the group of narrow ideal classes modulo $\mathfrak{f}$ of F of type (2.4). They are given by the following formula:

$$\chi_1((\mu))=(\operatorname{sgn}\mu')\xi_{0,1}(\mu),$$

$$\chi_2((\mu))=(\operatorname{sgn}\mu)\xi_{0,2}(\mu).$$

It is proved that the character χ_1 (resp. χ_2) corresponds to the quadratic extension $K_1=F(\sqrt{\varepsilon_0})$ (resp. $K_2=F(\sqrt{\varepsilon_0'})$) of F. The field K_1 is a biquadratic field with discriminant -400 and the class number of K_1 is 1. Set $\omega=\sqrt{\varepsilon_0}$. Then it is known that a system of generators for the group of units of K_1 is given by $\{\omega, 1+\omega\}$ (see tables of units in Bilevic [3]). For $\mathfrak{f}=(4)$, we have

$$R(\varepsilon, \mathfrak{f}^{-1})=\{(m+n\varepsilon)/4;\ m, n\in Z,\ 0<m\leqq 4,\ 0\leqq n<4,\ m^2+mn+n^2 \text{ is odd}\}.$$

The set $R(\varepsilon, \mathfrak{f}^{-1})$ consists of 4 cycles. More precisely, set

$$C_1=\left\{\frac{1}{4}, 1+\frac{\varepsilon}{4}, \frac{3+3\varepsilon}{4}\right\}$$

and

$$C_2=\left\{\frac{1+2\varepsilon}{4},\ \frac{2+3\varepsilon}{4},\ \frac{1+3\varepsilon}{4}\right\}.$$

Then the decomposition of $R(\varepsilon, \mathfrak{f}^{-1})$ into a disjoint union of cycles is given by

$$R(\varepsilon, \mathfrak{f}^{-1})=C_1\cup -C_1\cup C_2\cup -C_2 .$$

It is easy to see that $\tilde{C}_1=C_1$ and $\tilde{C}_2=-C_2$ (for notations, see § 2.4). Furthermore, for any $z\in C_1$, $\chi_1(\mathfrak{f}(z))=1$. Hence it follows from Corollaries 1 and 2 to Theorem 1 that

$$\varepsilon_0(1+\sqrt{\varepsilon_0})^2=F\Big(\frac{1}{4}, (1, \varepsilon)\Big)^4 F\Big(1+\frac{\varepsilon}{4}, (1, \varepsilon)\Big)^4 F\Big(\frac{3+3\varepsilon}{4}, (1, \varepsilon)\Big)^4$$

$$=F\Big(\frac{1}{4}, (1, \varepsilon')\Big)^4 F\Big(1+\frac{\varepsilon'}{4}, (1, \varepsilon')\Big)^4 F\Big(\frac{3+3\varepsilon'}{4}, (1, \varepsilon')\Big)^4 ,$$

where

$$F(z, (1, \varepsilon))=\frac{\Gamma_2(z, (1, \varepsilon))}{\Gamma_2(1+\varepsilon-z, (1, \varepsilon))},$$

$$F(z, (1, \varepsilon'))=\frac{\Gamma_2(z, (1, \varepsilon'))}{\Gamma_2(1+\varepsilon'-z, (1, \varepsilon'))},$$

Now we employ notations in § 1.3. For any positive integer n, the Dedekind zeta-function $\zeta_F(s)$ of $F=Q(\sqrt{5})$ is given by

$$\zeta_F(s)=\sum_{\substack{0<l\leq n\\ 0\leq m<n}} n^{-2s}\zeta\Big(s, (\varepsilon, \varepsilon'), \Big(\frac{l}{n}, \frac{m}{n}\Big)\Big). \tag{3.1}$$

Recall that ζ_F is holomorphic except for a simple pole at $s=1$ with residue $2\log\varepsilon_0/\sqrt{5}$ and satisfies the following functional equation:

$$\pi^{-s}\sqrt{5}^{s}\Gamma\Big(\frac{s}{2}\Big)^2\zeta_F(s)=\pi^{s-1}\sqrt{5}^{1-s}\Gamma\Big(\frac{1-s}{2}\Big)^2\zeta_F(1-s).$$

Thus

$$\Big\{\frac{d}{ds}\zeta_F(s)\Big\}_{s=0}=-\frac{\log\varepsilon_0}{2}.$$

It follows from Proposition 3 and the equality (3.1) for $n=1$ that

$$-\frac{\log\varepsilon_0}{2}=\log\Big\{\frac{\Gamma_2(1, (1, \varepsilon))\Gamma_2(1, (1, \varepsilon'))}{\rho_2((1, \varepsilon))\rho_2((1, \varepsilon'))}\Big\}+\frac{\varepsilon-\varepsilon'}{24}\log(\varepsilon)^{-2}.$$

It follows from Corollary to Proposition 2 that

$$\log\Big\{\frac{\Gamma_2(1, (1, \varepsilon'))}{\rho_2((1, \varepsilon'))}\Big\}-\log\Big\{\frac{\Gamma_2(\varepsilon, (1, \varepsilon))}{\rho_2((1, \varepsilon))}\Big\}=0.$$

