On the discrete periodic Toda flow

(and Gauß composition for quadratic forms)

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Plan: 1) Motivation: Periodic box-ball system & number theory

2) Main protagonist: Discrete periodic Toda system

Main Goal: 3) Linearization of dpToda via Mumford's Jacobian

4) pBB $\subset$ dpToda

5) Outlook (or a dream)

§1. Motivation: pBB & number theory

'02 Yura–Tokihiro "On a periodic soliton cellular automaton"

As abstract algorithm "equivalent" to arithmetic-harmonic mean algo.
(Y.T '02) ($\sqrt[2N]{\cdot}$)

It has essentially two sources

$q \to 0$ 6-vertex model, Bethe vectors, qKZ, ..
crystallization

tropicalization KP, KdV, Toda

Let V be 2-dim std rep. of $U_q(sl_2)$

For $N \in \mathbb{N}$, define the state space of pBB flow

$$C_N = \underset{\text{Kashiwara}}{\mathrm{Crystal}}(V^{\otimes N}) \simeq \{0,1\}^{\times N}$$

$|C_N| = 2^N$

$0 \hat{=} \square$ empty box
$1 \hat{=} \boxed{\bullet}$ box with ball

Kashiwara also defined action of extended affine Weyl group

$$W = \hat{W}(\hat{sl}_2) = \langle \underset{\text{ext.}}{\omega}, \underset{\text{aff.}}{s_0}, s_1 \rangle \curvearrowright C_N$$

involutions

$(W(sl_2) = \langle s_1 \rangle)$

Def: The periodic box-ball flow is defined by

$$B : C_N \longrightarrow C_N \quad , \quad B = \omega \circ s_1$$

$\boxed{N=11}$

Example:

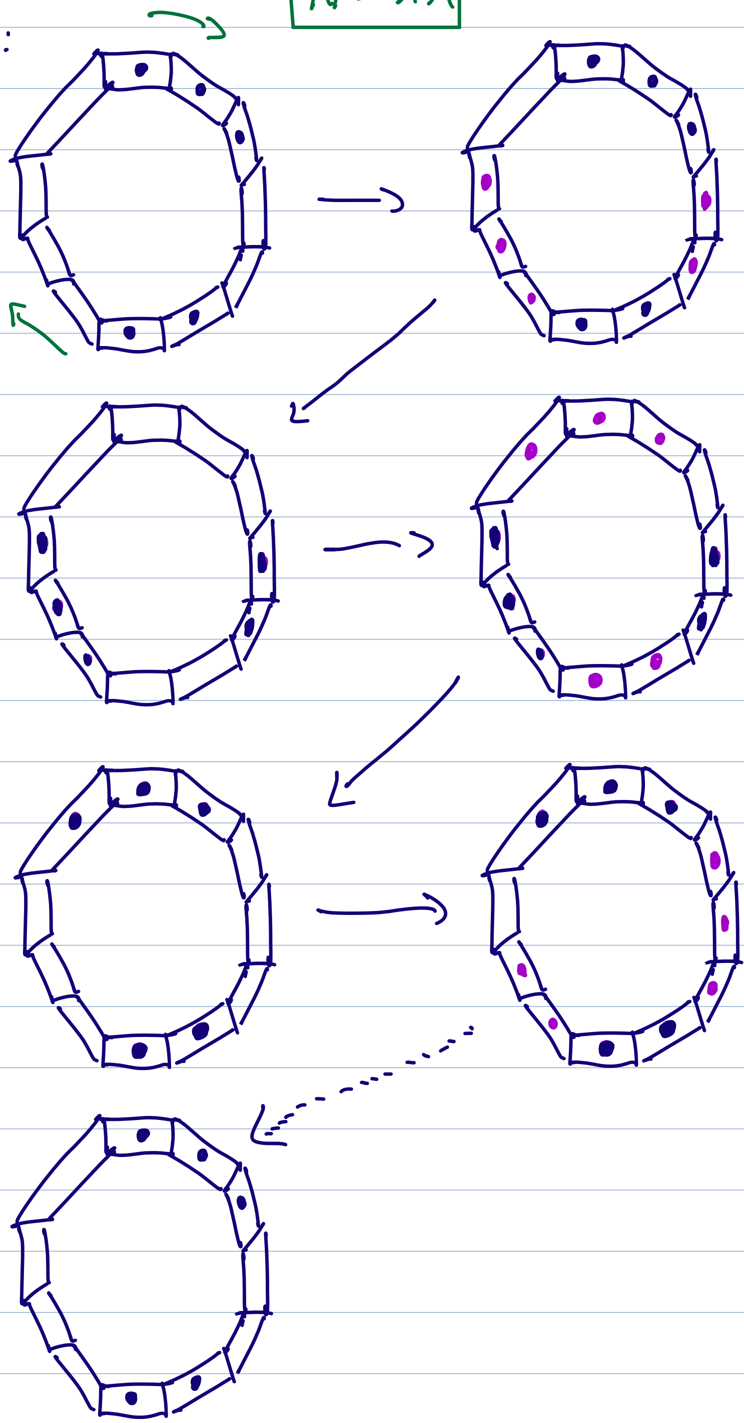
$t=0$:

$t=1$:

$t=2$:

$\vdots$

$t=33$



Def: (Fundamental cycle)

For $\alpha \in C_N$, the fundamental cycle (or period) of α is defined by

$$\operatorname{per}(\alpha) = \min\{ m \in \mathbb{N} \mid \mathcal{B}^m(\alpha) = \alpha \}$$

<u>Thm</u>: (Tokihiro-Mada, '05) (rough form)

$\forall N \ \exists \alpha_N \in C_N$ s.th.

$$\boxed{\operatorname{per}(\alpha_N) \approx e^{\psi(N)}}$$

[$\exists$ explicit, combinatorial formula for $\operatorname{per}(\alpha)$]

Here $\psi(N) = \sum_{p \leq N} \lfloor \log_p(N) \rfloor \log(p)$ is

Chebyshev's prime counting function.

<u>Fact</u>: Riemann hypothesis $\iff$ $\psi(N) = N + \mathcal{O}(N^{1/2} \log^2(N))$ as $N \to \infty$

<u>Rm</u>: Despite the seemingly simple appearance of pBB flow, it is complicated (or sufficiently interesting) enough to encode the RH!

(Recall: $\text{pBB} \triangleq \text{2N}\!\!\sqrt{\cdot}$) ("inverse Frobenius"?)

§1. Discrete periodic Toda flow

[Later: pBB c dpToda]

There are many different flavours of Toda systems, e.g.,

$\sim$ quantum cohomology / K-theory, Sato grassmannian, twistor theory, ...

quantum – classical

infinite – finite

periodic – open

continuous – discrete – tropical

Of central importance in integrable systems. ("Universal")

There are very interesting links with number theory, e.g.,

- Whittaker fcts (automorphic fcts) [Kostant, ..]
- (generalized) Γ-fcts [Gerasimov-Lebedev-Oblezin]
- regulators of number fields [Butler]
- Random matrix theory
- ....

Def: (Hirota)

The discrete periodic Toda flow$^{\text{for } n\in\mathbb{N}}$ is the birational map

$$\tilde{J} : \underset{\text{state space}}{R^{2n}} \longrightarrow R^{2n}$$

R a (nice) field, such as $\mathbb{R}$ or $\mathbb{C}$, but also $\mathbb{Q}(T)$.

$$(I^t, V^t) \mapsto (I^{t+1}, V^{t+1})$$

Coordinates $\quad I^t = (I_1^t, \dots, I_n^t)$

$V^t = (V_1^t, \dots, V_n^t)$

Discrete: $\quad t \in \mathbb{N}_0 = \{0, 1, \dots\}$

Periodic: $\quad I_{i+n}^t = I_i^t, \quad V_{i+n}^t = V_i^t \quad \forall i \in \{1, \dots, n\}$

defined by

$$\boxed{\begin{aligned} I_i^{t+1} &= I_i^t + V_i^t - V_{i-1}^{t+1} \\ V_i^{t+1} &= \frac{I_{i+1}^t V_i^t}{I_i^{t+1}} \end{aligned}} \qquad \text{(dpToda flow)}$$