Since $\Gamma_2(1, (1, \varepsilon))=\sqrt{2\pi/\varepsilon}$ and $\Gamma_2(\varepsilon, (1, \varepsilon))=\sqrt{2\pi}$,

$$\log\Big\{\frac{2\pi}{\rho_2((1, \varepsilon))^2\sqrt{\varepsilon}}\Big\}=\Big(\frac{\varepsilon-\varepsilon'}{12}-\frac{1}{4}\Big)\log\varepsilon.$$

Hence,

$$\rho_2((1, \varepsilon))=\sqrt{2\pi}(\varepsilon')^{(3+\sqrt{5})/24}\quad\Big(\varepsilon=\frac{3+\sqrt{5}}{2}\Big).$$

Furthermore, it follows from the equality (3.1) for general n that

$$-\frac{\log\varepsilon_0}{2}=\log\prod_{\substack{0<l\leq n\\ 0\leq m<n}}\Gamma_2\Big(\frac{l+m\varepsilon}{n}, (1, \varepsilon)\Big)\Gamma_2\Big(\frac{l+m\varepsilon'}{n}, (1, \varepsilon')\Big)$$

$$-n^2\log\{\rho_2((1, \varepsilon))\rho_2((1, \varepsilon'))\}+\frac{\varepsilon-\varepsilon'}{4}\log(\varepsilon)^{-2}\sum_{l,m}B_2\Big(\frac{l}{n}\Big),$$

where the summation with respect to (l, m) is over all pairs of integers which satisfy the inequalities $0<l\leqq n$ and $0\leqq m<n$. Thus we get

$$\prod_{\substack{0<l\leq n\\ 0\leq m<n}}\Gamma_2\Big(\frac{l+m\varepsilon}{n}, (1, \varepsilon)\Big)\Gamma_2\Big(\frac{l+m\varepsilon'}{n}, (1, \varepsilon')\Big)=(2\pi)^{n^2}\varepsilon^{(n^2-1)(3-\sqrt{5})/12}.$$

2. As in the previous example, set $F=Q(\sqrt{5})$, $\varepsilon_0=(1+\sqrt{5})/2$, and $\varepsilon=(3+\sqrt{5})/2$. Furthermore, put $\mathfrak{f}=(4-\sqrt{5})$. Then $\mathfrak{f}$ is a prime ideal of F with norm 11. Since $\varepsilon_0\equiv 8 \bmod \mathfrak{f}$, the group $(\mathfrak{o}(F)/\mathfrak{f})^*$ of invertible residue classes modulo $\mathfrak{f}$ is a cyclic group of order 10 generated by ε_0. The group $I(\mathfrak{f})/P(\mathfrak{f})$ of narrow ideal classes modulo $\mathfrak{f}$ is isomorphic to a cyclic group of order 2. Denote by ξ a character of $(\mathfrak{o}(F)/\mathfrak{f})^*$ given by the following formula: For $x\equiv\varepsilon_0^n \bmod \mathfrak{f}$, $\xi(x)=(-1)^n$. The group $I(\mathfrak{f})/P(\mathfrak{f})$ has only one character of type (2.4). It is given by

$$\chi((x))=\operatorname{sgn}(x')\xi(x).$$

The character χ corresponds to a quadratic extension $K=F(\sqrt{(3\sqrt{5}-1)/2})$. The discriminant of the field K is -275 and the class number of K is 1. The fundamental unit ε_0 of F remains to be a primitive unit in K. A table for units of quartic fields with $r_1=2$ and $r_2=1$ is given in [3]. In the table, K is described as a field generated over Q by a root ρ of the equation $\rho^4-\rho^3+2\rho-1=0$. A system of fundamental units of K is given as $\{\rho^3+2, 2\rho^3-\rho^2-\rho+3\}$. It is easy to see that one may put $\rho=\Big(\frac{1-\sqrt{5}}{2}+\sqrt{\frac{3\sqrt{5}-1}{2}}\Big)\Big/2$. Then $\varepsilon_0'=\rho^2/(\rho-1)$, $(\rho-1)(\rho^3+2)=-1$, $2\rho^3-\rho^2-\rho+3=\rho/(\rho-1)^2$. Hence, the group of units of K is generated by $\pm\varepsilon_0$ and by ρ. The set $R(\varepsilon, \mathfrak{f}^{-1})$ consists of two cycles C_1 and $C_2=-C_1$ where we put

$$C_1=\Big\{\frac{5+10\varepsilon}{11}, \frac{1+2\varepsilon}{11}, \frac{9+7\varepsilon}{11}, \frac{4+8\varepsilon}{11}, \frac{3+6\varepsilon}{11}\Big\},$$

$$C_2=\Big\{\frac{6+\varepsilon}{11}, \frac{10+9\varepsilon}{11}, \frac{2+4\varepsilon}{11}, \frac{7+3\varepsilon}{11}, \frac{8+5\varepsilon}{11}\Big\}.$$

It is easy to see that $\chi(C_1)=-\chi(C_2)=1$. Set

$$F(z)=\frac{\Gamma_2(z,(1,\varepsilon))}{\Gamma_2(1+\varepsilon-z,(1,\varepsilon))}\quad\left(\varepsilon=\frac{3+\sqrt{5}}{2}\right).$$

It follows from Corollary 1 and Corollary 2 to Theorem 1 that

$$F\left(\frac{5+10\varepsilon}{11}\right)F\left(\frac{1+2\varepsilon}{11}\right)F\left(\frac{9+7\varepsilon}{11}\right)F\left(\frac{4+8\varepsilon}{11}\right)F\left(\frac{3+6\varepsilon}{11}\right)$$

$$\times F\left(\frac{10+5\varepsilon}{11}\right)F\left(\frac{2+\varepsilon}{11}\right)F\left(\frac{7+9\varepsilon}{11}\right)F\left(\frac{8+4\varepsilon}{11}\right)F\left(\frac{6+3\varepsilon}{11}\right)$$

$$=\left(\frac{3+\sqrt{5}}{2}+\sqrt{\frac{3\sqrt{5}-1}{2}}\right)/2.$$

3. Set $F=\boldsymbol{Q}(\sqrt{21})$. The fundamental unit of F is totally positive and is given by $\varepsilon=(5+\sqrt{21})/2$. The class number of F is 1. Put $\mathfrak{f}=(3+\sqrt{21})/2$. Then the group $(\mathfrak{o}(F)/\mathfrak{f})^*$ of invertible residue classes modulo $\mathfrak{f}$ of F is a cyclic group of order 2. We note that $\varepsilon\equiv 1 \bmod. \mathfrak{f}$ and that the group of narrow ideal classes modulo $\mathfrak{f}$ of F is isomorphic to $\boldsymbol{Z}/(2)\times\boldsymbol{Z}/(2)$. Denote by ξ_0 the unique non-principal character of the group $(\mathfrak{o}(F)/\mathfrak{f})^*$. It is easy to see that there are exactly two characters of type (2.4) of the group of narrow ideal classes modulo $\mathfrak{f}$ of F. They are given by the following formula:

$$\chi_1((\mu))=\operatorname{sgn}(\mu')\xi_0(\mu),\qquad \chi_2((\mu))=\operatorname{sgn}(\mu)\xi_0(\mu).$$

The character χ_1 (resp. χ_2) corresponds to the quadratic extension

$$K_1=F\left(\sqrt{\frac{3+\sqrt{21}}{2}}\right)\quad\left(\text{resp. } K_2=F\left(\sqrt{\frac{3-\sqrt{21}}{2}}\right)\right).$$

Put $\mathfrak{a}_1=\mathfrak{o}(F)$ and $\mathfrak{a}_2=\mathfrak{f}=(3+\sqrt{21})/2$. Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ form a complete set of representatives for narrow ideal classes of F. By simple computations we see that $R(\varepsilon,(\mathfrak{a}_1\mathfrak{f})^{-1})=\{(1+2\varepsilon)/3,(2+\varepsilon)/3\}$ and $R(\varepsilon,(\mathfrak{a}_2\mathfrak{f})^{-1})=\{1/3, 1+\varepsilon/3, (2+2\varepsilon)/3, 2/3, 1+(2/3)\varepsilon, (1+\varepsilon)/3\}$. The set $R(\varepsilon,(\mathfrak{a}_1\mathfrak{f})^{-1})$ consists of two cycles $C_1^{(1)}=\{(1+2\varepsilon)/3\}$ and $C_2^{(1)}=\{(2+\varepsilon)/3\}$. The set $R(\varepsilon,(\mathfrak{a}_2\mathfrak{f})^{-1})$ also consists of two cycles

$$C_1^{(2)}=\left\{\frac{1}{3}, 1+\frac{\varepsilon}{3}, \frac{2+2\varepsilon}{3}\right\}\quad\text{and}\quad C_2^{(2)}=\left\{\frac{2}{3}, 1+\frac{2}{3}\varepsilon, \frac{1+\varepsilon}{3}\right\}.$$

We see that $C_2^{(1)}=-C_1^{(1)}=\tilde{C}_1^{(1)}$ and that $C_2^{(2)}=-C_1^{(2)}=-\tilde{C}_1^{(2)}$. For any $z\in C_1^{(2)}$, $\chi_1(\mathfrak{a}_2\mathfrak{f}(z))=1$ (for notations, see § 2.4). It is easy to see that ε remains to be a primitive unit in K_1. It is proved that the class number of K_1 is equal to 1. Choose a unit η of K_1 which satisfies conditions (2.11) and (2.12) for $K=K_1$. Then Corollary 1 and Corollary 2 to Theorem 1 imply that

$$\left|\frac{\eta}{\eta^\sigma}\right|=F\left(\frac{1}{3},(1,\varepsilon)\right)^2F\left(1+\frac{\varepsilon}{3},(1,\varepsilon)\right)^2F\left(\frac{2+2\varepsilon}{3},(1,\varepsilon)\right)^2,$$

where $F(z,(1,\varepsilon))=\frac{\Gamma_2(z,(1,\varepsilon))}{\Gamma_2(1+\varepsilon-z,(1,\varepsilon))}$ and σ is the conjugation with respect to F. Denote by X the right side of the above equality. A numerical computation (which involves a computer machine) show that $X+X^{-1}=|\eta/\eta^\sigma|+|\eta^\sigma/\eta|$ $=2.791287847\cdots$, while $\varepsilon=4.791287847\cdots$. Denote by τ an isomorphism from K_1 into K_2 which induces the non-trivial automorphism on F. Then $\tau(X+X^{-1})$ $=\pm(\eta^\tau/\eta^{\sigma\tau}+\eta^{\sigma\tau}/\eta^\tau)$. Since $\eta^\tau/\eta^{\sigma\tau}$ is a complex number of modulus 1, we see that $X+X^{-1}$ is an integer in F whose conjugate is in the interval $(-2, 2)$. Thus we infer that $X+X^{-1}=\varepsilon-2=(1+\sqrt{21})/2$.