One can show

$$I_i^{t+1} = I_i^t \, \frac{I_i^t I_{i-1}^t \cdots I_{i+2}^t + V_i^t I_{i-1}^t \cdots I_{i+2}^t + \dots + V_i^t V_{i-1}^t \cdots V_{i+2}^t}{I_{i-1}^t I_{i-2}^t \cdots I_{i+1}^t + V_{i-1}^t I_{i-2}^t \cdots I_{i+1}^t + \dots + V_{i-1}^t V_{i-2}^t \cdots V_{i+1}^t}$$

In particular, the dpToda flow is highly non-linear.

§3. Linearization of dpToda flow

<u>Idea</u>: Integrable flows are linearized on Jacobian of spectral curve.

<u>Better coordinates</u>: (Mumford - van Moerbeke) (~Lax formalism)

Define periodic Jacobi matrices (with spectral parameter z)

$$R^{an} \hookrightarrow \mathrm{Jac}_n^{per} = \left\{ L_t(z) = M_t(z) R_t(z) \;\middle|\; \begin{array}{l} M_t(z) = \begin{bmatrix} 1 & & & V_n^t/z \\ V_1^t & \ddots & 0 & \\ & \ddots & \ddots & \\ 0 & & V_{n-1}^t & 1 \end{bmatrix} \\ R_t(z) = \begin{bmatrix} I_1^t & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ z & & & I_n^t \end{bmatrix} \end{array} \right\}$$

$$L_t(z) = \begin{bmatrix} I_1^t + V_n^t & 1 & & & I_n^t V_n^t / z \\ I_1^t V_1^t & I_2^t + V_1^t & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & & \\ z & & & I_{n-1}^t V_{n-1}^t & I_n^t + V_{n-1}^t \end{bmatrix}$$

tridiagonal

+ two off-diagonal entries

factors switched ← NCG!

dpToda flow $\iff$ $\boxed{L_{t+1}(z) = R_t(z) M_t(z)}$

(fundamental matrix equation)

$\iff$ $L_{t+1}(z) = R_t(z) L_t(z) R_t(z)^{-1}$ (isospectral)

Spectral curve

$\mathrm{Jac}_n^{\mathrm{pes}}$

$\Downarrow$

(characteristic polynomial)

$L_t(z) \longmapsto \underbrace{C : z^2 + h(x) z - f(x)}_{\text{spectral curve}} = (-1)^{n-1} z \, |L_t(z) - x \mathbb{1}_n| = 0$

C is a real hyperelliptic curve ("Toda curve"), i.e., has 2 pts at ∞.

C does not depend on t (isospectral).

C is birational to $\hat{C} : y^2 - h(x)^2 - 4f(x) = 0$

$\big((x,z) \mapsto (x, 2z + h(x)) = (x,y) \big)$

$\rightsquigarrow$ C has genus $\boxed{g = n-1}$

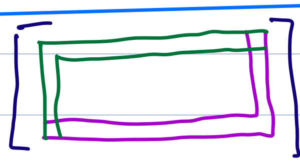
More explicitly:

$f(x) = f = -I_1 V_1 \cdots I_n V_n$ ← (does not depend on t! isospectral)

$h(x) = (I_n^t + V_{n-1}^t - x) \, u(x) - v(x) - w(x)$

$= (-1)^{n-1} \left[(I_n^t + V_{n-1}^t - x) \, |{}^0\mathcal{L}_1| - I_n^t V_n^t \, |{}^1\mathcal{L}_1| - I_{n-1}^t V_{n-1}^t \, |{}^0\mathcal{L}_2| \right]$

Notation: $\boxed{\mathcal{L} = L_t(z) - x \mathbb{1}_n} \in \mathbb{R}^{n \times n}$

$\mathbb{R}^{n-l-k \times n-l-k} \ni {}^{k}h_{\ell} =$ 

Delete first k rows + columns

Delete last ℓ rows + columns

Mumford's algebraic description of $\mathrm{Jac}(C)$

(Heavily used in cryptography)

Def: Let $C: z^2 + h(x)z - f = 0$ be a Toda curve

$$\mathrm{Jac}_{\text{Mumford}}(C) = \left\{ ([P,Q], d) \;\middle|\; \begin{array}{l} P \in \mathbb{R}[x] \text{ monic}, \deg P \leq g \\ Q \in \mathbb{R}[x]/(P), \; \deg Q < \deg P \\ \boxed{P \mid Q^2 + h(x)Q - f} \\ d \in \mathbb{N}, \text{ numerical conditions} \ldots \end{array} \right\}$$

(We will ignore param. d in this talk.)