Hence,

$$X=(\varepsilon-2+\sqrt{\varepsilon-1})/2=\frac{\varepsilon+\sqrt{\varepsilon-1}}{\varepsilon-\sqrt{\varepsilon-1}}.$$

Therefore, $\eta/\frac{1}{2}(\varepsilon+\sqrt{\varepsilon-1})$ is a unit in F. Thus,

$$F\left(\frac{1}{3},(1,\varepsilon)\right)^2F\left(1+\frac{\varepsilon}{3},(1,\varepsilon)\right)^2F\left(\frac{2+2\varepsilon}{3},(1,\varepsilon)\right)^2$$

$$=\left(\frac{1+\sqrt{21}}{2}+\sqrt{\frac{3+\sqrt{21}}{2}}\right)/2.$$

Moreover, we have seen that the group of units of K_1 is generated by $\pm\varepsilon$ and $(1/2)(\varepsilon+\sqrt{\varepsilon-1})$ $(\varepsilon=(5+\sqrt{21})/2)$.

4. Set $F=\boldsymbol{Q}(\sqrt{3})$. Then the fundamental unit of F is totally positive and is given $\varepsilon=2+\sqrt{3}$. The class number of F is 1. Put $\mathfrak{f}=(4(1+\sqrt{3}))$. Then the mapping $(l,m,n)\mapsto(-1)^l(2\sqrt{3}-3)^m\varepsilon^n$ establishes an isomorphic mapping from the group $\boldsymbol{Z}/(2)\times\boldsymbol{Z}/(2)\times\boldsymbol{Z}/(4)$ onto the group $(\mathfrak{o}(F)/\mathfrak{f})^*$ of invertible residue classes modulo $\mathfrak{f}$. The group of narrow ideal classes modulo $\mathfrak{f}$ of F is isomorphic to $\boldsymbol{Z}/(2)\times\boldsymbol{Z}/(2)\times\boldsymbol{Z}/(2)$. Denote by ξ_1 and ξ_2 characters of the group $(\mathfrak{o}(F)/\mathfrak{f})^*$ given as follows:

For $x\equiv(-1)^l(2\sqrt{3}-3)^m\varepsilon^n$ modulo $\mathfrak{f}$.

$$\xi_1(x)=(-1)^l$$

$$\xi_2(x)=(-1)^{l+m}.$$

There are four characters χ_1, χ_2, χ_3 and χ_4 of the group of narrow ideal classes modulo $\mathfrak{f}$ of F of type (2.4). They are given by the following formula:

$$\chi_1((x))=\operatorname{sgn}(x')\xi_1(x)$$

$$\chi_2((x))=\operatorname{sgn}(x')\xi_2(x)$$

$$\chi_3((x))=\operatorname{sgn}(x)\xi_1(x)$$

$$\chi_4((x))=\operatorname{sgn}(x)\xi_2(x).$$

The character χ_1 (resp. χ_4) corresponds to the quadratic extension $K_1=F(\sqrt{(1+\sqrt{3})(2+\sqrt{3})})$ (resp. $K_4=F(\sqrt{(1-\sqrt{3})(2-\sqrt{3})})$). The character χ_2 (resp. χ_3) corresponds to the quadratic extension $K_2=F(\sqrt{1+\sqrt{3}})$ (resp. $K_3=F(\sqrt{1-\sqrt{3}})$). The unit $\varepsilon=2+\sqrt{3}$ remains to be a primitive unit in K_1 and in K_2. Furthermore class numbers of K_1 and K_2 are both equal to 1. As a complete set of representatives for narrow ideal classes of F, we choose $\mathfrak{a}_1=\mathfrak{o}(F)$ and $\mathfrak{a}_2=(1+\sqrt{3})$. Then the set $R(\varepsilon, (\mathfrak{a}_1\mathfrak{f})^{-1})$ consists of four cycles $C_1^{(1)}$, $C_2^{(1)}$, $(-C_1^{(1)})$ and $(-C_2^{(1)})$, where we put

$$C_1^{(1)}=\left\{\frac{1+\varepsilon}{8}, \frac{7+5\varepsilon}{8}, \frac{3+3\varepsilon}{8}, \frac{5+7\varepsilon}{8}\right\},$$

$$C_2^{(1)}=\left\{\frac{5+\varepsilon}{8}, \frac{7+\varepsilon}{8}, \frac{7+3\varepsilon}{8}, \frac{5+3\varepsilon}{8}\right\}.$$

We have $\tilde{C}_1^{(1)}=C_1^{(1)}$ and $\tilde{C}_2^{(1)}=-C_2^{(1)}$. The set $R(\varepsilon, (\mathfrak{a}_2\mathfrak{f})^{-1})$ consists of eight cycles $C_1^{(2)}$, $C_2^{(2)}$, $C_3^{(2)}$, $C_4^{(2)}$, $(-C_1^{(2)})$, $(-C_2^{(2)})$, $(-C_3^{(2)})$ and $(-C_4^{(2)})$, where we put

$$C_1^{(2)}=\left\{\frac{1}{8}, 1+\frac{\varepsilon}{8}, \frac{7+4\varepsilon}{8}, \frac{4+7\varepsilon}{8}\right\},$$

$$C_2^{(2)}=\left\{\frac{3}{8}, 1+\frac{3\varepsilon}{8}, \frac{5+4\varepsilon}{8}, \frac{4+5\varepsilon}{8}\right\},$$

$$C_3^{(2)}=\left\{\frac{2+\varepsilon}{8}, \frac{7+6\varepsilon}{8}, \frac{2+7\varepsilon}{8}, \frac{1+6\varepsilon}{8}\right\},$$

$$C_4^{(2)}=\left\{\frac{3+2\varepsilon}{8}, \frac{6+3\varepsilon}{8}, \frac{5+2\varepsilon}{8}, \frac{6+5\varepsilon}{8}\right\},$$

We see that $\tilde{C}_1^{(2)}=C_1^{(2)}$, $\tilde{C}_2^{(2)}=C_2^{(2)}$, $\tilde{C}_3^{(2)}=-C_3^{(2)}$ and $\tilde{C}_4^{(2)}=-C_4^{(2)}$. Furthermore, $\chi_1(C_1^{(1)})=-1$, $\chi_1(C_1^{(2)})=\chi_1(C_2^{(2)})=1$, $\chi_2(C_1^{(1)})=\chi_2(C_1^{(2)})=\chi_2(C_2^{(2)})=1$.

Now let η_1 (resp. η_2) be a unit of K_1 (resp. K_2) which satisfies the conditions (2.11) and (2.12) for $K=K_1$ (resp. K_2). To simplify notations set

$$F(z)=F(z, (1, \varepsilon))=\frac{\Gamma_2(z, (1, \varepsilon))}{\Gamma_2(1+\varepsilon-z, (1, \varepsilon))},$$

$$X=F\left(\frac{1+\varepsilon}{8}\right)^2 F\left(\frac{7+5\varepsilon}{8}\right)^2 F\left(\frac{3+3\varepsilon}{8}\right)^2 F\left(\frac{5+7\varepsilon}{8}\right)^2$$

$$Y=F\left(\frac{1}{8}\right)^2 F\left(1+\frac{\varepsilon}{8}\right)^2 F\left(\frac{7+4\varepsilon}{8}\right)^2 F\left(\frac{4+7\varepsilon}{8}\right)^2$$

$$\times F\left(\frac{3}{8}\right)^2 F\left(1+\frac{3\varepsilon}{8}\right)^2 F\left(\frac{5+4\varepsilon}{8}\right)^2 F\left(\frac{4+5\varepsilon}{8}\right)^2.$$

Then Corollary 1 and Corollary 2 to Theorem 1 imply that

$$|\eta_1/\eta_1^{\sigma_1}|=X^{-1}Y \quad \text{and} \quad |\eta_2/\eta_2^{\sigma_2}|=XY,$$

where σ_1 (resp. σ_2) is the non trivial automorphism of K_1 (resp. K_2) with respect to F. Numerical computations (which involve a computer machine) show that

$$X^{-1}Y+XY^{-1}=12.92820323\cdots \quad \text{and} \quad XY+X^{-1}Y^{-1}=42.78460969\cdots,$$

$$\text{while} \quad \sqrt{3}=1.7320508075\cdots.$$

Both $X^{-1}Y+XY^{-1}$ and $XY+X^{-1}Y^{-1}$ are integers of F whose conjugates are in the interval $(-2, 2)$. Hence we obtain $X^{-1}Y+XY^{-1}=6+4\sqrt{3}$ and $XY+X^{-1}Y^{-1}=22+12\sqrt{3}$.

Hence,

$$X^{-1}Y=3+2\sqrt{3}+2\sqrt{5+3\sqrt{3}}$$
$$=(2+\sqrt{3}+\sqrt{5+3\sqrt{3}})/(2+\sqrt{3}-\sqrt{5+3\sqrt{3}})$$

and

$$XY=(2+\sqrt{3})(4+\sqrt{3}+2\sqrt{3}\sqrt{1+\sqrt{3}})$$
$$=(\sqrt{3}+\sqrt{1+\sqrt{3}})/(\sqrt{3}-\sqrt{1+\sqrt{3}}).$$

Thus we see that $\eta_1/(2+\sqrt{3}+\sqrt{5+3\sqrt{3}})$ and $\eta_2/(\sqrt{3}+\sqrt{1+\sqrt{3}})$ are both units of F. Hence, the group of units of K_1 (resp. K_2) is generated by $\pm\varepsilon$ and $2+\sqrt{3}+\sqrt{5+3\sqrt{3}}$ (resp. $\sqrt{3}+\sqrt{1+\sqrt{3}}$).