Example: $\sum_{i=1}^{g} (x_i, z_i)$ divisor on C (without multiplicities)

$\updownarrow$

$[P,Q] \in \mathrm{Jac}_{\text{Mum}}(C)$ s.th.

$P(x) = \prod_{i=1}^{g} (x - x_i)$

$Q(x_i) = z_i \quad \forall i \in \{1, \ldots, g\}$

Surprisingly, the addition law $\boxplus$ on $\mathrm{Jac}_{\text{Mum}}(C)$ is given by Gauß' composition law for quadratic forms $/\mathbb{R}$.

Reminder: Inspired by Bhargava's famous work on Gauß composition $/\mathbb{Z}$, one has the following generalization:

Thm: (Wood '11)

$$\mathrm{PrimQuad}(R,D) \overset{1:1}{\longleftrightarrow} \mathrm{Pic}\left(R[y]/(y^2-D)\right)$$

$$aX^2+2bXY+cY^2 \mathrel{\hat{=}} [a,2b,c] \longmapsto (a, y-b)$$

In our set-up: $a=P(x)$, $b=Q(x)$, i.e., $[a,b]\in \mathrm{Jac}_{Mum}(y^2-D=0)$

More concretely Gauß composition consists of two steps:

1) Composition step $+_c$

2) Reduction step $\boxplus$

In our framework, we have the following recipe:

Let $[P_1,Q_1],[P_2,Q_2]\in \mathrm{Jac}_{Mum}(C)$.

1) $[\tilde{P},\tilde{Q}] = [P_1,Q_1]+_c[P_2,Q_2]$

Find $S\in R[x]$ monic, $f_1,f_2\in R[x]$, s.th.

$$S=\gcd(P_1,P_2)=f_1P_1+f_2P_2$$

Then $[\tilde{P}, \tilde{Q}] = [P_1 P_2 / s^2 \, , \, (f_1 P_1 Q_2 + f_2 P_2 Q_1)/s \bmod \tilde{P}]$

Composition

2) $[P, Q] = [P_1, Q_1] \boxplus [P_2, Q_2]$

$$= \left[(\tilde{Q}^2 + h\tilde{Q} - f)/\tilde{p} \text{ made monic}, \, -(\tilde{Q} + h) \bmod P \right]$$

Reduction

(Shows the genius of Gauß!)

Ex: • $[0] = ([1, 0], 0) \in \mathrm{Jac}_{\mathrm{Mum}}(C)$ (ignoring d)

• $-[P, Q] = [P, \, -(h + Q) \bmod P] \in \mathrm{Jac}_{\mathrm{Mum}}(C)$

Last ingredient: eigenvector map Ψ

Denote $\mathcal{L} = \{ \mathcal{L}_t(z) \}_{t \in \mathbb{N}_0}$ (orbit under $\mathcal{T}$)

Def: The eigenvector map Ψ is defined by

$$\Psi : \mathcal{L} \longhookrightarrow \mathrm{Jac}_{\mathrm{Mum}}(C)$$

$$\mathcal{L}_t(z) \longmapsto [u(x), v(x)]$$

$$= \left[(-1)^{n-1} \left| {}^0\mathcal{L}_1 \right| , \, (-1)^{n-1} I_n^t v_n^t \left| {}^1\mathcal{L}_1 \right| \right]$$

deg = n-1 deg = n-2

Lemma: Ψ is well-defined and injective

Proof: Main point $u \mid v^2 + hv - f$ follows from "tridiagonal" structure of $L_t(z)$.

Rm: $[u(x), v(x)]$ encodes eigenvector of $L_t(z)$ with eigenvalue x !