5. Set $F=\mathbf{Q}(\sqrt{10})$. The fundamental unit is $\varepsilon_0=3+\sqrt{10}$ and the fundamental totally positive unit is $\varepsilon=\varepsilon_0^2=19+6\sqrt{10}$. The class number of F is 2. Put $\mathfrak{f}=(4)$. Then the mapping: $(l, m)\mapsto(-1)^l\varepsilon_0^m$ establishes an isomorphic mapping from the group $\mathbf{Z}/(2)\times\mathbf{Z}/(4)$ onto the group $(\mathfrak{o}(F)/\mathfrak{f})^*$ of invertible residue classes modulo $\mathfrak{f}$ of F. Set $\mathfrak{p}_5=(5, \sqrt{10})$. Then the integral ideal $\mathfrak{p}_5$ is not principal and $\mathfrak{p}_5^2=(5)$, $5\equiv1 \pmod{\mathfrak{f}}$. Thus we see that the group $I(\mathfrak{f})/P(\mathfrak{f})$ of narrow ideal classes modulo $\mathfrak{f}$ of F is isomorphic to $\mathbf{Z}/(2)\times\mathbf{Z}/(2)\times\mathbf{Z}/(2)$. For $x\equiv(-1)^l\varepsilon_0^m$ modulo $\mathfrak{f}$, put

$$\xi_1(x)=(-1)^{l+m} \quad \text{and} \quad \xi_2(x)=(-1)^l.$$

There are four characters of the group $I(\mathfrak{f})/P(\mathfrak{f})$ of type (2.4). They are given by the following formula: For $\mathfrak{a}=\mathfrak{p}_5^n(x)$ ($n=0, 1$, $x\in F^*$ is prime to $\mathfrak{f}$),

$$\chi_1(\mathfrak{a})=\operatorname{sgn}(x')\xi_1(x)(-1)^n$$
$$\chi_2(\mathfrak{a})=\operatorname{sgn}(x')\xi_1(x)$$
$$\chi_3(\mathfrak{a})=\operatorname{sgn}(x)\xi_2(x)(-1)^n$$
$$\chi_4(\mathfrak{a})=\operatorname{sgn}(x)\xi_2(x).$$

It is proved that the character χ_1 (resp. χ_3) corresponds to the quadratic extension $K_1=F(\sqrt{\varepsilon_0})$ (resp. $K_3=F(\sqrt{\varepsilon_0'})$) of F and that the character χ_2 (resp. χ_4)

corresponds to the quadratic extension $K_2=F(\sqrt{2\varepsilon_0})$ (resp. $K_4=F(\sqrt{2\varepsilon_0'})$) of F.

Now we will show that the class numbers of K_1 and K_2 are both equal to 2. The discriminants of K_1 and K_2 are both equal to $4^2\times(40)^2$ and the Minkowski constants for K_1 and K_2 are smaller than 0.12. Hence, in each ideal class of K_1 or K_2, there exists an integral ideal whose norm is smaller than $0.12\times 160=19.2$. Prime ideals in F with norm not greater than 19 are $\mathfrak{p}_{13}=(13, 6+\sqrt{10})$, $\mathfrak{p}_{13}'=(13, 6-\sqrt{10})$, $\mathfrak{p}_5=(5, \sqrt{10})$, $\mathfrak{p}_3=(3, 1+\sqrt{10})$, $\mathfrak{p}_3'=(3, 1-\sqrt{10})$ and $\mathfrak{p}_2=(2, \sqrt{10})$. In K_1, $\mathfrak{p}_2=(\varepsilon_0-1+2\sqrt{\varepsilon_0})$ and $\mathfrak{p}_2=\mathfrak{B}_2^2$, where we put $\mathfrak{B}_2=(\mathfrak{p}_2, \sqrt{\varepsilon_0}-1)$. It is easy to see that $\mathfrak{B}_2$ is not a principal ideal in K_1. In K_1, the principal ideals $\mathfrak{p}_3=\mathfrak{p}_2((\sqrt{10}-2)/2)$ and $\mathfrak{p}_5=\mathfrak{p}_2(\sqrt{10}/2)$ remain to be prime, while $\mathfrak{p}_3'$, $\mathfrak{p}_{13}$ and $\mathfrak{p}_{13}'$ split into products of different primes in the following manner:

$$\mathfrak{p}_3'=(\mathfrak{p}_3', \sqrt{\varepsilon_0}-1)(\mathfrak{p}_3', \sqrt{\varepsilon_0}+1),$$

$$\mathfrak{p}_{13}=(\mathfrak{p}_{13}, \sqrt{\varepsilon_0}-6)(\mathfrak{p}_{13}, \sqrt{\varepsilon_0}+6),$$

$$\mathfrak{p}_{13}'=(\mathfrak{p}_{13}', \sqrt{\varepsilon_0}-3)(\mathfrak{p}_{13}', \sqrt{\varepsilon_0}+3).$$

Now it is easy to see that

$$(\sqrt{\varepsilon_0}-1)=\mathfrak{B}_2(\mathfrak{p}_3', \sqrt{\varepsilon_0}-1),$$

$$(\sqrt{10}+\sqrt{\varepsilon_0})=(\mathfrak{p}_3', \sqrt{\varepsilon_0}+1)(\mathfrak{p}_{13}, \sqrt{\varepsilon_0}-6)$$

and

$$(\sqrt{\varepsilon_0}-3)=\mathfrak{B}_2(\mathfrak{p}_{13}', \sqrt{\varepsilon_0}-3).$$

Hence, prime ideals $(\mathfrak{p}_3', \sqrt{\varepsilon_0}-1)$, $(\mathfrak{p}_{13}, \sqrt{\varepsilon_0}-6)$ and $(\mathfrak{p}_{13}', \sqrt{\varepsilon_0}-3)$ are all in the same class as $\mathfrak{B}_2$. Since $\mathfrak{p}_3$, $\mathfrak{p}_{13}$ and $\mathfrak{p}_{13}'$ are all principal ideals in K_1, we have seen that all ideals in K_1 are either principal or equivalent to $\mathfrak{B}_2$. Hence the class number of K_1 is equal to 2. In K_2, $\mathfrak{p}_2=(\sqrt{2\varepsilon_0})=\tilde{\mathfrak{B}}_2^2$, where we put $\tilde{\mathfrak{B}}_2=(\mathfrak{p}_2, \sqrt{\varepsilon_0-2}-1)$ $(2\varepsilon_0=(\varepsilon_0-2)((\varepsilon_0-1)/3)^2)$. Since $\mathfrak{p}_2$ is not principal in F, $\tilde{\mathfrak{B}}_2$ is not principal in K_2. In K_2, principal ideals $\mathfrak{p}_3'$, $\mathfrak{p}_{13}$, $\mathfrak{p}_{13}'$ all remain to be prime in K_2, while $\mathfrak{p}_3$ and $\mathfrak{p}_5$ split into products of different primes in the following manner;

$$\mathfrak{p}_3=(\mathfrak{p}_3, \sqrt{2\varepsilon_0}-1)(\mathfrak{p}_3, \sqrt{2\varepsilon_0}+1)$$

$$\mathfrak{p}_5=(\mathfrak{p}_5, \sqrt{2\varepsilon_0}-1)(\mathfrak{p}_5, \sqrt{2\varepsilon_0}+1).$$

Now it is easy to see that

$$(\sqrt{2\varepsilon_0}+1)=(\mathfrak{p}_3, \sqrt{2\varepsilon_0}+1)(\mathfrak{p}_5, \sqrt{2\varepsilon_0}+1)$$

and

$$(\sqrt{\varepsilon_0-2}-1)=\tilde{\mathfrak{B}}_2(\mathfrak{p}_5, \sqrt{2\varepsilon_0}+1).$$

Hence, $(\mathfrak{p}_5, \sqrt{2\varepsilon_0}+1)$, $(\mathfrak{p}_3, \sqrt{2\varepsilon_0}+1)$ and $\tilde{\mathfrak{B}}_2$ are in the same ideal class in K_2.

Since $\mathfrak{p}_3$ and $\mathfrak{p}_5$ are both principal in K_2, we see that all ideals in K_2 are

either principal or equivalent to $\tilde{\mathfrak{B}}_2$. Hence, the class number of K_2 is equal to 2. The fundamental unit ε_0 of F remains to be primitive in K_2. On the other hand, ε_0 is the square of a primitive unit $\sqrt{\varepsilon_0}$ in K_1. Set $\mathfrak{a}_1=\mathfrak{o}(F)$ and $\mathfrak{a}_2=\mathfrak{p}_2=(2, \sqrt{10})$. Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ form a complete set of representatives for narrow ideal classes of F. We see that the set $R(\varepsilon, (\mathfrak{a}_1\mathfrak{f})^{-1})$ consists of twelve cycles:

$$R(\varepsilon, (\mathfrak{a}_1\mathfrak{f})^{-1})=\bigcup_{k=1}^{6} C_k^{(1)}\cup\bigcup_{k=1}^{6}(-C_k^{(1)}) \quad \text{(for notations see § 2.4)},$$

where we put

$$C_1^{(1)}=\left\{\frac{2+\varepsilon}{12}, \frac{11+4\varepsilon}{12}, \frac{8+7\varepsilon}{12}, \frac{5+10\varepsilon}{12}\right\},$$

$$C_2^{(1)}=\left\{\frac{1+2\varepsilon}{12}, \frac{4+11\varepsilon}{12}, \frac{7+8\varepsilon}{12}, \frac{10+5\varepsilon}{12}\right\},$$

$$C_3^{(1)}=\left\{\frac{1}{4}, 1+\frac{\varepsilon}{4}, \frac{3+2\varepsilon}{4}, \frac{2+3\varepsilon}{4}\right\}.$$

We omit the descriptions of the remaining cycles $C_4^{(1)}$, $C_5^{(1)}$ and $C_6^{(1)}$. For these omitted three cycles, the relation $\tilde{C}=-C$ holds. Hence, they may be neglected so far as the evaluation of L-functions at $s=1$ is concerned (see Corollary 2 to Theorem 1). We note that $\tilde{C}_1^{(1)}=C_2^{(1)}$ and $\tilde{C}_3^{(1)}=C_3^{(1)}$. It is easy to see that $\chi_1(C_k^{(1)})=\chi_2(C_k^{(1)})=1$ for $k=1, 2, 3$.