Finally, our main

Theorem: (Y.) Define $\underline{D = \big([\,x, \overbrace{(-1)^n I_1 \cdots I_n}^{\text{does not depend on } t}\,], 2\big)} \in \mathrm{Jac}_{\mathrm{Mum}}(C)$.

Then the diagram

$$
\begin{tikzcd}
\mathcal{L} \arrow[r, hook, "\Psi"] \arrow[d, "\mathcal{J}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Jac}_{\mathrm{Mum}}(C) \arrow[d, "\cdot \boxplus D"] \\
\mathcal{L} \arrow[r, hook, "\Psi"] & \mathrm{Jac}_{\mathrm{Mum}}(C)
\end{tikzcd}
$$

commutes.

i.e.,
$$\boxed{\mathcal{J}^n\big(\underbrace{L_0(z)}\big) = \Psi\big(L_0(z)\big) \boxplus n \cdot D.}$$
$$\qquad\quad \overset{\shortparallel}{L_n(z)}$$

Rm: 1) This give a new and completely algebraic linearization of dpToda flow.

2) Earlier linearizations (Iwao '08) have a more analytic flavour and use the theory of Θ-fcts on $\mathrm{Jac}(C)$. (Difficult to use with computers. Magma contains $\mathrm{Jac}_{\mathrm{Mum}}(C)$)

Proof: (Sketch)

Need to show

$$\Psi(\mathfrak{T}(I^t, V^t)) = \Psi(I^t, V^t) \boxplus D$$

$$\Psi(R_t(z) M_t(z)) \qquad\qquad [u(x), v(x)] \boxplus D$$

$$[\underline{u}(x), \underline{v}(x)] \qquad\qquad [\bar{u}(x), \bar{v}(x)]$$

$$\boxed{\begin{aligned}[\hat{u}, \hat{v}] &= [u(x), v(x)] \underline{+_C} D \\ &= [x\, u(x), -I_n^t u + v]\end{aligned}}$$ Composition

$$\boxed{\begin{aligned}\bar{u}(x) &= (\tilde{v}^2 + h\tilde{v} - f) / I_n^t \tilde{u} \\ \bar{v}(x) &= -(\tilde{v} + h) \bmod \bar{u}\end{aligned}}$$ Reduction

Enough to show $\boxed{\underline{u} \mid \tilde{v}^2 + h\tilde{v} - f}$.

Follows from key identity $\boxed{\tilde{v} = -I_n^t u + v = -I_n^t \underline{u} + \underline{w}}$, which follows from induction on n. $\square$

$\Big[$ Recall $h(x) = (I_n^t + V_{n-1}^t - x)\, u - v - w \quad \sim L_t(z) = M_t(z) R_t(z)$

invariant under $\mathcal{J}$

$= (I_n^t + V_n^t - x)\, \underline{u} - \underline{v} - \underline{w} \Big] \quad \sim L_{t+1}(z) = R_t(z) M_t(z)$

(nice interplay)

Rm: For the first time dpToda flow related to Gauß composition. Quite interesting how well they intersect...

§4. pBB ⊂ dpToda

$\exists$ map $\eta : C_N \hookrightarrow \overset{G \curvearrowright}{\mathbb{N}^{2n}} \hookrightarrow \mathbb{Q}(T)^{2n} \curvearrowleft G$

(well-defined)

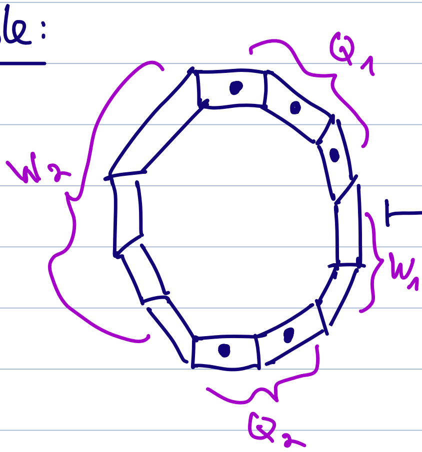
pBB state space — tropical periodic Toda — discrete periodic Toda

$\mathcal{B}^t(\alpha) = \alpha^t \longmapsto (Q_1^t, \dots, Q_n^t, W_1^t, \dots, W_n^t) \longmapsto (I_1^t, \dots, I_n^t, V_1^t, \dots, V_n^t)$

$Q_i^t \mapsto T^{Q_i^t}$

$W_i^t \mapsto T^{W_i^t}$

Example:



$\longmapsto ([3,2],[2,4]) \longmapsto ([T^3, T^2],[T^2, T^4])$

$N = 11$ $\quad n = 2$ $\quad n = 2$

[Rm: n depends on α !]