The set $R(\varepsilon, (\mathfrak{a}_2\mathfrak{f})^{-1})$ consists of thirty six cycles:

$$R(\varepsilon, (\mathfrak{a}_2\mathfrak{f})^{-1})=\bigcup_{k=1}^{18} C_k^{(2)}\cup\bigcup_{k=1}^{18}(-C_k^{(2)}),$$

where we put

$$C_1^{(2)}=\left\{\frac{17+\varepsilon}{48}, \frac{47+7\varepsilon}{48}, \frac{41+25\varepsilon}{48}, \frac{23+31\varepsilon}{48}\right\},$$

$$C_2^{(2)}=\left\{\frac{1+17\varepsilon}{48}, \frac{7+47\varepsilon}{48}, \frac{25+41\varepsilon}{48}, \frac{31+23\varepsilon}{48}\right\},$$

$$C_3^{(2)}=\left\{\frac{9+\varepsilon}{16}, \frac{15+15\varepsilon}{16}, \frac{1+9\varepsilon}{16}, \frac{7+7\varepsilon}{16}\right\},$$

$$C_4^{(2)}=\left\{\frac{29+13\varepsilon}{48}, \frac{35+43\varepsilon}{48}, \frac{5+37\varepsilon}{48}, \frac{11+19\varepsilon}{48}\right\},$$

$$C_5^{(2)}=\left\{\frac{13+29\varepsilon}{48}, \frac{19+11\varepsilon}{48}, \frac{37+5\varepsilon}{48}, \frac{43+35\varepsilon}{48}\right\},$$

$$C_6^{(2)}=\left\{\frac{3+3\varepsilon}{16}, \frac{13+5\varepsilon}{16}, \frac{11+11\varepsilon}{16}, \frac{5+13\varepsilon}{16}\right\}.$$

We omit the descriptions of the remaining twenty-four cycles $\bigcup_{k=7}^{18} C_k^{(2)}\cup\bigcup_{k=7}^{18}(-C_k^{(2)})$. For these omitted cycles, the relation $\tilde{C}=-C$ holds. Hence they may be neglected so far as the computations of $L_F(1, \chi_i)$ $(i=1, 2)$ are concerned.

We note that $\tilde{C}_1^{(2)}=C_2^{(2)}$, $\tilde{C}_3^{(2)}=C_3^{(2)}$, $\tilde{C}_4^{(2)}=C_5^{(2)}$, $\tilde{C}_6^{(2)}=C_6^{(2)}$. It is easy to see that $\chi_1(C_k^{(2)})=-\chi_2(C_k^{(2)})=1$ for $k=1, \cdots, 6$. Now set

$$F(z)=\frac{\Gamma_2(z, (1, \varepsilon))}{\Gamma_2(1+\varepsilon-z, (1, \varepsilon))} \qquad (\varepsilon=19+6\sqrt{10})$$

$$X=\prod_{k=1}^{3} \prod_{z\in C_k^{(1)}} F(z)^2$$

and

$$Y=\prod_{k=1}^{6} \prod_{z\in C_k^{(2)}} F(z)^2 .$$

Furthermore denote by η_1 (resp. η_2) a unit of K_1 (resp. K_2) which satisfies the conditions (2.11) and (2.12) for $K=K_1$ (resp. K_2). Let σ_1 (resp. σ_2) be the non-trivial automorphism of K_1 (resp. K_2) with respect to F. Then it follows from Corollaries 1 and 2 to Theorem 1 that

$$\left(\frac{\eta_1}{\eta_1^{\sigma_1}}\right)^2=(XY)^4, \qquad \left(\frac{\eta_2}{\eta_2^{\sigma_2}}\right)^2=\left(\frac{X}{Y}\right)^2 .$$

Numerical computations (which involve a computer machine) show that

$$(XY)^2+(XY)^{-2}=2629.50750\cdots$$

and

$$\frac{X}{Y}+\frac{Y}{X}=100.5964425\cdots .$$

Both $(XY)^2+(XY)^{-2}$ and $(X/Y)+(Y/X)$ are integers in F whose conjugates are in the interval $(-2, 2)$. Hence we conclude that

$$(XY)^2+(XY)^{-2}=1314+416\sqrt{10}=(26+8\sqrt{10})^2-2$$

and

$$\frac{X}{Y}+\frac{Y}{X}=50+16\sqrt{10} .$$

Thus, we have

$$XY=13+4\sqrt{10}+(3+\sqrt{10})\sqrt{8(\sqrt{10}-1)}$$

$$=\varepsilon_0(\varepsilon_0-2)+2(\varepsilon_0-1)\sqrt{\varepsilon_0}$$

and

$$\frac{X}{Y}=25+8\sqrt{10}+4(3+\sqrt{10})\sqrt{1+\sqrt{10}}$$

$$=\frac{\sqrt{\varepsilon_0-2}+2}{\sqrt{\varepsilon_0-2}-2} \qquad (\varepsilon_0=3+\sqrt{10}) .$$

It follows that both η_1/XY and $\eta_2/(\sqrt{\varepsilon_0-2}+2)$ are units of F. Hence the group of units of K_1 (resp. K_2) are generated by $\pm\sqrt{\varepsilon_0}$ (resp. $\pm\varepsilon_0$) and $\varepsilon_0(\varepsilon_0-2)+2(\varepsilon_0-1)\sqrt{\varepsilon_0}$ (resp. $2+\sqrt{\varepsilon_0-2}$).

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