But invariant under $\mathcal{B}$!

Observation: (Inoue-Kuniba-Takagi '12)

$$\begin{tikzcd} \mathcal{C}_N \arrow[r, hook, "\eta"] \arrow[d, "\mathfrak{Z}"'] & \mathbb{Q}(T)^{2n} \arrow[d, "\mathfrak{T}"] \\ \mathcal{C}_N \arrow[r, hook, "\eta"] & \mathbb{Q}(T)^{2n} \end{tikzcd}$$

does NOT commute in general!

But: Define cyclic shift operator

$$\sigma : (I_i^t, V_i^t) \mapsto (I_{i+1}^t, V_{i+1}^t)$$ (indices mod n)

Lemma: Spectral curve C is invariant under $\sigma \curvearrowright L_t(z)$.

Prop: (IKT '12 + Y.)

The diagram

T-adic valuation

$$\begin{tikzcd} \mathcal{C}_N \arrow[r, hook, "\eta"] \arrow[d, "\mathfrak{Z}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Q}(T)^{2n}/\sim_\sigma \arrow[r, "\nu_T"] \arrow[d, "\mathfrak{T}"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{N}^{2n}/\sim_\sigma \arrow[d, "\mathfrak{T}_{trop}"] \\ \mathcal{C}_N \arrow[r, hook, "\eta"] & \mathbb{Q}(T)^{2n}/\sim_\sigma \arrow[r, "\nu_T"] & \mathbb{N}^{2n}/\sim_\sigma \end{tikzcd}$$

commutes.

Question:

How does the cyclic action σ act on $\mathrm{Jac}_{\mathrm{Mum}}(C)$?

Define the subgroup

n-torsion part of Jac

$$Z_n = \langle ([1,0],2) \rangle \leq \mathrm{Jac}_{\mathrm{Mum}}(C)[n].$$

("generated" by action of σ on $\mathrm{Jac}_{\mathrm{Mum}}(C)$)

Then we have

Theorem: The following diagram is commutative

$$\begin{tikzcd}
C_N \arrow[r, "\psi \circ \eta"] \arrow[d, "\mathfrak{B}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Jac}_{\mathrm{Mum}}(C)/Z_n \arrow[d, "\cdot \boxplus D"] \\
C_N \arrow[r, "\psi \circ \eta"] & \mathrm{Jac}_{\mathrm{Mum}}(C)/Z_n
\end{tikzcd}$$

i.e., pBB flow can be expressed via Gauß composition for quadratic forms!
(up to n-torsion)

§5. Outlook (or a dream)

Recently, Devalapurkar (2404.09853) studied recent work of Ben-Zvi-Sakellaridis-Venkatesh on "relative geometric Langlands" in the "non-group" set-up and observed that in the "simplest" example $(PGL_2^{\times 3}/PGL_2^{\mathrm{diag}})$ (spherical varieties)

Bhargava's reformulation of Gauß composition of quadratic forms via "Bhargava cubes" appears naturally.

Very roughly speaking, he observed that

Gauß composition (à la Bhargava) (a weighted version or derived)

$\Big\updownarrow$ derived geometric Satake (Bezrukavnikov–Finkelberg '08)

tensor product on $Sh^*_{PGL_2[[t]]}(Gr_{PGL_2}; \mathbb{Q})$

certain sheaves on affine Grassmannian of PGL_2

Our hope/dream: Our set-up can be lifted to the derived/weighted set-up of derived geom. Satake

non-trivial

IF POSSIBLE $\rightsquigarrow$ Completly new description of pBB & dpToda flows!

Thank you very much
for your interest and patience !